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Classification of Holomorphic Functions as Pólya Vector Fields via Differential Geometry

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Abstract: We review Pólya vector fields associated to holomorphic functions as an important pedagogical tool for making the complex integral understandable to the students, briefly mentioning its use in other dimensions. Techniques of differential geometry are then used to refine the study of holomorphic functions from a metric (Riemannian), affine differential or differential viewpoint. We prove that the only nontrivial holomorphic functions, whose Pólya vector field is torse-forming in the canonical geometry of the plane, are the special Möbius transformations of the form $f(z) = b(z + d)^{-1}$. We define and characterize several types of affine connections, related to the parallelism of Pólya vector fields. We suggest a program for the classification of holomorphic functions, via these connections, based on the various indices of nullity of their curvature and torsion tensor fields.

Keywords: holomorphic functions; Pólya vector fields; Möbius transformations; torse-forming vector fields; harmonic functions

1. Introduction

Student's learning of Complex Analysis benefits greatly from physical interpretations, as advocated from the "begining" by Felix Klein [1] when writing about the corresponding work done by Riemann. Klein's starting point was to interpret a complex function $F(z)$, with its Cartesian components harmonic functions, as a *complex potential*, and then differentiate to get the 1-form $F'(z)dz$ in order to look at its pair of orthogonal distributions, determining the flows of the associated vector fields.

On the other hand, with the integration tool in it, Pólya associated to a complex function $f(z) = u + iv$ a vector field $V_f = <u, -v>$, which can be interpreted as the velocity field of a fluid flow, and then consider its integrability, in order to find such complex potentials. His purpose was definitely pedagogical in nature, to help students make sense of complex integrals [2,3].

Indeed, the function $f(z)$ has also a (canonically associated) 1-form $f(z)dz$, and the two components of its complex integral represent the work and flux of the Pólya vector field [3]:

$$f(x + iy) = u(x,y) + iv(x,y), \quad \int_C f(z)dz = \int_C V_f \cdot \vec{T}ds + i\int_C V_f \cdot \vec{N}ds,$$

where $\vec{T}$ and $\vec{N}$ are the tangent, and respectively the normal associated to the integration contour.

This two starting points are dual, in the sense differentiation and integration are "inverse" operations, related by the Fundamental Theorem of Path Integrals; in physics terms, it relates vector fields and potentials.

$$\underline{Potentials} \qquad\qquad \underline{VectorFields/DifferentialForms}$$

$$\text{``Klein''} \underset{d}{\overset{\int}{\leftrightarrows}} \text{``Pólya''}$$

$$F(z) \xrightarrow{\hspace{3cm}} F'(z) = f(z) = <u,v>$$

$$\left\| \qquad\qquad \textit{F. Th. Line Integrals} \qquad\qquad \downarrow \right.$$

$$\int_C f(z)dz \xleftarrow{\hspace{1.5cm}\int\hspace{1.5cm}} V_f = <u(x,y), -v(x,y)>$$
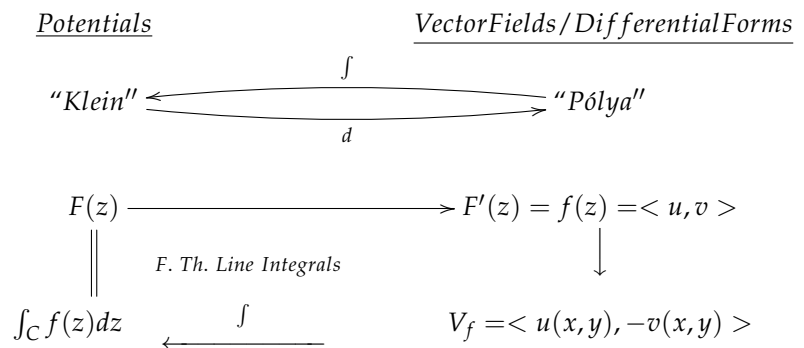
In this article, we review the above formalism as a bridge between real or complex analytic tools and some differential geometric tools on the plane $\mathbb{R}^2$. Our main underlying purpose is to use the later, in order to build the setting for future classifications of specific complex objects, for example the holomorphic functions. Secondly, we make further steps in deciphering the dynamical meaning of some properties of holomorphic functions (beyond the previous diagram), using invariants provided by Differential Geometry.

In Section 2, we consider the Möbius transformations and their associated Pólya vector fields and we try to unravel some physical interpretations behind this correspondence. Beyond their intrinsic role in the geometries of the plane, the Möbius transformations are quite appropriate as test functions in the big family of holomorphic functions [4,5]. For arbitrary complex functions, we define a specific new correlation, in terms of the Lie bracket of their associated Pólya vector fields; a new integer invariant provides a classification of the holomorphic functions (Remark 6).

In Section 3, as a new point, we document Klein's approach with its connection with Pólya's approach, together with the *symplectic geometry* interpretation: *the Pólya vector field is the Hamiltonian vector field associated to its (local) complex potential*, in Section 4. The fact that *Complex Geometry* is intertwined with *Symplectic Geometry* comes as no surprise, as in our case the complex plane has both structures: it is a *Kähler manifold* (a complex manifold with a Hermitean metric $<,>= g + i\omega$, which is essentially a pair of a real metric and a symplectic form).

Some techniques of differential geometry for the classification of the Pólya vector fields, hence the underlying analytic and meromorphic functions, from a global point of view, are developed in Section 4. We prove that the only nontrivial holomorphic functions, whose Pólya vector field is torse-forming in the canonical geometry of the plane, are the special Möbius transformations of the form $f(z) = \frac{b}{z+d}$. The torse-forming property of vector fields was intensively studied since [6], as a relaxation of the properties of parallelism, concurrency and/or recurrency; it is associated to motions of solenoidal-like type in Riemannian or affine differential dynamical models.

Then we define and characterize several types of affine connections, related to the parallelism of Pólya vector fields. We suggest a program for the classification of holomorphic functions, via these connections, based on the various indices of nullity of their curvature and torsion tensor fields. This is part of a more general theory, started in [7–9], which geometrizes the vector fields through families of associated affine connections or semi-Riemannian metrics. These families are differential invariants, not only affine differential or metric ones; moreover, refinements can be made, by using derived geometric objects such as curvature tensor fields, torsion tensors fields and their contracted tensor fields.

In the concluding Section 5, summarizing the results, we point towards a natural extension of this work in the context of hyperKähler manifolds.

2. Pólya Vector Fields Associated to Complex Functions

Consider a complex function $f : \mathbb{C} \to \mathbb{C}$, $f = f(z)$, with $z = x + yi$ and $f(x, y) = u(x, y) + v(x, y)i$. Here u and v are the real and the imaginary parts of f. When f is defined only on a domain $D(f)$ in $\mathbb{C}$, formulas must be adapted accordingly.

Definition 1. *The Pólya vector field associated to the complex function f is $V_f = u\partial_x - v\partial_y$, or just simply denoted $< u, -v >$ as a vector attached to the point (x, y) of the plane R^2, in its canonical basis.*

As mentioned in the introduction, Pólya's idea was to make use of vector calculus, to give a more intuitive interpretation of the associated complex integral.

Remark 1. *(i) Reiterating this approach, we note that the Pólya vector field in 1D, i.e., the values $f(x)$ of a function viewed as a displacement $<f(x)>$ attached to the point $x \in R$, is perhaps a good "preamble" for the student to be ready for the complex integral case: what is a vector field on the line, what work is, etc.*

More importantly, it shows how functions of one variable can be graphed using just one dimension, although we usually graph them in the plane, i.e., in fact representing the relation $\{(x, y)|x \in D(f), y = f(x)\}$. This would be justified for general relations which might happen not to be functions, e.g., circle, while in what regards complex functions, fails due to the "lack" of enough dimensions!

If f is holomorphic, then it satisfies the Cauchy-Riemann equations $u_x = v_y$ and $u_y + v_x = 0$. In this case, the Cauchy Integral Theorem ensures, interpreted in the sense of Pólya, that V_f is solenoidal (i.e., divergence-free) and irrotational (i.e., its curl is identical zero).

(ii) Note that from the point of view of conformal *geometry, "all Pólya vector fields are created equal": there is a conformal transformation which "straightens the flow", i.e., transforms the flow V_f into the standard flow V_1 (horizontal, from left to right, with normalized speed 1); the global aspects are left aside, being associated with critical points/zeros and poles of the complex function.*

(iii) One can loosely reverse the direction of this "magnifying/classifying" instrument, loosely saying, for Cosmology oriented readers, that the "free fall" (harmonic flow in the original conformal class of metric) is analyzed in a "gravitational field" (change of frame), to "see how it bends".

Remark 2. *Consider the Möbius transformation (MT) $f(z) = (az + b)/(cz + d)$ where the complex numbers $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$ and $d = d_1 + id_2$ satisfy $ad - bc \neq 0$. Denote $z = x + iy$ and define the real valued functions*

$$A := a_1x - a_2y + b_1 \ , \ B := a_1y + a_2x + b_2,$$

$$C := c_1x - c_2y + d_1 \ , \ D := c_1y + c_2x + d_2.$$

Then $u(x, y) := Ref(x, y)$ and $v(x, y) := Imf(x, y)$ can be expressed as

$$u(x, y) = \frac{AC + BD}{C^2 + D^2}, \ \ v(x, y) = \frac{BC - AD}{C^2 + D^2} \ .$$

The associated Pólya vector field V_f has the components rational functions of x and y. As u and v are harmonic functions, we can derive the potential Φ of V_f, such that $\nabla\Phi = V_f$. Namely, from $\frac{\partial\Phi}{\partial x} = u$ and $\frac{\partial\Phi}{\partial y} = -v$, we get $\Phi = \Phi(x, y)$. Particular cases (omitting an obvious additive constant):

(i) $f(z) = z + b$ *provides* $\Phi(x, y) = \frac{x^2}{2} - \frac{y^2}{2} + b_1x - b_2y$;

(ii) $f(z) = az$ *provides* $\Phi(x, y) = a_1(\frac{x^2}{2} - \frac{y^2}{2}) - a_2xy$;

(iii) $f(z) = \frac{1}{z+d}$ *provides* $\Phi(x, y) = \frac{1}{2}ln((x + d_1)^2 + (y + d_2)^2)$;

(iv) *If $c = 0$, we may suppose $d = 1$; then* $\Phi(x, y) = a_1(\frac{x^2}{2} - \frac{y^2}{2}) - a_2xy + b_1x - b_2y$;

(v)   *If $a = 0$, we may suppose $b = 1$; then* $\Phi(z) = \frac{1}{c}\log(cz + d)$.

But wait! While this is a quiz opportunity for Calculus I students, why not integrate the simple fraction to get the *complex potential*, using a CAS for instance. (There is no question mark here; this is just a rhetorical question !):

$$Wolfram\ Alpha\ In : Integral\ of\ (a*z+b)/(c*z+d),$$

$$Out : (acz + (bc - ad)log(d + cz))/c^2,$$

since we are not needing the Pólya interpretation at this point.

While the complex potential function of a Möbius transformation is easy to compute, it might be a qualitative understanding, "à la Klein" that we are after. The level curves (equipotentials) are closed curves, of "magnetic type", obtained by conjugation via $g(z)$ (conformal transformation) of the multiplier M_z with the associated characteristic constant k of f; this will be explained in what follows.

2.1. The Klein/Physical Interpretation of Mobius Transformations

It is well known that any MT with distinct fixed points z_1, z_2 is conjugate via the transformation $g(z) = (z - z_1)/(z - z_2)$, which "straightens" the fixed points to antipodal positions, the North and South poles, if viewed via the stereographic projection [10]:

$$g(z) := \frac{z - z_1}{z - z_2}\ ,\ gfg^{-1} = M_k\ ,\ M_k(z) = kz\ ,\ f'(z_1) = k = 1/f'(z_2) \in\ C^*.$$

Now conjugation is not only an MT, but also has nice algebraic properties, revealed by the following diagram:

$$C^* \underset{Diag.}{\overset{Conj((z_1,z_2)\to(N,S))}{\rightleftarrows}} SL(2,C).$$
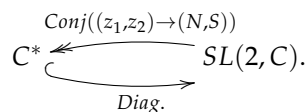
The *characteristic constant* $k = re^{i\theta}$ of f is the amplitwist at the fixed points: amount of rotation θ and scaling r.

The Pólya VF of the multipliers M_k have meridians and parallels as streamlines and equipotentials. By conjugation, the "electric charges" at the origin and infinity are mapped conformally to the fixed points of the non-parabolic MT f.

Note that, once the Pólya interpretation is applied, the electric-magnetic/harmonic and harmonic conjugation rotational symmetry by $90°$ is "broken": the streamlines join the fixed points, which now play the roles of electric charges ("ends of line fields"), while the equipotentials circling them, play the role of magnetic lines of "force" (closed line fields).

The "EM force field" interpretation is slight abuse of language, since the equations we are dealing with are first order, suited for the velocity field of a fluid flow interpretation; yet it can be "forced" in this context, if regarding the plane as a symplectic manifold T^*R, as mentioned elsewhere.

Remark 3. *One may define an "electric" and "magnetic" charge at this point, by comparison with Coulomb's law and Ampere's law. Moreover, one can relate the conjugation to* rapidity *in a 1 + 1 relativistic model, to visualize how a 2D-Lorentz transformation (in the T^*R symplectic framework), "transforms" the pure Coulomb electric force into a combined electric-and-magnetic force field.*

2.2. Correlated Complex Functions

We denote $\mathcal{POL}$ the (real) vector space of Pólya vector fields associated to complex functions and $\mathcal{POLH}$ its vector subspace of Pólya vector fields associated to holomorphic functions.

While $\mathcal{POL}$ is a Lie algebra with respect to Lie bracket, the subspace $\mathcal{POLH}$ is not. Let us transfer the Lie bracket on the functions themselves, and then compare with the Poisson bracket later on.

Let f and h be two complex functions and denote

$$f * h := \bar{f} h_z - \bar{h} f_z + f h_{\bar{z}} - h f_{\bar{z}} \, ,$$

where the indices mean derivative w.r.t. z and the overline is complex conjugation, as usual. It is easy to check that we have

$$[V_f, V_h] = V_{f*h} \, . \tag{1}$$

This formula allows to transport the Lie bracket on vector fields to a bracket on complex functions, and the Pólya vector field map becomes a Lie algebra homomorphism:

$$V : (\mathcal{COM}, *) \to (\mathcal{X}(\mathbb{R}^2), [,]).$$

Unfortunately, it does not induce a Lie algebra isomorphism between $(\mathcal{POL}, [,])$ and $(\mathcal{HOL}, *)$. Indeed, if moreover f and h are holomorphic functions, then

$$f * h := \bar{f} h_z - \bar{h} f_z,$$

may not be holomorphic as well.

In all the previous considerations, f and h are defined on the whole complex plane; in particular, if they are also holomorphic, then they are entire functions. In case f and h are defined on smaller subsets, the theory must be refined accordingly.

Remark 4. *The above operation $*$ is considered separately with respect to the two independent variables x, y:*

$$\{f, g\}_x = f g_x - g f_x \, , \quad \{f, g\}_y = f g_y - g f_y \quad ,$$

as if $C = T^* R$ is a symplectic space and $x = q, y = p$ are conjugate variables.

Remark 5. *Consider the differential equation $dy/dx = P(x, y)/Q(x, y)$. If it is integrable, then it can be reduced to the form [11]:*

$$dy/dx = \frac{F G_x - G F_x}{F G_y - G F_y} = \frac{\{F, G\}_x}{\{F, G\}_y},$$

expressible in terms of the above 1D-Poisson bracket. The associated distribution is the kernel of the 1-form $P(x, y)dx - Q(x, y)dy$, familiar to the students of Calculus III when introduced to line integrals and Green's Theorem. The relation to the holomorphic distributions studied loc. cit. will be addressed elsewhere.

Example 1. *As simple examples, we mention: $f * f = 0$ (obviously); $f * \bar{f} = -2i Im(f f_z + \bar{f} f_{\bar{z}})$.*

2.3. Correlated Holomorphic Functions

Consider two complex differentiable functions f and $\tilde{f}$. Their Poisson bracket is defined by

$$\{f, \tilde{f}\} = f_x \tilde{f}_y - f_y \tilde{f}_x,$$

or, equivalently,

$$\{f, \tilde{f}\} = -2i(f_z \tilde{f}_{\bar{z}} - f_{\bar{z}} \tilde{f}_z) \quad .$$

When f and $\tilde{f}$ are holomorphic, $\{f, \tilde{f}\} = 0$.

Definition 2. *We say the holomorphic functions f and $\tilde{f}$ are correlated if their Pólya vector fields commute via the Lie bracket, i.e., $[V_f, V_{\tilde{f}}] = 0$.*

Proposition 1. *Consider two holomorphic functions $f(x,y) = u(x,y) + v(x,y)i$ and $h(x,y) = \tilde{u}(x,y) + \tilde{v}(x,y)i$. Then the following assertions are equivalent: (i) f and h are correlated; (ii) $f * h = 0$; (iii) $\overline{f}h' = f'\overline{h}$; (iv) the following system of PDEs is satisfied:*

$$u\tilde{u}_x - v\tilde{u}_y = \tilde{u}u_x - \tilde{v}u_y, \quad u\tilde{u}_y + v\tilde{u}_x = \tilde{u}u_y + \tilde{v}u_x.$$

Remark 6. *The set $\mathcal{V}^f$ of Pólya vector fields of the holomorphic functions correlated with a given function f is a vector subspace of $\mathcal{POL}$.*

The dimension of $\mathcal{V}^f$ classifies the holomorphic functions in a family of classes, indexed with the set of positive integers.

2.4. The Canonical Frame Associated to a Complex Function

We shall suppose f (respectively V_f) is a nowhere zero complex function (respectively vector field). Denote by $\mathcal{D}_f$ the one-dimensional distribution spanned by V_f and by $\mathcal{D}_f^\perp$ its complementary (one-dimensional) orthogonal distribution. Then $T\mathbb{R}^2 = \mathcal{D}_f \oplus \mathcal{D}_f^\perp$ and we got a field of orthogonal frames depending on f. Obviously, $\mathcal{D}_f^\perp$ is spanned by $V_f^\perp := -V_{if} = v\partial_x + u\partial_y$. Alternatively and dually, $\mathcal{D}_f$ and $\mathcal{D}_f^\perp$ may be defined as kernels of the canonical associated one-forms

$$\alpha_f := vdx + udy, \quad \alpha_f^\perp := udx - vdy.$$

We may write

$$f(z)dz = \alpha_f^\perp + i\alpha_f \quad , \quad 2\overline{f}(z)\partial_z = V_f - iV_f^\perp.$$

Remark 7. *We may write the previous formulas in complex coordinates $(z, \overline{z})$ as well:*

$$V_f = f\partial_{\overline{z}} + \overline{f}\partial_z \quad , \quad iV_f^\perp = f\partial_{\overline{z}} - \overline{f}\partial_z,$$

$$2i\alpha_f = fdz - \overline{f}d\overline{z} \quad , \quad 2\alpha_f^\perp = fdz + \overline{f}d\overline{z}.$$

Remark 8. *The function f is holomorphic if and only if α_f and $\alpha_f^\perp$ are closed (i.e., if and only if V_f and $V_f^\perp$ are irrotational). Locally, or on simply connected domains, this is equivalent to α_f and $\alpha_f^\perp$ being exact (i.e., if and only if V_f and $V_f^\perp$ being gradients).*

For a holomorphic function f we have

$$[V_f, V_f^\perp] = V_{-2if_z\overline{f}}$$

and the same formula holds (unespectedly) for arbitrary complex functions also. It follows that $[V_f, V_f^\perp] = 0$ for (and only for) anti-holomorphic functions.

3. Hamiltonian Framework for Holomorphic Functions

Consider $\mathbb{R}^2$ with canonical coordinates (x, y) and the symplectic form $\omega = dx \wedge dy$. If $H : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, then its associated Hamiltonian vector field is $X_H := H_y\partial_x - H_x\partial_y$. This vector field is dual to the one-form dH (w.r.t. ω), via the relation $dH = \omega(X_H, \cdot)$.

3.1. Pólya Vector Fields as Hamiltonian Vector Fields

Theorem 1. *Suppose f is a holomorphic function. With the notations of the previous paragraph, denote F an integral of α_f, i.e., $dF = \alpha_f$. Then:*

(i)   The Pólya vector field is precisely the associated Hamiltonian vector field:

$$X_F := F_y \partial_x - F_x \partial_y = V_f.$$

(ii)   If $F^\perp$ is an integral of $\alpha_f^\perp$, then the associated Hamiltonian vector field $X_{F^\perp}$ is precisely $-V_f^\perp$.

Proof. The details are left for the reader to check.   □

Recall the relation to Poisson bracket:

$$\{F, G\} := F_x G_y - F_y G_x, \qquad [X_F, X_G] = X_{\{F,G\}}.$$

In terms of Pólya vector fields V_f (holomorphic case) and associated complex potentials $F' = d/dz(F) = f$:

$$[V_f, V_g] = [X_F, X_G] = X_{\{F,G\}} = V_{\{F,G\}'} = V_{\{f,G\}} + V_{\{F,g\}}.$$

Recall that, since $\{F, G\} = dF(X_G) = X_G(F)$, we also have, for example:

$$\{f, G\} = X_G(f) = V_g(f) = Re(g) f_x + iIm(g) f_y.$$

In view of the operation $*$, we further have:

$$f * g = \{f, G\} + \{F, g\},$$

which may be thought of as "integration by parts".

3.2. Hamiltonian Mechanics Interpretation

Recall that Pólya's interpretation of the complex integral was in terms of the flow of a fluid, and its circulation and flux. Now R^2 viewed as a symplectic space allows to interpret the Complex Analysis in terms of Hamiltonian mechanics, as follows.

Assume canonical coordinates $q = x$ and $p = y$ are chosen, to be more specific. Even more, being global, one can then interpret them as coordinates on T^*R, the cotangent bundle of the real line.

Now consider the complex potential $F(z)$ (locally always exists, for $f(z)$ holomorphic), as defining the Hamiltonian of a dynamical system.

Klein's interpretation of Cauchy-Riemann equations for $F(z) = U(x, y) + iV(x, y)$, which is by now a standard when teaching complex analysis, is that the harmonic function U is the velocity potential of a fluid flow, with gradient $\nabla U = V_f$, with its integral lines the streamlines and level curves of U the equipotential lines [1,12].

Alternatively, since $V_f = X_F$, the streamlines in $R^2 = T^*R$ may be interpreted as the Hamiltonian flow on the *real line*, with F the complexification of the energy.

This allows to provide an application of Complex Analysis to Newton's Mechanics in 1D, which apriori is a 2nd order differential equation framework.

4. Classifications of Pólya Vector Fields on the Complex Plane

Pairing a vector field X with a connection ∇, and yielding another vector field $\nabla_X Y$, can be compared with a "scattering process", or the application of a filter when studying and/or observing how a system "reacts" under a certain class of interactions. We will use this two ways: fixing a class of connections, deriving information regarding a class of vector fields; or the other way around, as follows.

4.1. Recurrent, Concurrent, Parallel and Torse Forming Pólya Vector Fields

We shall suppose f, hence V_f, is holomorphic and nowhere zero. The main result is the following.

Theorem 2. *Denote ∇^0 the Levi-Civita connection of the canonical Euclidean metric on the complex plane, identified with $\mathbb{R}^2$. Then, w.r.t. ∇^0, we have :*

(i) *V_f is recurrent iff V_f is parallel iff V_f is concurrent iff V_f is auto-parallel iff f is constant;*

(ii) *V_f is torse-forming iff f is a Möbius transformation of the form $f(z) = \frac{b}{z+d}$, with complex parameters b, d such that $d \neq 0$.*

Proof. (i) We calculate

$$\nabla^0_{V_f} V_f = (uu_x - vu_y)\partial_x + (vu_x + uu_y)\partial_y.$$

It is easy to see that V_f is auto-parallel if and only if f is a constant function. Similarly, we get the other equivalent properties.

We recall that: V_f is *recurrent* if there exists a 1-form α on $\mathbb{R}^2$ such that for every $X \in \mathcal{X}(\mathbb{R}^2)$

$$\nabla^0_X V_f = \alpha(X)V_f;$$

V_f is *concurrent* if there exists a differentiable function $h : \mathbb{R}^2 \to \mathbb{R}$ such that for every $X \in \mathcal{X}(\mathbb{R}^2)$

$$\nabla^0_X V_f = hX.$$

(ii) Suppose V_f is *torse forming*, i.e., there exists a differentiable function $h : \mathbb{R}^2 \to \mathbb{R}$ and a 1-form α on $\mathbb{R}^2$ such that for every $X \in \mathcal{X}(\mathbb{R}^2)$

$$\nabla^0_X V_f = \alpha(X)V_f + hX .$$

We calculate

$$u_x = h + \alpha_1 u \ , \ v_x = \alpha_1 v \ , \ u_y = \alpha_2 u \ , \ v_y = -h + \alpha_2 v.$$

It follows

$$\alpha_1 = -\frac{2hu}{u^2 + v^2} \quad , \quad \alpha_2 = \frac{2hv}{u^2 + v^2}$$

and u and v are solutions of an ODEs system of the form

$$u_x = -h \, (u^2 - v^2)(u^2 + v^2)^{-1}, \, v_x = -h \, 2uv(u^2 + v^2)^{-1} \tag{2}$$

(The right hand sides are just the cartesian components of the h-multiple of $-f^2/\bar{f}^2$.) We eliminate h in (2) and obtain

$$2uv \, u_x = (u^2 - v^2)v_x, \tag{3}$$

written as

$$\left(\frac{u^2 + v^2}{v}\right)_x = 0, \left(\frac{u^2 + v^2}{u}\right)_y = 0. \tag{4}$$

(Alternatively *Re* $\partial_x \, (f^3/\bar{f}^2) = 0$.) We integrate and obtain

$$u(x,y) = \frac{a^2(y)b(x)}{a^2(y) + b^2(x)} \, v(x,y) = \frac{a(y)b^2(x)}{a^2(y) + b^2(x)}, \tag{5}$$

satisfying $b^2 a_y + a^2 b_x = 0$.

As a and b do not depend on x and y, respectively, it follows that there exist real constants $\epsilon \in \mathbb{R}^*$ and $x_0, y_0 \in \mathbb{R}$ such that

$$a(y) = -\epsilon(y + y_0)^{-1} \ , \ b(x) = \epsilon(x + x_0)^{-1}.$$

We get

$$u(x,y) = \epsilon \frac{x + x_0}{(x + x_0)^2 + (y + y_0)^2},$$

$$v(x,y) = -\epsilon \frac{y + y_0}{(x + x_0)^2 + (y + y_0)^2}.$$

Replacing (x, y) with z, we get

$$f(z) = \frac{\epsilon}{z + z_0}.$$

We remark that multiplication with a complex constant does not change the properties of f, so we may consider $\epsilon \in \mathbb{C}^*$. □

Example 2. *(i) For $h(x, y) = (x^2 + y^2)^{-1}$ and $\alpha = -2h(x, y)(xdx + ydy)$, we conclude that the Pólya vector field associated to the function $f(z) = \frac{1}{z}$ is torse-forming. Here $u(x, y) := Ref(x, y) = x(x^2 + y^2)^{-1}$ and $v(x, y) := Imf(x, y) = -y(x^2 + y^2)^{-1}$.*

(ii) Consider the functions: $f(z) = sin(z)$ with $u(x, y) = sin(x)cosh(y)$ and $v(x, y) = cox(x)sinh(y)$; $f(z) = cos(z)$, with $u(x, y) = cos(x)cosh(y)$ and $v(x, y) = -sin(x)sinh(y)$; $f(z) = e^z$, with $u(x, y) = e^x cosy$ and $v(x, y) = e^x siny$. In all these cases, V_f is not torse-forming.

4.2. Connections and Metrics Associated to a Given Pólya Vector Field

Consider a nowhere vanishing holomorphic function f. We will use the Pólya vector field V_f as a filter, to distinguish the following classes of connections and metrics.

Denote by $\mathcal{C}(\mathbb{R}^2, V_f)$ the set of affine connections ∇ such that the trajectories of V_f are autoparallel w.r.t. them, i.e., $\nabla_{V_f} V_f = 0$.

Denote by $\mathcal{C}_{par}(\mathbb{R}^2, V_f)$ the set of affine connections such that V_f is parallel w.r.t. them, i.e., $\nabla_X V_f = 0$, for every vector field X on $\mathbb{R}^2$.

Denote by $Riem(\mathbb{R}^2, V_f)$ the set of Riemannian metrics such that the trajectories of V_f are geodesics w.r.t. them.

Remark 9. *If we think of the Pólya vector field's flow as "free falling" (incompressible and irrotational), then, in particular, this addresses the problem of finding the connections or the metrics admitting a gravitational field consistent with this particular type of "free-fall".*

Theorem 3. *With the previous notations, we have:*

(i) $\mathcal{C}(\mathbb{R}^2, V_f)$ *is the set of connections whose coefficients satisfy the following (compatible) algebraic system*

$$uu_x - vu_y + u^2\Gamma^1_{11} + v^2\Gamma^1_{22} - uv\Gamma^1_{12} - uv\Gamma^1_{21} = 0,$$

$$vu_x + uu_y + u^2\Gamma^2_{11} + v^2\Gamma^2_{22} - uv\Gamma^2_{12} - uv\Gamma^2_{21} = 0.$$

(ii) $\mathcal{C}_{par}(\mathbb{R}^2, V_f)$ *is the set of connections whose coefficients satisfy the following (compatible) algebraic system*

$$u\Gamma^1_{11} - v\Gamma^1_{12} = -u_x \ , \ u\Gamma^2_{11} - v\Gamma^2_{12} = v_x \ , \ u\Gamma^1_{21} - v\Gamma^1_{22} = -u_y \ , \ u\Gamma^2_{21} - v\Gamma^2_{22} = v_y.$$

Any such connection ∇ satisfies $div^\nabla V_f = 0$.

(iii) $Riem(\mathbb{R}^2, V_f)$ *is the set of Riemannian metrics whose coefficients form a positively defined matrix and satisfy the following (compatible) system*

$$ug_{11} - vg_{12} = U, \ ug_{12} - vg_{22} = V,$$

where U and V are two differentiable functions, solutions of

$$Uu - Vv = 1 \ , \ uU_x - vU_y = -u_xU + v_xV \ , \ uV_x - vV_y = -u_yU + v_yV$$

If, moreover, the metric $g \in Riem(\mathbb{R}^2, V_f)$ is Hermitian, then it is Kählerian, and of the form $g_{jk} = (u^2 + v^2)^{-1}\delta_{jk}$, with $u(x,y) = b(x)(a^2(y) + b^2(x))^{-1}$ and $v(x,y) = a(y)(a^2(y) + b^2(x))^{-1}$. Its Kähler potential $F = F(x,y)$ is a solution of the Poisson equation

$$\Delta F = 2a^2 + 2b^2,$$

where the Laplacian is calculated w.r.t. the Euclidean metric. The conformal transformation from the Euclidean metric to g is given by $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\phi(x,y) = (\alpha(x,y), \beta(x,y))$, where $\alpha + i\beta$ is holomorphic or anti-holomorphic and there exists a function $B = B(x,y)$ such that

$$d\alpha = \frac{1}{\sqrt{a^2 + b^2}}(\cos(B)dx + \sin(B)dy).$$

(iv)   *For a connection ∇, the property $div^\nabla V_f = 0$ is equivalent with*

$$u(\Gamma_{11}^1 + \Gamma_{21}^2) = v(\Gamma_{12}^1 + \Gamma_{22}^2)$$

and the property $div^\nabla V_f^\perp = 0$ is equivalent with

$$v(\Gamma_{11}^1 + \Gamma_{21}^2) = -u(\Gamma_{12}^1 + \Gamma_{22}^2).$$

Theorem 4. *Denote $h(x,y) = ln(u^2(x,y) + v^2(x,y))$. Then:*

(i)   *$\mathcal{C}(\mathbb{R}^2, V_f) \cap \mathcal{C}(\mathbb{R}^2, V_f^\perp)$ is the set of connections given by*

$$2(u^2 - v^2)\Gamma_{11}^1 = 2uv(\Gamma_{12}^1 + \Gamma_{21}^1) - (u^2 - v^2)h_x,$$

$$2(u^2 - v^2)\Gamma_{22}^1 = -2uv(\Gamma_{12}^1 + \Gamma_{21}^1) - (u^2 - v^2)h_x,$$

$$2(u^2 - v^2)\Gamma_{11}^2 = 2uv(\Gamma_{12}^2 + \Gamma_{21}^2) - (u^2 - v^2)h_y,$$

$$2(u^2 - v^2)\Gamma_{22}^2 = -2uv(\Gamma_{12}^2 + \Gamma_{21}^2) - (u^2 - v^2)h_y.$$

(ii)   *The symmetric connections in $\mathcal{C}(\mathbb{R}^2, V_f) \cap \mathcal{C}(\mathbb{R}^2, V_f^\perp)$ are given by*

$$2(u^2 - v^2)\Gamma_{11}^1 = 4uv\Gamma_{12}^1 - (u^2 - v^2)h_x,$$

$$2(u^2 - v^2)\Gamma_{22}^1 = -4uv\Gamma_{12}^1 - (u^2 - v^2)h_x,$$

$$2(u^2 - v^2)\Gamma_{11}^2 = 4uv\Gamma_{12}^2 - (u^2 - v^2)h_y,$$

$$2(u^2 - v^2)\Gamma_{22}^2 = -4uv\Gamma_{12}^2 - (u^2 - v^2)h_y.$$

(iii)   *Denote $D := 2(u^2 + v^2)^2$. Suppose, moreover, uv is nowhere vanishing. Then there exists a unique symmetric connection in $\mathcal{C}(\mathbb{R}^2, V_f) \cap \mathcal{C}(\mathbb{R}^2, V_f^\perp)$, which makes V_f and $V_f^\perp$ divergence-free; its coefficients are*

$$\Gamma_{11}^1 = (u^2 - v^2)\{-(u^2 - v^2)h_x + 2uvh_y\}D^{-1},$$

$$\Gamma_{22}^2 = (u^2 - v^2)\{-(u^2 - v^2)h_y - 2uvh_x\}D^{-1},$$

$$\Gamma_{12}^1 = (u^2 - v^2)\{(u^2 - v^2)h_y + 2uvh_x\}D^{-1},$$

$$\Gamma_{12}^2 = (u^2 - v^2)\{(u^2 - v^2)h_x - 2uvh_y\}D^{-1},$$

$$\Gamma_{22}^1 = \{-2uv(u^2 - v^2)h_y - [4u^2v^2 + (u^2 + v^2)^2]h_x\}D^{-1},$$

$$\Gamma_{11}^2 = \{2uv(u^2 - v^2)h_x - [4u^2v^2 + (u^2 + v^2)^2]h_y\}D^{-1}.$$

(This unique connection will be called the rectificant connection associated to f and will be denoted by $^f\nabla$.)

If $uv = 0$, then $\Gamma_{11}^1 = \Gamma_{22}^1 = -\frac{1}{2}h_x$, $\Gamma_{11}^2 = \Gamma_{22}^2 = -\frac{1}{2}h_y$ and Γ_{12}^1, Γ_{12}^2 are arbitrary.

Theorem 5. *There exists a unique connection in $\mathcal{C}_{par}(\mathbb{R}^2, V_f) \cap \mathcal{C}_{par}(\mathbb{R}^2, V_f^\perp)$; its coefficients are given by*

$$-\Gamma_{21}^2 = \Gamma_{22}^1 = \Gamma_{12}^2 = \Gamma_{11}^1 = -\frac{uu_x + vv_x}{u^2 + v^2} \;,\; -\Gamma_{22}^2 = \Gamma_{12}^1 = -\Gamma_{11}^2 = -\Gamma_{21}^1 = \frac{vv_y + uu_y}{u^2 + v^2}.$$

(This connection will be called the modular connection associated to the function f and will be denoted by ∇^f.)

Remark 10. *For any holomorphic function f, Theorem 5 associates a unique and canonical linear connection ∇^f, whose geometry is perfectly adapted to the behaviour of f. Moreover, the modular connection is simpler than the rectificant connection $^f\nabla$ and allows easier computations of the main geometric invariants. If we know ∇^f, then $| f |$ can be determined by integration and exponentiation, and hence f is determined locally, modulo a multiplicative constant.*

Further properties of this connection are addressed next. Denote $h(x,y) = \frac{1}{2}ln(u^2(x,y) + v^2(x,y))$. This means $Re[\ln f(z)]$, hence h, is a harmonic function.

(i)   The coefficients of the connection ∇^f may be written in a simplified form

$$-\Gamma_{11}^1 = -\Gamma_{22}^1 = -\Gamma_{12}^2 = \Gamma_{21}^2 = h_x \;,\; -\Gamma_{22}^2 = -\Gamma_{11}^2 = -\Gamma_{21}^1 = \Gamma_{12}^1 = h_y.$$

(ii) This connection ∇^f is symmetric if and only if f is constant. In this case, it coincides with the canonical affine connection ∇ of $\mathbb{R}^2$. Moreover, ∇ is precisely the symmetric connection associated to ∇^f.

(iii) The modular connection makes V_f and $V_f^\perp$ divergence-free.

(iv) The connection ∇^f is flat: this follows from $R^f(\partial_x, \partial_y)V_f = 0$ and $R^f(\partial_x, \partial_y)V_f^\perp = 0$. Alternatively, by direct computation, we get $R^f(\partial_x, \partial_y)\partial_x = (h_{xx} + h_{yy})\partial_y$, $R^f(\partial_x, \partial_y)\partial_y = -(h_{xx} + h_{yy})\partial_x$ and we use the harmonicity of h.

(v)   Its torsion is completely characterized by $T^f(\partial_x, \partial_y) = 2(h_y\partial_x - h_x\partial_y)$. The covariant derivative of T^f w.r.t. ∇^f is given by

$$(\nabla_{\partial_x}^f T^f)(\partial_x, \partial_y) = 2[h_{xy}\partial_x - (h_x^2 + h_y^2 + h_{xx})\partial_y],$$

$$(\nabla_{\partial_y}^f T^f)(\partial_x, \partial_y) = 2[(h_x^2 + h_y^2 + h_{yy})\partial_x - h_{xy}\partial_y].$$

We can rewrite the precedings formulas in invariant form. We remark that $T^f(X, Y) = 2dh(Y)X - 2dh(X)Y$ and $trace T^f = 2dh$. It follows that ∇^f is a semi-symmetric connection and that the Hessian of h w.r.t. ∇^f is symmetric, where $Hess_h^{\nabla^f}(X, Y) = (\nabla_X^f dh)(Y)$. Moreover

$$(\nabla_Z^f T^f)(X, Y) = 2Hess_h^{\nabla^f}(Z, Y)X - 2Hess_h^{\nabla^f}(Z, X)Y.$$

Moreover, higher order covariant derivatives of T^f reduce to higher order covariant derivatives of dh, as

$$(\nabla_{Z_1,...,Z_k}^{f,k} T^f)(X, Y) = 2(\nabla_{Z_1,...,Z_k}^{f,k} dh)(Y)X - 2(\nabla_{Z_1,...,Z_k}^{f,k} dh)(X)Y,$$

for any non-negative integer k and any vector fields $X, Y, Z_1, \ldots, Z_k$. Further calculations support the conjecture that some higher order covariant derivative of T^f vanishes if and only if the torsion tensor field vanishes (i.e., f is a constant function).

Example 3. *(i) For the system in Theorem 3, (i), a family of particular solutions is the following:*

$$\Gamma^1_{12} + \Gamma^1_{21} = 0 \ , \ \Gamma^2_{12} + \Gamma^2_{21} = 0.$$

$$\Gamma^1_{11} = \Gamma^1_{22} = \frac{vu_y - uu_x}{u^2 + v^2} \ , \ \Gamma^2_{11} = \Gamma^2_{22} = -\frac{vu_x + uu_y}{u^2 + v^2}.$$

Such a connection is symmetric if and only if $\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{12} = \Gamma^2_{21} = 0$, *so it is unique.*

(ii) For the system in Theorem 3, (ii), a particular solution is the following:

$$\Gamma^1_{11} = \frac{-uu_x}{u^2 + v^2} \ , \ \Gamma^1_{12} = \frac{vu_x}{u^2 + v^2} \ , \ \Gamma^2_{11} = \frac{uv_x}{u^2 + v^2} \ , \ \Gamma^2_{12} = \frac{-vv_x}{u^2 + v^2},$$

$$\Gamma^1_{21} = \frac{-uu_y}{u^2 + v^2} \ , \ \Gamma^1_{22} = \frac{vu_y}{u^2 + v^2} \ , \ \Gamma^2_{21} = \frac{uv_y}{u^2 + v^2} \ , \ \Gamma^2_{22} = \frac{-vv_y}{u^2 + v^2}.$$

We remark that this connection is allways non-symmetric.

(iii) For the system in Theorem 3, (ii), another particular solution is the following:

$$\Gamma^1_{11} = \frac{-u_x}{u} \ , \ \Gamma^1_{12} = 0 \ , \ \Gamma^2_{11} = \frac{-u_y}{u} \ , \ \Gamma^2_{12} = 0,$$

$$\Gamma^1_{21} = 0 \ , \ \Gamma^1_{22} = \frac{-v_x}{v} \ , \ \Gamma^2_{21} = 0 \ , \ \Gamma^2_{22} = \frac{-v_y}{v}.$$

We remark that this connection is symmetric. Moreover, $div^\nabla (V_f) = 0$.
We calculate

$$R(\partial_x, \partial_y)\partial_x = \left(\frac{u_{xy}u - u_x u_y}{u^2} - \frac{u_y v_x}{uv} \right)\partial_x + \left(\frac{u_{yy}u - (u_y)^2}{u^2} - \frac{u_y v_y}{uv} \right)\partial_y,$$

$$R(\partial_x, \partial_y)\partial_y = \left(-\frac{v_{xx}v - (v_x)^2}{v^2} + \frac{u_x v_x}{uv} \right)\partial_x + \left(-\frac{v_{yx}v - v_y v_x}{v^2} + \frac{u_y v_x}{uv} \right)\partial_y.$$

As $R(\partial_x, \partial_y)V_f = 0$, *we get the (curvature) index of nullity at least one. This index is two if and only if* $R(\partial_x, \partial_y)V_f^\perp = 0$, *i.e.,* R *identically vanish. This case is characterized by the following PDEs*

$$\frac{u_{xy}u - u_x u_y}{u} - \frac{u_y v_x}{v} = 0, \quad \frac{u_{yy}u - (u_y)^2}{u} - \frac{u_y v_y}{v} = 0,$$

$$-\frac{v_{xx}v - (v_x)^2}{v} + \frac{u_x v_x}{u} = 0, \quad -\frac{v_{yx}v - v_y v_x}{v} + \frac{u_y v_x}{u} = 0.$$

We skip the study of the solutions of this PDEs, which provides a family of (probably) interesting particular holomorphic functions, and we return to the general case.

The Ricci tensor field has the components

$$(Ric)_{11} = -\frac{u_{yy}u - (u_y)^2}{u^2} + \frac{u_y v_y}{uv}, \quad (Ric)_{22} = -\frac{v_{xx}v - (v_x)^2}{v^2} + \frac{u_x v_x}{uv},$$

$$(Ric)_{12} = \frac{v_{yx}v - v_y v_x}{v^2} - \frac{u_y v_x}{uv}, \quad (Ric)_{21} = \frac{u_{xy}u - u_x u_y}{u^2} - \frac{u_y v_x}{uv}.$$

In general, Ric is not symmetric: it happens if and only if

$$\left(\frac{v_y}{v} \right)_x = \left(\frac{u_x}{u} \right)_y.$$

(iv) Consider the particular case when $u(x,y) = x$ and $v(x,y) = y$. Then, for the system in Theorem 3, (iii), a particular solution is the following:

$$g_{12} = 0 \, , \, g_{11} = \frac{1}{2x^2} \, , \, g_{22} = \frac{1}{2y^2}.$$

Here, the intermediary functions are $U(x,y) = \frac{1}{2x}$ and $V(x,y) = -\frac{1}{2y}$.

(v) A particular solution for the case in Theorem 3, (iii): consider a Möbius transformation $f(z) = \frac{1}{cz+d}$, for arbitrary fixed real numbers c and d. Denote $a(y) = cy$ and $b(x) = cx + d$. Then the metric $g_{jk} = (a^2 + b^2)\delta_{jk}$ is Kähler and belongs to $Riem(\mathbb{R}^2, V^f)$.
These are the only Möbius transformations with this property. (Compare with Theorem 2, ii.)
(vi) Suppose $u \neq 0$. For the system in Theorem 3, (ii), the general solution is the following:

$$\Gamma_{11}^1 = \frac{v}{u}\Gamma_{12}^1 - \frac{u_x}{u} \, , \, \Gamma_{11}^2 = \frac{v}{u}\Gamma_{12}^2 + \frac{v_x}{u},$$

$$\Gamma_{21}^1 = \frac{v}{u}\Gamma_{22}^1 - \frac{u_y}{u} \, , \, \Gamma_{21}^2 = \frac{v}{u}\Gamma_{22}^2 + \frac{v_y}{u}.$$

This family of connections depends on four arbitrary parameter functions.
Suppose, moreover, that we impose the connections be symmetric. Then, the general solution is

$$\Gamma_{11}^1 = \frac{v^2}{u^2}\Gamma_{22}^1 - \frac{v}{u^2}u_y - \frac{u_x}{u} \, , \, \Gamma_{11}^2 = \frac{v^2}{u^2}\Gamma_{22}^2 + \frac{v}{u^2}v_y + \frac{v_x}{u}$$

$$\Gamma_{12}^1 = \frac{v}{u}\Gamma_{22}^1 - \frac{u_y}{u} \, , \, \Gamma_{12}^2 = \frac{v}{u}\Gamma_{22}^2 + \frac{v_y}{u}$$

This family of connections depends on two arbitrary parameter functions.

4.3. The Classification Project

In view of Theorems 3–5, we can classify the nowhere vanishing holomorphic functions, w.r.t. the following new invariants:

(Step 1.) Consider the linear connections and metrics associated to V_f, each one or their set as a whole.

(Step 2.) Consider their curvature tensor fields R^f and torsion tensor fields T^f, together with all their covariant derivatives.

(Step 3.) Then their various indices of nullity, like a huge barcode, will classify V_f, hence f.

This classification is similar to that of polynomial functions, by imposing some of their higher derivative to vanish. The difference is that, in our case, the choice of the (covariant) derivative is more flexible and subtle, and the direction of the classical derivative is replaced by a quiver of scattered directions. (For a similar but more general classification project, see [9].)

Example 4. *Consider a nowhere vanishing holomorphic function f, its Pólya vector field and ∇^f the associated modular connection, as in Theorem 5 and the Remark 9. This connection is flat, so all its indices of nullity associated to R^f and its covariant derivatives are maximal. Instead, the torsion tensor field or at least one the first two of its covariant derivatives are null for, and only for, constant functions f. We conjecture, as above, that if some higher order covariant derivative of T^f is null, then f is constant. Such a condition seems to be too strong for holomorphic functions which are rigid, so we suggest to appeal to nullity indices.*

For example: there exists a non-null vector field X such that $\nabla_X^f T^f = 0$ if and only if

$$h_{xy}^2 = (h_x^2 + h_y^2 + h_{xx})(h_x^2 + h_y^2 + h_{yy}).$$

As h is harmonic, this is equivalent to

$$(h_x^2 + h_y^2)^2 = h_{xx}^2 + h_{xy}^2. \tag{6}$$

Equation (6) is Monge-Ampère-like, because the right side is $(-\det \text{Hess}_h)$, where the Hessian is calculated with the canonical affine connection of $\mathbb{R}^2$.

In terms of the magnitude of the gradient $g := ||\text{grad } h||^2 = h_x^2 + h_y^2$ of the function $h = \frac{1}{2}\ln(u^2 + v^2)$ (Theorem 4), Equation (6) becomes

$$||\text{grad } g||^2 = 4g^3.$$

Geometrically, it follows that the vector field $\text{grad }(g^{-\frac{1}{2}})$ lies on the unit circle.

Note that the index of nullity for the covariant derivative of T^f is measured by a polynomial PDE of the gradient of h. It is quite plausible that the indices of nullity of higher order covariant derivatives of T^f are controlled by similar polynomial PDEs, of higher degrees.

A standard argument finds the solution of Equation (3):

$$h = \int C^{-1}\cos(\psi)dx + \int \{C^{-1}\sin(\psi) - \frac{\partial}{\partial y}[\int C^{-1}\cos(\psi)dx]\}dy + k_2,$$

where

$$C(x,y) = \int \cos(\varphi(x,y))dx + \int \{\sin(\varphi(x,y)) - \frac{\partial}{\partial y}[\int \cos(\varphi(x,y))dx]\}dy + k_1,$$

the functions $\varphi = \varphi(x,y)$ and $\psi = \psi(x,y)$ are arbitrary and k_1, k_1 are arbitrary real numbers. A particular solution is $h(x,y) = \frac{1}{2}\ln((x+a)^2 + (y+b)^2)$, for arbitrary real numbers a and b, corresponding to $|f(z)| = e^{\frac{1}{2}|z+a+ib|^2}$.

5. Conclusions

The use of Pólya's associating vector fields to functions, and applying Vector Calculus interpretations for the complex integral has pedagogical benefits, and is quite economical in its use of required dimensions.

We may extend the definition of associated Pólya vector fields, and the subsequent study, from holomorphic functions f to meromorphic functions, and even further, to arbitrary functions $f : \mathbb{C} \to \mathbb{C}$. Very few details will change but the main ideas will subsist.

The same formalism can be used in 3D to study functions of three variables making use of the representation of rotations via quaternions.

Moreover, one can extend the theory from Kähler manifolds to hyperKähler manifolds, with applications to physics, e.g., to the presentations of Electromagnetism that make use of quaternions.

In this article, we contribute to the popularization of Pólya's pedagogical lesson in Complex Analysis considering smooth function, beyond the holomorphic functions case. We used the wealth of metrics, including those from the conformal class of the complex structure, to derive information regarding the possible "free flow" Pólya vector fields, and, hence, better understanding conformal mappings from the metric viewpoint.

The notable case is that of Pólya vector fields associated to Möbius transformations. The main result is the characterization of the recurrent, concurrent and torse-forming cases.

In the other direction, we refined the classification of connections, using Möbius transformations as a filter, and deriving several algebraic systems of PDEs satisfied by the corresponding Christoffel coefficients. As an example, for each holomorphic function there are unique connections with prescribed properties: the rectificant and modular connections (Theorem 4, iii and Theorem 5). We pointed out how to use these (or any other similar) connections for classifications of holomorphic functions, through differential geometric invariants (the nullity indices of the curvature and torsion vector fields).

Further work is suited for undergraduate projects, to be reported elsewhere.

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