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**Creation of multiple electron-positron pairs in arbitrary fields**

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We examine the spontaneous breakdown of the matter vacuum triggered by an external force of arbitrary strength and spatial and temporal variations. We derive a nonperturbative framework that permits the computation of the complete time evolution of various multiple electron-positron pair probabilities. These time-dependent probabilities can be computed from a generating function as well as from solutions to a set of ratelike equations with coupling constants determined by the single-particle solutions to the time-dependent Dirac equation. This approach might be of relevance to the planned experiments to observe for the first time the laser-induced breakdown process of the vacuum.

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**I. INTRODUCTION**

The creation of matter-antimatter pairs is one of the oldest and probably most frequently discussed example of an intrinsically quantum field theoretical (QFT) process. Anderson's experimental observation of the positron [1] confirmed Dirac's original prediction based on his famous relativistic quantum spinor equation [2,3]. In 1951, Schwinger [4] proposed a nonperturbative theory to describe the long-time behavior of the pair-production process. However, his work considered only the limiting case of a constant and time-independent electric force field.

The spontaneous breakdown process has been widely studied theoretically since Schwinger. Early works by Nikishov [5], Hansen and Ravndal [6], and Holstein [7,8] used the Sauter potential [9] to extend Schwinger's work to compute the rate constant characteristic of the long-time behavior for spatially inhomogeneous but time-independent external force fields. Analytical and numerical estimations of the leading-order pair-creation rate in spatially inhomogeneous fields were also obtained using the instanton approximation of the world line path-integral formulation of quantum field theory [10–13]. The complete time-dependence of the breakdown process of the vacuum beyond decay rates for inhomogeneous and time-dependent electric fields can be described phenomenologically by the Vlasov-Maxwell equations [14–16], but a distinction between single and multiple pair probabilities is difficult with this approach.

Generalizations of the Schwinger formula due to effects of the finite temporal extent of focused laser pulses have been addressed in several works [17–19]. The authors focused on the nonadiabatic correction to the temporal extent of the laser field and neglected the spatial nonuniformity of the laser beam in the creation region. This permitted them to compute the correction to the Schwinger formula due to temporal effects.

About two decades ago, experiments based on heavy-ion collisions were conducted at various laboratories and some numerical efforts were put forth to obtain a better understanding of the underlying mechanisms in pair production [20,21]. The injection of heavy ions at various energies had the potential of differentiating various relativistic processes [22]. Unfortunately the unavoidable presence of nuclear re-

actions and the associated internal structure of high-Z nuclei involved in the ultrarelativistic collisions at high energies have made the theoretical and experimental comparisons challenging. This kind of indirect observation has certainly been partially responsible for early controversies based on these experiments. Today, it is widely agreed that an unambiguous experimental observation of the vacuum decay and the corresponding creation of electron-positron pairs triggered solely by a strong photon field are still lacking.

An alternative avenue to study the ultrarelativistic processes has come into view due to the continued advancement of laser-based systems with extremely high power. These systems will have the ability to deliver large energies within short temporal pulse durations at high repetition rates. These future light sources will make it possible for the first time to observe the direct conversion of photons into matter in the form of electron-positron pairs. First laser-based experiments that created single electron-positron pairs were carried out in 1997. These pioneering experiments by Burke *et al.* at SLAC [23] reported on the first detection of positrons using a relativistic electron beam. In this nonlinear Compton scattering process, the observed positrons were created in the electron-laser interaction but an observation of a single electron-positron pair triggered *directly* from a laser field without supplied heavy ion or electron beams has not been reported yet.

Recently, these laser systems have pushed the frontier intensity steadily up to the value of  $2 \times 10^{22}$  W/cm<sup>2</sup>. [24] Multinational research centers have been formed to develop high power laser systems [25]. One can speculate that in the next decade, they will reach a level triggering nonperturbative quantum electrodynamic processes. In addition, these systems have the advantage of relatively easy control of the spatial and temporal field profiles that might permit a better control of these interactions than possible in nuclear collisions. However, a computational procedure that can take the ultrashort temporal interaction nature into account is quite challenging. We would need an exact time-dependent description for the vacuum, single-, and multiparticle probabilities for external force fields with arbitrary space-time dependence. This would require an approach that goes beyond the usual descriptions of the long-time behavior by a single rate constant.

This paper makes an attempt to fill this gap. The approach is based on the recently developed large-scale computer

simulations of the strong-field process, which we refer to as computational quantum field theory. An initial analysis of this approach has already provided some insights into various conceptual problems such as the Klein paradox [26–29], the Zitterbewegung [30–32], the electron localization, and the dynamics of the formation of bound states in supercritical fields [33]. We will present an exact theoretical framework to calculate quantities to describe the (multiparticle) pair-production processes. From a computational point of view, the method is based on the complete set of the single-particle wave-function solutions and the link between multipair quantum field state probabilities and the particle coincidence counts. In the special limit of a supercritical constant field and asymptotic long times, the theory reproduces the famous Schwinger’s pair-production rate [4]. However, we point out that Schwinger’s theoretical limit might be an ill-defined proposition for the planned laser-based experiments that will use ultrashort laser pulses and therefore require a theory with full spatial and temporal resolutions beyond a single rate.

The paper is organized as follows. In Sec. II, we discuss the connection between measured particle counts and probabilities of QFT states and derive a general solution for the QFT probabilities. In Sec. III, we investigate the special case of the pair-production rate for supercritical static fields and compare it to the Schwinger pair-production rate. In Sec. IV, we analyze the time dependence of the multipair probabilities for the long-time regime in supercritical fields and examine a cascadelike mechanism in which states with a higher particle number get excited consecutively from those with a lower number. We finish with a short summary and an outlook on future work.

## II. PARTICLE DETECTION COUNT AND THE QFT STATE PROBABILITY

The Dirac Hamiltonian [3] can be written (in atomic units) as

$$h = c\boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}/c) + eV + \beta c^2. \quad (2.1)$$

The electric charge  $e$  is either  $-1$  a.u. (for the electron) or  $1$  a.u. (for the positron). Here  $V$  and  $\mathbf{A}$  are parts of the electromagnetic four-potential,  $A_\mu = (V, \mathbf{A})$ , which fulfill the Lorentz condition

$$\boldsymbol{\nabla} \cdot \mathbf{A} + (1/c) \partial V / \partial t = 0. \quad (2.2)$$

The electric field  $\mathbf{E}$  (which determines the force from the potential) can be written as

$$\mathbf{E} = -\boldsymbol{\nabla}V - (1/c) \partial \mathbf{A} / \partial t. \quad (2.3)$$

The symbols  $\beta$  and  $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$  in Hamiltonian (2.1) are the Dirac matrices defined as

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad (2.4)$$

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The Dirac equation  $i\partial|\phi\rangle/\partial t = h|\phi\rangle$  for the single-particle state  $|\phi\rangle$  has a formal solution in terms of the time evolution operator  $u(t) = T_o \exp(-i\int h dt)$ , which, in principle, contains all the information about the quantum-mechanical system. As  $h$  is time dependent in general,  $T_o$  denotes the required time-ordering operator.

We will show below that the information contained in the evolution of all single-particle states can be used to obtain the time evolution of the QFT state probability that describes multiparticle dynamics. We define the matrix elements  $u_{p,n} \equiv \langle p|u(t)|n\rangle$  between the positive and negative-energy Dirac eigenstates  $|p\rangle$  and  $|n\rangle$  of the field-free Hamiltonian with  $V = \mathbf{A} = 0$ . Similarly, we define  $u_{p,p'}(t)$  and  $u_{n,n'}(t)$ . These matrix elements can be obtained by evolving (numerically) [34–38] each possible initial eigenstate of the field-free Hilbert space under the full interaction in Eq. (2.1). Each evolved state is then projected on every field-free eigenstate.

As the number of particles in the pair-creation process changes, the Dirac equation has to be viewed from a quantum field theoretical perspective. In QFT, one of the key quantities is the field operator  $\hat{\Psi}$ . Its time evolution is described by the Heisenberg equation of motion with the commutator  $[\hat{\Psi}(t), \hat{H}]_- = i\partial\hat{\Psi}(t)/\partial t$ , where  $\hat{H}$  denotes the field theoretical Hamiltonian. In the Appendix, we show that the time evolution of  $\hat{\Psi}$  can be obtained equivalently from the Dirac equation,  $i\partial\hat{\Psi}/\partial t = h\hat{\Psi}$ . In other words, due to this dual nature, the field operator  $\hat{\Psi}$  can be viewed as a hybrid of an operator and a state. This is also obvious if we introduce the destruction and creation operators  $\hat{b}_p(t)$  and  $\hat{d}_n^\dagger(t)$  for positive-energy states  $|p\rangle$  and negative-energy states  $|n\rangle$  and expand the field operator as

$$\hat{\Psi}(t) = \sum_p \hat{b}_p(t)|p\rangle + \sum_n \hat{d}_n^\dagger(t)|n\rangle. \quad (2.5)$$

Using  $i\partial\hat{\Psi}/\partial t = h\hat{\Psi}$ , one can show that these operators evolve in time according to  $\hat{b}_p(t) = \sum_{p'} \hat{b}_{p'}(t) u_{p,p'}(t) + \sum_n \hat{d}_n^\dagger(t) u_{p,n}(t)$  and  $\hat{d}_n^\dagger(t) = \sum_p \hat{b}_p(t) u_{n,p}(t) + \sum_{n'} \hat{d}_{n'}^\dagger(t) u_{n,n'}(t)$ . This indicates that their solutions to the Heisenberg equation of motion with  $\hat{H}$  can be obtained equivalently from the matrix elements  $u_{p',p}(t)$ ,  $u_{n',n}(t)$ , and  $u_{p',n}(t)$ , which can be computed via the time evolution of the single-particle Dirac equation as discussed above. The solution Eq. (2.5) can be used to calculate various quantum field theoretical observables. For example, the average number density  $\rho(z_1, \dots, z_n, t)$  of events to detect simultaneously electrons at spatial locations  $z_1, \dots, z_n$  at any time  $t$ , if the system is initially in the quantum field theoretical state  $|\phi_0\rangle$ , is [39]

$$\rho(z_1, \dots, z_n, t) = \langle\langle \phi_0 | \hat{\Psi}_e^\dagger(z_1, t) \cdots \hat{\Psi}_e^\dagger(z_n, t) \times \hat{\Psi}_e(z_n, t) \cdots \hat{\Psi}_e(z_1, t) | \phi_0 \rangle\rangle / n! \quad (2.6)$$

We use the notation  $|\dots\rangle\rangle$  to distinguish the quantum field theoretical states, which in general can be superpositions of states with different numbers of particles (such as the vacuum), from the single-particle states  $|\dots\rangle$ , which are usually associated with the unitary quantum mechanics without second quantization. The subscript  $e$  for the field operator  $\hat{\Psi}_e(z, t)$  denotes the electronic portion [40] of the field operator  $\hat{\Psi}(z, t)$ , which may be expressed as  $\hat{\Psi}_e(z, t) \equiv \sum_p \hat{b}_p(t) \langle z | p \rangle$ . In our particular situation, the initial state  $|\phi_0\rangle\rangle$  for pair-creation process is the vacuum state  $|\phi_0\rangle\rangle = |0\rangle\rangle$ .

The number density  $\rho(z_1, \dots, z_n, t)$  can be used to compute the average number of finding  $n$  electrons at time  $t$ , defined as the multiple integral

$$\begin{aligned} C_n(t) &\equiv \int dz_1 \cdots \int dz_n \rho(z_1, \dots, z_n, t) \\ &= \langle\langle \phi_0 | \int dz_1 \cdots \int dz_n \hat{\Psi}_e^\dagger(z_1, t) \cdots \hat{\Psi}_e^\dagger(z_n, t) \hat{\Psi}_e(z_n, t) \cdots \\ &\quad \times \hat{\Psi}_e(z_1, t) / n! | \phi_0 \rangle\rangle \\ &= \langle\langle \phi_0 | : \left[ \int dz \hat{\Psi}_e^\dagger(z, t) \hat{\Psi}_e(z, t) \right]^n / n! : | \phi_0 \rangle\rangle. \quad (2.7) \end{aligned}$$

Here,  $:\dots:$  represents normal ordering in the field operators, i.e.,  $:\hat{\Psi}_e \hat{\Psi}_e^\dagger := \hat{\Psi}_e^\dagger \hat{\Psi}_e$ , as well as reverse sequencing of the coordinates in the operators  $\hat{\Psi}_e$  with respect to the sequence of coordinates of the Hermitian adjoint operators  $\hat{\Psi}_e^\dagger$ , i.e.,  $:\hat{\Psi}_e^\dagger(z_1) \hat{\Psi}_e^\dagger(z_2) \hat{\Psi}_e(z_1) \hat{\Psi}_e(z_2) := \hat{\Psi}_e^\dagger(z_1) \hat{\Psi}_e^\dagger(z_2) \hat{\Psi}_e(z_2) \hat{\Psi}_e(z_1)$ . As the electron and positrons always appear in pairs, this counting number is the same for detecting particle pairs. It is important to note that  $0 \leq C_n(t) < \infty$ , for example,  $C_1 = \int dz_1 \rho(z_1, t)$  is equal to  $n$  for an  $n$ -pair state. Similarly, the average number of two-pair coincidence events is given by the binomial coefficient  $C_2 = \binom{n}{2}$  for an  $n$ -pair state. As the functions  $\rho(z_1, \dots, z_n, t)$  are not probability densities,  $\rho(z_1)$  cannot be viewed as the marginal density associated with the joint density  $\rho(z_1, z_2)$ , i.e.,  $\int dz_2 \rho(z_1, z_2) \neq \rho(z_1)$ .

If we insert the solution for  $\hat{\Psi}$  in terms of the time-dependent creation and annihilation operators into Eq. (2.7), we obtain

$$\begin{aligned} C_n(t) &= \int dp_1 \cdots \int dp_n \int dn_1 \cdots \int dn_n \sum_P \sigma_P u_{p_1, n_1}^\dagger \cdots u_{p_n, n_n}^\dagger \\ &\quad \times u_{p_n', n_n} \cdots u_{p_1', n_1} \langle\langle 0 | \hat{d}_1^\dagger \cdots \hat{d}_n^\dagger \hat{d}_n' \cdots \hat{d}_1' | 0 \rangle\rangle / n! \\ &= \int dp_1 \cdots \int dp_n \sum_P \sigma_P A_{p_1, p_1'} \cdots A_{p_n, p_n'} / n!. \quad (2.8) \end{aligned}$$

In this equation, the symbol  $\sum_P$  denotes the summation over all permutations of the second (primed) indices in each  $A$ , starting from the natural sequence of indices  $p_1, p_2, \dots, p_n$ .

For an even (or odd) number of permutations, the symbol  $\sigma_P$  is 1 (or  $-1$ ). For example, for  $n=2$ , we have the determinant  $\sum_P \sigma_P A_{p_1, p_1'} A_{p_2, p_2'} = A_{p_1, p_1'} A_{p_2, p_2'} - A_{p_1, p_2'} A_{p_2, p_1'}$ . The corresponding sign of  $\sigma_P$  [41] follows from the fermionic anti-commutator relationship among the creation and destruction operators. We have also used  $\hat{b}_p | 0 \rangle\rangle = 0$  to arrive at this expression. Also, in the last step, we have defined

$$A_{p_1, p_2}(t) = \sum_N u_{p_1, n} u_{p_2, n}^*, \quad (2.9)$$

where  $p_1$  and  $p_2$  correspond to positive field-free energy eigenstates and the summation is over all the negative-energy states labeled with  $n$ .

We should note an interesting similarity between the Eq. (2.8) and an expression derived by Hencken and co-workers [42,43]. In their original paper from 1995, the authors compute the complex amplitude for the creation of  $N$  pairs and express it as a summation of various quantum paths, each represented by the amplitude for only a single pair. This  $S$ -matrix-like expression is valid asymptotically for  $t \rightarrow \infty$ . Both expressions contain the summation and integration and involve the vacuum expectation values over products of fermion creation and annihilation operators of arbitrary high order. In our case, the functions  $A_{p_1, p_2}$  have to be obtained numerically and are different from the  $s^{+-}$ . Furthermore, we would like to point out that Ref. [42] computes the complex amplitude for  $N$ -pair production, while our Eq. (2.8) is for the positive average number of  $n$  particles.

The complicated expression in Eq. (2.8) can be simplified further if we introduce the following quantities as traces over the positive-energy manifold

$$\begin{aligned} D_1 &\equiv \sum_p A_{p, p}(t), \\ D_2 &\equiv \sum_{p_1, p_2} A_{p_1, p_2}(t) A_{p_2, p_1}(t), \\ D_3 &\equiv \sum_{p_1, p_2, p_3} A_{p_1, p_2}(t) A_{p_2, p_3}(t) A_{p_3, p_1}(t), \\ &\quad \dots \\ D_m &\equiv \sum_{p_1, \dots, p_m} A_{p_1, p_2}(t) A_{p_2, p_3}(t) \cdots A_{p_m, p_1}(t), \quad (2.10) \end{aligned}$$

With this notation, the summations of the permutations in Eq. (2.8) can be regrouped into different partitions [44]. Finally, we obtain

$$\begin{aligned} C_N(t) &= \sum_{m=1}^{\infty} (-1)^{N+m} / m! \sum_{\ell_1}^N \cdots \sum_{\ell_m=1}^N \prod_{i=1}^m (D_{\ell_i} / \ell_i) \\ &\quad \times \delta_{(\ell_1 + \ell_2 + \cdots + \ell_m), N}. \quad (2.11) \end{aligned}$$

As an example, the lowest four counting numbers reduce to

$$C_1(t) = D_1,$$

$$C_2(t) = (D_1^2 - D_2) / 2!,$$

$$C_3(t) = (D_1^3 - 3D_1D_2 + 2D_3)/3!,$$

$$C_4(t) = (D_1^4 - 6D_1^2D_2 + 8D_1D_3 + 3D_2^2 - 6D_4)/4!. \tag{2.12}$$

Until now, we have just considered the average number of events with  $N$ -pair coincidences  $C_N$ . The contributions to each number can come from different QFT states. For example,  $C_1=1$  can be realized by a system that is 100% in the single-pair state. The same count can also characterize a different state that is a 50%-50% superposition of the vacuum and a double-pair state. This example shows that different states can have the same average number of coincidence counts. The entire set of  $n$ -pair probabilities  $P_n$  and the average  $m$ -pair counts  $C_m$  are directly related to each other

$$C_m(t) = \sum_{n=m} n! / [(n-m)! m!] P_n. \tag{2.13}$$

This relationship can be obtained in a straightforward way if we apply Eq. (2.7) in the Schrödinger picture to a superposition of Fock states with different particle numbers and amplitude  $\sqrt{P_n}$ . Note that the QFT state probabilities  $P_n$  are mutually exclusive as  $\sum_{n=0} P_n = 1$ , while, as noted above, the sum over the  $m$ -pair counts  $\sum_{m=1} C_m$  can take arbitrary values between zero (for the vacuum) and infinity (for the long-time limit of supercritical potentials). The linear equation above can be easily inverted and gives the QFT probability of finding the system in an  $m$ -pair state  $P_m$  from the set of all  $C_n$  according to

$$P_m(t) = \sum_{n=m} (-1)^{n+m} n! / [(n-m)! m!] C_n. \tag{2.14}$$

After we insert Eq. (2.11) into Eq. (2.14), the resulting expression for the vacuum probability  $P_0$  can be summed up analytically and leads to a remarkably simple exponential form

$$P_0(t) = 1 - \sum_n (D_n/n) + (1/2!) \sum_{n_1, n_2} (D_{n_1}/n_1)(D_{n_2}/n_2) - \dots = \exp[-\sum_n (D_n/n)]. \tag{2.15}$$

Similarly, for  $m=1$ , we obtain

$$P_1(t) = \sum_n (D_n) - (1/2!) \sum_{n_1+n_2 \geq 2} (D_{n_1}/n_1)(D_{n_2}/n_2)(n_1+n_2) + \dots \tag{2.16}$$

The general probability of an  $m$ -pair state can be also obtained from the same procedure

$$P_m(t) = (-1)^{m+1} \left\{ \sum_{n=m} D_n [(n-1)! / m! / (n-m)!] - (1/2!) \sum_{n_1+n_2 \geq m+1} (D_{n_1}/n_1)(D_{n_2}/n_2) \times [(n_1+n_2)! / (n_1+n_2-m)! / m!] + \dots \right\}. \tag{2.17}$$

In fact, one can construct a generating function  $F(x)$ ,

$$F(x) = 1 - 1/1! \sum_n (1+x)^n D_n/n + (1/2!) \times \sum_{n_1, n_2} (1-x)^{n_1} + (1-x)^{n_2} (D_{n_1}/n_1)(D_{n_2}/n_2) + \dots = \exp[-\sum_n (1-x)^n D_n/n], \tag{2.18}$$

from which each probability  $P_m(t)$  can be obtained as  $P_m(t) = (1/m!) d^m F(x) / dx^m |_{x=0}$ . We obtain the central result of our approach

$$P_m(t) = (1/m!) (d^m/dx^m) \exp[-\sum_n [D_n(1-x)^n/n]] |_{x=0}, \tag{2.19}$$

which allows us to evaluate the probability of any  $m$ -pair QFT state. This result may be generalized to any initial state  $|\phi_0\rangle$  and in terms of field operators

$$P_m(t) = \langle\langle \phi_0 | : (1/m!) (d^m/dx^m) \exp \left[ (x-1) \int dz \hat{\Psi}_e^\dagger(z,t) \hat{\Psi}_e(z,t) \right] |_{x=0} : | \phi_0 \rangle \rangle, \tag{2.20}$$

where  $: \dots :$  represent again the normal order and reverse sequencing in spatial variables of the field operator with respect to the Hermitian adjoint field operators. This procedure lays the foundation for computing the time dependence of the pair-creation process for arbitrary external fields. This relationship is the main connection between the QFT probabilities and the transition matrix elements of the single-particle evolution operator  $u$ . It is also clear from Eq. (2.19) that we only need to calculate the functions  $D_n$  to obtain the QFT probabilities  $P_m$ . The functions  $D_n$  can be computed, according to Eqs. (2.10) and (2.9), directly from the single-particle evolution operator  $u$  associated with  $h$  of Eq. (2.1) and its matrix elements  $u_{p,n}$ . It is important to note that the operator  $u$  is unitary as the result of the hermiticity of the generator  $h$ . It is therefore associated with norm-preserving processes while the particle creation process obviously does not preserve the norm in a quantum-mechanical sense.

As the probabilities Eq. (2.19) contain the entire time dependence ranging from early time transients to possibly steady-state long-time regimes, they are in general complicated functions that depend on the particular temporal and spatial forms of the external force fields. However, in the long-time limit, these probabilities become universal in the sense that special turn-on details of the forces become irrelevant for the decay rates. This permits us to compute analytical expressions for these rates in the asymptotic long-time limit as discussed in Sec. III.

Even though we have the full solutions for the probabilities, the way they are coupled with respect each other is quite interesting from a dynamical point of view. If we take the time derivative of  $P_m(t)$  in Eq. (2.19), we find that the resulting right-hand side can be expressed in terms of the probabilities associated with a lower number of pairs,  $dP_m(t)/dt = \sum_{j=0}^m (-1)^{j+1} \gamma_j P_{m-j}(t)$ , where the (time-dependent)

coupling coefficients are defined as  $\gamma_j = \sum_n dD_n/dt (n-1)!/(n-j)!/j!$ . To obtain a better picture, we summarize here the first five equations:

$$dP_0(t)/dt = -\gamma_0 P_0, \quad (2.21a)$$

$$dP_1(t)/dt = -\gamma_0 P_1 + \gamma_1 P_0, \quad (2.21b)$$

$$dP_2(t)/dt = -\gamma_0 P_2 + \gamma_1 P_1 - \gamma_2 P_0, \quad (2.21c)$$

$$dP_3(t)/dt = -\gamma_0 P_3 + \gamma_1 P_2 - \gamma_2 P_1 + \gamma_3 P_0, \quad (2.21d)$$

$$dP_4(t)/dt = -\gamma_0 P_4 + \gamma_1 P_3 - \gamma_2 P_2 + \gamma_3 P_1 - \gamma_4 P_0. \quad (2.21e)$$

It is important to note that this set of equations is exact. Usually, the exact quantum dynamics must be described in terms of complex state amplitudes. It is therefore quite remarkable that these probabilities are sufficient to describe all coherent and incoherent mechanisms of the pair-creation dynamics for any time and arbitrary external forces. At the same time, we have to keep in mind that the time-dependent coupling coefficients were obtained from the solutions to the single-particle Dirac equation.

As the total probability is conserved,  $\sum_n P_n = 1$ , the sum over all right-hand sides of Eqs. (2.21) has to vanish to be consistent. In other words, we require the term  $\sum_n dP_n(t)/dt = \sum_n \sum_j (-1)^j \gamma_j = \sum_n \sum_j (-1)^j dD_n/dt (n-1)!/(n-j)!/j!$  to vanish. This is obviously true for any function  $D_n$  as  $\sum_{j=0}^{n-1} (-1)^j/(n-j)!/j! = 0$ .

While each of the functions  $D_n$  is positive at any time, the coupling constants  $\gamma_j$  can be positive or negative as they are related to the sum of the time derivatives of the  $D_n$ . An exception occurs for very early times where each decay rate  $\gamma_j$  is positive. This exception follows from the fact that the original matrix elements  $u_{p,n}$  grow linearly in time  $u_{p,n} \propto t$  and as a result, we have the simple scaling  $D_n \rightarrow \lambda_n t^{2n}/2$  with positive prefactors  $\lambda_n$ . This translates directly into the early time scaling  $\gamma_0 = \gamma_1 = \lambda_1 t$  and  $\gamma_n = \lambda_n t^{2n-1}$  for  $n=2, 3, 4, \dots$

It might seem at first that the set of coupled Eqs. (2.21) in its general form suggests a monotonic flow of probabilities of states with a lower number of pairs to states with a higher number. This would also (incorrectly) suggest a complete absence of pair-annihilation processes. However, if we replace the  $P_0$  on right-hand side of Eq. (2.21) with  $1 - \sum_{n=1} P_n$ , we see that these probabilities for  $n \neq 0$  can act as an effective source term and represent a decay from the probabilities  $P_n$  back to the vacuum state, e.g.,  $dP_0(t)/dt = -\gamma_0 + \gamma_0 \sum_{n=1} P_n$ . This suggests that the coupling  $\gamma_0$  can be related also to the pair-annihilation rate with respect to all other probabilities.

A similar interpretation is possible if we replace  $\gamma_0$  in the same equation with  $\gamma_0 = \gamma_1 - \gamma_2 + \gamma_3 - \dots$  leading to  $dP_0(t)/dt = \sum_{j=1} (-1)^j \gamma_j P_0$ . This shows that the overall vacuum decay represents the sum of all decay channels into the probabilities  $P_n$  with  $n \geq 1$ . In Sec. IV, we will discuss the specific implications of Eq. (2.21) for the long-time regime of supercritical forces.

### III. SUPERCRITICAL FORCES AND THE CONNECTION WITH THE SCHWINGER LIMIT

In this section, we show how our formalism can be applied to compute the vacuum decay rate for static supercritical scalar potentials. This special case can also serve as a consistency check as we can compare our results to the well-known case considered originally by Schwinger [4] for the vacuum decay rate of an infinitely extended constant force field. The external field is characterized here by a scalar potential  $V(z)$  that is chosen as a general potential step along the  $z$  direction, where for  $z \ll 0$  the potential vanishes and in some limited range around  $z=0$  it is ramped up to a constant value  $V_0 (> 2c^2)$ , which is maintained for  $z \gg 0$ . The corresponding force would be localized around  $z=0$ .

Since we are interested in the long-time behavior, we only need to consider the asymptotic behavior of the evolution operator and the corresponding matrix elements  $u_{p,n}$ . This behavior can be obtained formally by inserting the complete “in-field” eigenstates  $|\mathbf{k}\rangle$  (the eigenstates of the full Hamiltonian  $h$  with  $h|\mathbf{k}\rangle = E_{\mathbf{k}}|\mathbf{k}\rangle$ ) into the definition for the  $u_{p,n}$ ,

$$u_{p,n} = \langle p|u|n\rangle = \sum_{\mathbf{k}} \exp(-iE_{\mathbf{k}}t) \langle p|\mathbf{k}\rangle \langle \mathbf{k}|n\rangle. \quad (3.1)$$

Here  $\sum_{\mathbf{k}}$  represents summation over the complete basis  $|\mathbf{k}\rangle$  of the Hamiltonian (2.1). It turns out that only those states  $|\mathbf{k}\rangle$  that have an energy in the range  $c^2 < E_{\mathbf{k}} < V_0 - c^2$  contribute to the supercritical pair-production process. This is the range in which electron-positron pairs are created. Our analysis below will show that in this range, the  $u_{p,n}$  contain an energy delta function. This makes expressions such as  $D_1 = \sum_{p,n} |u_{p,n}|^2$  to increase in the long-time limit while outside this range, the contributions of  $u_{p,n}$  become constant and are therefore negligible.

In order to compute the two scalar products  $\langle p|\mathbf{k}\rangle \langle \mathbf{k}|n\rangle$ , it is advantageous to perform the required integrations in coordinate space. We will argue below that without loss of generality, we can assume that the supercritical potential is just a simple step function  $V(z) = V_0 \theta(z)$ , with the unit-step function  $\theta(z) = (|z|+z)/|z|$  and an amplitude  $V_0$  that is larger than  $2c^2$ . As the energy  $E_{\mathbf{k}}$  in this range is twofold degenerate, we can arbitrarily choose two orthogonal superposition states. We label one as the “left-going” state  $|\mathbf{k}^L\rangle$  and the other one as “right-going” state  $|\mathbf{k}^R\rangle$ . These states are normalized according to  $\langle \mathbf{k}^L|\mathbf{k}'^L\rangle = \langle \mathbf{k}^R|\mathbf{k}'^R\rangle = \delta(k_x - k'_x) \delta(k_y - k'_y) \delta(k - k')$ . Their spatial representation is

$$\langle \mathbf{r}|\mathbf{k}^R\rangle = [R_1(z)\theta(-z) + R_2(z)\theta(z)]\Psi_{\perp}(x, y, k_{\perp}), \quad (3.2a)$$

$$\langle \mathbf{r}|\mathbf{k}^L\rangle = [L_1(z)\theta(-z) + L_2(z)\theta(z)]\Psi_{\perp}(x, y, k_{\perp}), \quad (3.2b)$$

where we label the three-dimensional momentum  $(k_x, k_y, k)$  and denote the transverse momentum by  $k_{\perp}^2 = k_x^2 + k_y^2$ . The energy is  $E_{\mathbf{k}} = \sqrt{[c^4 + c^2(k^2 + k_{\perp}^2)]}$ . The plane-wave solutions in the transverse direction are  $\Psi_{\perp}(x, y, k_{\perp}) \equiv e^{-i(k_x x + k_y y)}/(2\pi)$ . The four solutions along the  $z$  direction are defined as

$$R_1(z) = N(\mathbf{k})[e^{ikz}\phi_+(\mathbf{k}) + re^{-ikz}\phi_+(-\mathbf{k})], \quad (3.3a)$$

$$R_2(z) = N(\mathbf{k})se^{-ikz}\phi_-(\tilde{\mathbf{k}}), \quad (3.3b)$$

$$L_1(z) = -M(\tilde{k})e_{\tilde{k}}^-/e_{\tilde{k}}s e^{-ikz}\phi_+(-\mathbf{k}), \quad (3.3c)$$

$$L_2(z) = M(\tilde{\mathbf{k}})[e^{ikz}\phi_-(\tilde{\mathbf{k}}) + r e^{-ikz}\phi_-(\tilde{\mathbf{k}})]. \quad (3.3d)$$

We have defined  $e_{\mathbf{k}} \equiv k/(c^2 + E_{\mathbf{k}})$  and the normalization factors  $N(\mathbf{k}) \equiv \sqrt{[(E_{\mathbf{k}} + c^2)/(4\pi E_{\mathbf{k}})]}$ . For the quantities characterizing the states under the potential step, we have introduced for notational simplicity a shifted energy,  $\tilde{E}_{\mathbf{k}} = V_0 - E_{\mathbf{k}}$ , an effective momentum  $\tilde{k} \equiv \sqrt{[(V_0/c)^2 + k^2 - 2(V_0/c)\sqrt{(c^2 + k^2 + k_{\perp}^2)}]}$ ,  $e_{\tilde{\mathbf{k}}} \equiv \tilde{k}/(c^2 + \tilde{E}_{\tilde{\mathbf{k}}})$ , and the normalization factor  $M(\tilde{k}) \equiv \sqrt{[(\tilde{E}_{\tilde{\mathbf{k}}} + c^2)/(4\pi\tilde{E}_{\tilde{\mathbf{k}}})]}\sqrt{[(k\tilde{E}_{\tilde{\mathbf{k}}})/(\tilde{k}E_{\tilde{\mathbf{k}}})]}$ . Note that  $k$  and  $\tilde{k}$  are chosen to be positive. We also define with  $\phi_+(\mathbf{k})$  and  $\phi_-(\mathbf{k})$  the spinor components of the field-free wave function, usually associated with positive and negative free-energy states.

The two most important parameters are the transmission amplitude  $s(\mathbf{k})$  and the reflection amplitude  $r(\mathbf{k})$ . Due to the hermiticity of  $h$  and the corresponding norm-conserving dynamics, the two are linked by the continuity equation  $1 = r^2 + s^2 e_{\tilde{\mathbf{k}}}^-/e_{\mathbf{k}}$ .

Using a similar set of abbreviations, the spatial representations for the force-free energy eigenstates with positive and negative energies  $E_p = \sqrt{[c^4 + c^2 p^2 + c^2 p_{\perp}^2]}$  and  $E_n = -\sqrt{[c^4 + c^2 n^2 + c^2 n_{\perp}^2]}$  are given by

$$\langle \mathbf{r} | p \rangle = N(\mathbf{p}) e^{ipz} \phi_+(\mathbf{p}) \Psi_{\perp}(x, y, p_{\perp}), \quad (3.4a)$$

$$\langle \mathbf{r} | n \rangle = N(\mathbf{n}) e^{inz} \phi_-(\mathbf{n}) \Psi_{\perp}(x, y, n_{\perp}), \quad (3.4b)$$

where we define  $(p_x, p_y, p)$  and  $(n_x, n_y, n)$  as the momenta of the states  $|p\rangle$  and  $|n\rangle$ .

Next we use the expressions of Eqs. (3.2) and (3.4) to perform the scalar products in coordinate space  $\langle p | \mathbf{k}^L \rangle \langle \mathbf{k}^L | n \rangle + \langle p | \mathbf{k}^R \rangle \langle \mathbf{k}^R | n \rangle$ . The derivations for these four matrix elements are very similar to each other, so we just illustrate this for  $\langle p | \mathbf{k}^L \rangle$  as an example. We first insert the unit operator in its coordinate representation  $\langle p | \mathbf{k}^L \rangle = \int \int \int dr \langle p | \mathbf{r} \rangle \langle \mathbf{r} | \mathbf{k}^L \rangle = \int \int \int dr \langle p | \mathbf{r} \rangle [L_1(z)\theta(-z) + L_2(z)\theta(z)] \Psi_{\perp}(x, y, k_{\perp})$ . The two integrations over the transverse directions can be performed leading to  $\langle p | \mathbf{k}^L \rangle = \delta(p_x - k_x) \delta(p_y - k_y) \{N(\mathbf{p}) \int dz e^{ipz} \phi_+(\mathbf{p}) [L_1(z)\theta(-z) + L_2(z)\theta(z)]\}$ . The second integrand  $\phi_+(\mathbf{p}) L_2(z)\theta(z)$  vanishes identically due to the orthogonality of the spinors  $\phi_-(\tilde{\mathbf{k}})$  and  $\phi_+(-\mathbf{k})$ . The remaining integral over the negative half- $z$  space can be done leading to the final result for  $\langle p | \mathbf{k}^L \rangle$ . Following similar steps, we obtain the other matrix elements required to compute  $\langle p | \mathbf{k} \rangle \langle \mathbf{k} | p \rangle$ ,

$$\begin{aligned} \langle p | \mathbf{k}^L \rangle &= -\delta(p_x - k_x) \delta(p_y - k_y) s \{ \delta(k + p) / 2 \\ &\quad + iP[1/(k + p)] / (2\pi) \} (e_{\tilde{\mathbf{k}}}^- / e_{\mathbf{k}}) \\ &\quad \times \sqrt{[k\tilde{E}_{\tilde{\mathbf{k}}}(1 + e_{\tilde{\mathbf{k}}}^-)] / [\tilde{k}E_{\tilde{\mathbf{k}}}(1 + e_{\tilde{\mathbf{k}}}^-)]}, \end{aligned} \quad (3.5a)$$

$$\begin{aligned} \langle \mathbf{k}^L | n \rangle &= \delta(n_x - k_x) \delta(n_y - k_y) \{ [\delta(\tilde{k} - n) + r\delta(\tilde{k} + n)] / 2 \\ &\quad - iP[1/(\tilde{k} - n) - r/(\tilde{k} + n)] / (2\pi) \} \sqrt{[k\tilde{E}_{\tilde{\mathbf{k}}}] / [\tilde{k}E_{\tilde{\mathbf{k}}}]}, \end{aligned} \quad (3.5b)$$

$$\begin{aligned} \langle p | \mathbf{k}^R \rangle &= \delta(p_x - k_x) \delta(p_y - k_y) s \{ [\delta(k - p) + r\delta(k + p)] / 2 \\ &\quad - iP[1/(k - p) - r/(k + p)] / (2\pi) \}, \end{aligned} \quad (3.5c)$$

$$\begin{aligned} \langle \mathbf{k}^R | n \rangle &= \delta(n_x - k_x) \delta(n_y - k_y) t \{ \delta(\tilde{k} + n) / 2 \\ &\quad + iP[1/(\tilde{k} + n)] / (2\pi) \} \sqrt{[1 + e_{\tilde{\mathbf{k}}}^2] / [1 + e_{\tilde{\mathbf{k}}}^2]}, \end{aligned} \quad (3.5d)$$

where  $P[\dots]$  represent Cauchy principal part resulting from  $\int_0^{\infty} dz \exp(ikz) = \pi\delta(k) + iP(1/k)$ .

As the final step, we perform the integration over the restricted energy range,  $\sum_{\mathbf{k}} \exp(-iE_{\mathbf{k}}t) \langle p | \mathbf{k} \rangle \langle \mathbf{k} | n \rangle$ . The restricted summation  $\sum_{\mathbf{k}}$  in Eq. (3.1) amounts to the three-dimensional integration over the momentum  $\sum_{\mathbf{k}} = \int \int \int dk_x dk_y dk$ , where  $-\infty < k_x < \infty$ ,  $-\infty < k_y < \infty$ , and  $0 < k < \infty$  and the constraint  $k_x^2 + k_y^2 + k^2 < (V_0 - c^2)^2 - c^4$ . When  $V_0$  is large, the first two integrations over  $dk_x dk_y$  will give rise to  $\delta(p_x - n_x) \delta(p_y - n_y)$ . For the third integral over  $k$ , we do not have such a delta function in momentum space as the two momenta  $k$  and  $\tilde{k}$  are different. However, the relations between  $\tilde{k}$  and  $n_z$  and between  $k$  and  $p_z$  lead the requirement that the energies  $E_p$  and  $E_n$  must be related to each other,  $E_p = E_n + V_0$ . By changing the integration over  $dk$  to  $dE_{\mathbf{k}}$  and noting the product of principal parts will lead to a delta function and should be integrated with care [45], we obtain

$$u_{pn} = \exp(-iE_p t) \delta(E_p - E_n + V_0) \delta(p_x - n_x) \delta(p_y - n_y) \sqrt{[T(\mathbf{p})]}, \quad (3.6)$$

where  $T$  is defined as  $T(\mathbf{p}) = s^2(\mathbf{p}) e_{\tilde{\mathbf{k}}}^- / e_{\mathbf{k}}$ . Note that  $T(\mathbf{p})$  is the transmission coefficient of the corresponding single-particle scattering problem for a supercritical step whose strength is larger than the energy of the incoming electron.

We note that the derivation leading to Eq. (3.6) has been obtained as the long-time limit of an exact theory. It is therefore the first proof of an old conjecture by Hund [46] about significance of the transmission coefficient for the pair-creation process. A similar relationship was also suggested by Hansen and Ravndal [6] and Dombey and Calogeracos [27] who made two assumptions. They confined the dynamics by allowing only specific directions of the particle fluxes on both sides of the potential barrier and required the presence or absence of certain modes in the steady state. Under these two assumptions, Hansen and Ravndal computed the expectation value of the average particle number in the outgoing vacuum state, while Dombey and Calogeracos determined the charge current.

Furthermore, it is important to note that the transmission coefficient is less than unity,  $0 < T(\mathbf{p}) < 1$ . Similar heuristic derivations used a basis set of nonscattering states leading to negative transmission coefficients and reflection coefficients that are larger than unity. It was attempted to justify these results in terms of the language of quantum field theory. It was argued that a reflection larger than unity corresponds to the reflected electron as well as the created electron due to the pair-creation process and the negative transmission coefficient can be interpreted as describing the flux of particles of opposite charge (positrons) evolving under the barrier. In our

opinion, it is not very satisfactory to try to associate purely quantum-mechanical data with intrinsically quantum field theoretical processes.

With the establishment of Eq. (3.6), we are ready to compute  $D_1 = \sum_{p,n} |u_{p,n}|^2$  in the limit of  $t \rightarrow \infty$ . For a large spatial extent in  $x$  and  $y$  (denoted by  $L_x$  and  $L_y$ ), we obtain

$$D_1 = \int dE_p \int dp_x \int dp_y \int dE_n \int dn_x \int dn_y |u_{p,n}|^2 = tL_xL_y/(8\pi^3) \int \int \int dE_p dp_x dp_y T(\mathbf{k}). \quad (3.7)$$

Note that three of the six delta functions in  $|u_{p,n}|^2$  were used to carry out the three integrations  $\int dE_n \int dn_x \int dn_y$  resulting in three delta functions with zero argument. These were then replaced by  $\delta(0) = t/(2\pi)$  for the energy and  $\delta(0) = L_x/(2\pi)$  or  $L_y/(2\pi)$  for the momentum delta functions. If we take into account of the spin degree of freedom, an additional factor of 2 will be added to the expression. To carry out the integrations over the energy, one should keep in mind that  $E_p$  depends on  $p_x$  and  $p_y$ , via  $p = \sqrt{[E_p^2 - c^4 - (p_x^2 + p_y^2)c^2]}/c$ , while  $\tilde{k}(p)$  is determined via  $E(\mathbf{p}) = \tilde{E}(\tilde{\mathbf{p}})$ . Similarly, from Eq. (3.6) we can also show (we include from now on the additional factor of 2 for the spin)

$$D_n = tL_xL_y/(4\pi^3) \int \int \int dE_p dp_x dp_y T^n(\mathbf{p}). \quad (3.8)$$

We should note that so far, we have derived the relationship between the long-time behavior of the  $D_n$  and the powers of the single-particle transmission coefficient  $T^n(\mathbf{p})$  only for the special potential  $V(z)$  with a step at  $z=0$ . Had we chosen a more general smooth form, it would introduce into the derivation a new spatial region where the potential is ramped up. However, the corresponding additional contributions to the scalar products  $\langle p|\mathbf{k} \rangle$  and  $\langle n|\mathbf{k} \rangle$  would arise from integrals over the finite region with nonsingular integrands. Compared to the infinite contributions  $\delta(E_p - E_n + V_0)$  obtained from the two semi-infinite regions where the potential is either zero or constant, the finite contributions are therefore negligible in the long-time limit. We can therefore replace  $T(\mathbf{p})$  in Eq. (3.8) with the corresponding quantity characteristic of any potential.

We can now use the main result Eq. (3.8) to determine the vacuum decay rate  $\gamma$  for the vacuum decay  $P_0(t) = \exp(-\gamma t)$ ,

$$\gamma = L_xL_y/(4\pi^3) \int \int \int dE_p dp_x dp_y \sum_n (1/n) T^n(k). \quad (3.9)$$

Note that this general expression applies to any potential  $V(z)$ .

To apply this expression to the special case discussed by Schwinger, we consider the Sauter potential  $V(z) = V_0 [1 + \tanh(z/W)]$  because its transmission coefficient is available analytically [7,9,36]

$$T(p) = \{ \cosh[\pi W(p + \tilde{p})] - \cosh[\pi W(p - \tilde{p})] \} / \{ \cosh[\pi W V_0/c] - \cosh[\pi W(p - \tilde{p})] \}. \quad (3.10)$$

In the limit of a large extension of the force field,  $W \rightarrow \infty$ , this coefficient simplifies to

$$T(p) = \exp[\pi W(p + \tilde{p} - V_0/c)]. \quad (3.11)$$

If we assume that  $T(p)$  is a flat top function (which is only true for large  $A = V_0/(2W) = V_0/L_z$ ) and expand  $p + \tilde{p} - V_0/c$  around  $E = V_0/2$  only to the 0th order, i.e.,  $T(p) = \exp(-\pi c^3/A)$ , then it leads to the following vacuum decay per unit time:

$$\gamma = L_xL_y/(4\pi^3) \int_0^{2AW} dE \sum_n 1/n \exp(-n\pi c^3/A) \int dp_x \int dp_y \times \exp[-n\pi c(p_x^2 + p_y^2)/A] = L_xL_yL_z/(4c\pi^3) A^2 \sum_n (1/n^2) \exp(-n\pi c^3/A). \quad (3.12)$$

Note that since we took the limit of  $W \rightarrow \infty$  with  $V_0 = 2AW$ , the integration range of the energy can be changed from  $(c^2, V_0 - c^2)$  to  $(0, V_0)$ . The integrations over  $p_x$  and  $p_y$  were extended to  $-\infty$  and  $\infty$ .

From the expression above, it is clear that if the spatial extensions of the potentials  $L_xL_y$  and  $L_z (=2W$  here) approach infinity, the vacuum decay rate diverges. This means the vacuum probability would be 0 for an extended potential as considered by Schwinger. In this case, we need to define the decay rate per unit volume  $\Gamma \equiv \gamma/L_xL_yL_z$  and the vacuum decay becomes exponential in time

$$P_0 = \exp(-\Gamma L_xL_yL_z t), \quad (3.13)$$

with  $\Gamma = 1/(4c\pi^3) A^2 \sum_n (1/n^2) \exp(-n\pi c^3/A)$ . This rate is exactly Schwinger's result [4], but it is now derived as the long-time limit of an *ab initio* theory; an approach that is different from the usual *S*-matrix theory.

Having established a more general framework, we can also provide corrections to the Schwinger limit due to the inhomogeneities of the field. In the derivation leading to Eq. (3.12), we expanded  $p + \tilde{p} - V_0/c$  around  $E = V_0/2$  and considered the leading term to be independent of the energy. This is not entirely correct, in fact, a more accurate approximation is

$$\Gamma = 1/(4\pi^3 L_z) \int_0^{AL_z} dE \sum_n 1/n \exp(-n\pi c^3) \{ A[1 - (E/V_0 - 1/2)^2] \} \int dp_x \int dp_y (1/n) \times \exp(-n\pi c(p_x^2 + p_y^2)) / \{ A[1 - (E/V_0 - 1/2)^2] \}. \quad (3.14)$$

If we now perform the limit  $W \rightarrow \infty$ , the integration ranges of  $p_x$  and  $p_y$  are from  $-\infty$  to  $\infty$  and we obtain

$$\Gamma_\infty = 1/(4\pi^3 c L_z) \int_0^{AL_z} dEA [1 - 4(E/V_0 - 1/2)^2] \times \sum_n \exp\{-n\pi c^3 / \{A[1 - 4(E/V_0 - 1/2)^2]\}\} / n^2. \quad (3.15)$$

This decay rate per unit volume  $\Gamma_\infty$  reduces to the usual Schwinger rate  $\Gamma$  only if we could discard the term  $(E/V_0 - 1/2)^2$  and kept only the zeroth order term of the  $T(p)$ .

However, in the large width limit, the exact decay rate  $\Gamma$  can be recovered from the Schwinger formula if it is averaged over the spatial profile of the Sauter force field [9], assumed to be dependent on the position,  $A(z) \equiv -dV(z)/dz$ . The profile averaged rate  $\Gamma_{\text{av}}$ , defined as  $\lim_{W \rightarrow \infty} \frac{1}{2W} \int_{-W}^W dz A(z)^2 / (4\pi^3 c) \sum_{n=1}^{\infty} n^{-2} \exp[-n\pi c^3 / |A(z)|]$ , is identical to Eq. (3.15) for the Sauter potential using the relationship  $A(z) = A\{V^{-1}[V(z)]\} = A\{1 - 4[V(z)/V_0 - 1/2]^2\}$  and  $V_0 \gg c^2$ .

Since numerical calculations are generally computationally quite efficient in one spatial dimension and permit a comparison to analytical expressions, we rederive the corresponding one-dimensional version for  $\Gamma$ ,

$$\Gamma = 1/(\pi L) \int_0^{AL} dE \sum_n 1/n \exp(-n\pi c^3 / \{A[1 - (E/V_0 - 1/2)^2]\}). \quad (3.16)$$

This expression can be used to obtain the final approximation for both large and small  $A$ .

As an interesting side note, we would like to add that in the Schwinger limit  $D_n = A^2 / (4c\pi^3 n) t \exp(-n\pi c^3 / A)$ , the generating function Eq. (2.18) reduces to

$$F^{(3D)}(x) = \exp\left\{-L_x L_y L_z A^2 / (4c\pi^3) t \int_0^{\varepsilon(1-x)} dy [\ln(1-y)]/y\right\}, \quad (3.17)$$

where  $\varepsilon = \exp(-\pi c^3 / A)$ . One may then calculate all multipair probabilities  $P_m^{(3D)}$  via  $(1/m!) d^m F^{(3D)} / dx^m|_{x=0}$ . For the case of a one-dimensionally extended field, the corresponding expression is even simpler, as the corresponding generating function takes the form

$$F^{(1D)}(x) = (1 - \varepsilon + \varepsilon x)^{ALt/\pi} \quad (3.18)$$

and therefore

$$P_m^{(1D)}(t) = (1/m!) [\varepsilon / (1 - \varepsilon)]^m (ALt/\pi) (1 - \varepsilon)^{ALt/\pi} \times \prod_{j=1}^{m-1} (ALt/\pi - j). \quad (3.19)$$

Note that while many independent works have verified the expression for  $P_0(t)$  obtained by Schwinger, higher-order probabilities  $P_m(t)$  have not been reported before.

#### IV. CASCADELIKE DYNAMICAL GROWTH OF THE PROBABILITIES FOR MULTIPLE PAIRS

At the end of Sec. II, we showed that the time dependences of the  $m$ -pair probabilities  $P_m$  are given by the solutions Eq. (2.19). To obtain a better insight into their growth patterns, we also derived that these probabilities satisfy a set of linear rate equations where the coupling constants  $\gamma_j$  are obtained from the solutions to the Dirac equation Eq. (2.1) via the functions  $D_n$ . For the special case of supercritical force fields that become asymptotically independent of time, we have shown in the previous section [Eq. (3.8)] that the functions  $D_n$  become simply a linear function of time,  $D_n \rightarrow \chi_n t$ . If we use this expression, the coupling coefficients  $\gamma_j$  defined as  $\gamma_j = \sum_n dD_n/dt (n-1)! / (n-j)! / j!$  become time independent and take the form  $\sum_n \chi_n (n-1)! / (n-j)! / j!$ . For weak fields, the coefficients  $\chi_n$  decrease rapidly with increasing  $n$  and we can neglect  $\chi_n$  for  $n > 1$ . As a result, we obtain for the coupling constants  $\gamma_1 \approx \gamma_0 \approx \chi_1$  and  $\gamma_j \approx 0$  for  $j \geq 2$ . The rate Eqs. (2.21) reduce then to a set of cascadelike equations controlled by a single effective decay rate  $\chi_1$ ,

$$dP_0(t)/dt = -\chi_1 P_0, \quad (4.1a)$$

$$dP_1(t)/dt = -\chi_1 P_1 + \chi_1 P_0, \quad (4.1b)$$

$$dP_2(t)/dt = -\chi_1 P_2 + \chi_1 P_1, \quad (4.1c)$$

$$dP_3(t)/dt = -\chi_1 P_3 + \chi_1 P_2, \quad (4.1d)$$

....

In this cascadelike sequential process,  $P_0$  decays first, then  $P_1$  becomes populated and decays, and then  $P_2$  is excited following its decay. This sequence of the excitation and decay of the probabilities is the main mechanism to create states with higher number of particles in steady supercritical force fields. It might be interesting to examine under which conditions the alternative process of a direct excitation to multiparticle states is possible. A similar question has been discussed in strong-field multiphoton ionization physics where the transition from sequential to direct multiple ionization has been studied theoretically and also experimentally [47–50].

As Eqs. (4.1) are valid only in the long-time regime, they should not be solved with the physical initial condition  $P_n(t=0) = \delta_{n,0}$ . Instead, we need to solve these equations with the initial conditions  $P_n(t=T_{\text{on}}) \equiv p_n$ . Here  $p_n$  denote the probabilities after the turn-on time  $T_{\text{on}}$  of the supercritical field. As a result, we obtain for  $t > T_{\text{on}}$  the solutions

$$P_n(t) = \exp[-\chi_1(t - T_{\text{on}})] \sum_{m=0}^n [\chi_1(t - T_{\text{on}})]^m / m! p_{n-m}. \quad (4.2)$$

In a separate work [51], we will compare the predicted cascadelike time dependence of the probabilities  $P_n$  to exact *ab initio* solutions obtained from the Dirac equation.

### V. BRIEF SUMMARY AND FUTURE WORK

We have developed a quantum field theoretical formalism that allows us to compute the time-dependent multipair-production probability. The approach is very general and applies to external fields with arbitrary space and time variations. It is based on the connection between the  $n$ -pair creation probability and the average number of  $n$ -particle coincidence events. This framework makes the connection with the static field limit developed originally by Schwinger but it also permits new investigations for space-time varying fields. Even though the direct experimental observation of the spontaneous decay of the vacuum is presently lacking, the concept of multiparticle coincidence counts seems to be of relevance to possible observations in the future. The  $n$ -pair quantum field theoretical probability gives us a direct theoretical measure for the extent to which the vacuum has decayed.

While most experimental information about the creation of electron-positron pairs is available from heavy-ion collisions, the experimentally available laser intensities have steadily increased to make also light-matter creation experiments possible in the future. The current maximum laser intensities have not yet reached a sufficiently high value for a direct comparison to Schwinger's formula, but just recently the Paris-Michigan collaboration [24] reported on having reached the unprecedented peak intensity of  $2 \times 10^{22}$  W/cm<sup>2</sup> in the focal spot. Recent theoretical studies have demonstrated that the space or time variations in the force field may effectively lower the threshold for the light-assisted pair creation. More detailed calculations will be needed to estimate the field conditions for these processes. Our technique requires the (repeated) numerical solution to the time-dependent Dirac equation. This task has become feasible in reduced spatial dimension but it becomes numerically very involved if there are no simplifying symmetries involved. For instance, the presence of the magnetic field component of the laser requires a fully three-dimensional relativistic calculation.

So far, the only laser-related experimental evidence of pair-creation has been reported at SLAC [23] based on the initial presence of a relativistic electron beam. Our present work assumes the vacuum as the initial state but a generalization of our technique to initial states describing a high-energy fermion may be possible. Also, in our model, we have not considered the interaction between the created fermions. To include this possibly relevant interaction is very difficult and, at the moment, the only estimates for the relevance of interfermionic forces can be obtained from classical-mechanical-based multiparticle simulations.

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### APPENDIX

Here, we prove that if the electron-positron field operator fulfills the Heisenberg equation of motion  $i \partial \hat{\Psi}(t) / \partial t$

$= [\hat{\Psi}(t), \hat{H}(t)]_-$ , then  $\hat{\Psi}(t)$  can be obtained equivalently as a solution to the usual Dirac equation with the Hamiltonian  $h$  of Eq. (2.1),  $i \partial \hat{\Psi}(t) / \partial t = h(t) \hat{\Psi}(t)$ . The quantum field theoretical Hamiltonian  $\hat{H}$  is related to the usual (quantum-mechanical) Dirac Hamiltonian via  $\hat{H} = \sum_{\alpha} \sum_{\beta} \hat{b}_{\alpha}^{\dagger} \hat{b}_{\beta} h_{\alpha\beta}$ , where  $h_{\alpha\beta} = \langle \alpha | h | \beta \rangle$  denotes the usual single-particle scalar products. The states  $|\alpha\rangle$  are arbitrary and form a complete basis such that  $\sum_{\alpha} |\alpha\rangle \langle \alpha|$  is the unit operator. The field operator can be expanded in this basis, leading to  $\hat{\Psi}(t) = \sum_{\gamma} \hat{b}_{\gamma} |\gamma\rangle$ .

If we insert this expansion into the commutator required for the Heisenberg equation of motion, we obtain

$$\begin{aligned} [\hat{\Psi}(t), \hat{H}(t)]_- &= \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \hat{b}_{\gamma} |\gamma\rangle \hat{b}_{\alpha}^{\dagger} \hat{b}_{\beta} h_{\alpha\beta} \\ &\quad - \sum_{\alpha} \sum_{\beta} \hat{b}_{\alpha}^{\dagger} \hat{b}_{\beta} h_{\alpha\beta} \sum_{\gamma} \hat{b}_{\gamma} |\gamma\rangle \\ &= \sum_{\alpha} \sum_{\beta} \sum_{\gamma} [\hat{b}_{\gamma} \hat{b}_{\alpha}^{\dagger} \hat{b}_{\beta} - \hat{b}_{\alpha}^{\dagger} \hat{b}_{\beta} \hat{b}_{\gamma}] h_{\alpha\beta} |\gamma\rangle. \end{aligned} \quad (A1)$$

If we use the fermionic anticommutator relationship  $\hat{b}_{\alpha}^{\dagger} \hat{b}_{\beta} = \delta_{\alpha\beta} - \hat{b}_{\beta} \hat{b}_{\alpha}^{\dagger}$ , this expression simplifies to

$$\begin{aligned} [\hat{\Psi}(t), \hat{H}(t)]_- &= \sum_{\alpha} \sum_{\beta} \sum_{\gamma} [\hat{b}_{\gamma} (\delta_{\alpha\beta} - \hat{b}_{\beta} \hat{b}_{\alpha}^{\dagger}) \\ &\quad - (\delta_{\alpha\beta} - \hat{b}_{\beta} \hat{b}_{\alpha}^{\dagger}) \hat{b}_{\gamma}] h_{\alpha\beta} |\gamma\rangle \\ &= \sum_{\alpha} \sum_{\beta} \sum_{\gamma} [-\hat{b}_{\gamma} \hat{b}_{\beta} \hat{b}_{\alpha}^{\dagger} + \hat{b}_{\beta} \hat{b}_{\alpha}^{\dagger} \hat{b}_{\gamma}] h_{\alpha\beta} |\gamma\rangle. \end{aligned} \quad (A2)$$

Furthermore, if we use  $\hat{b}_{\beta} \hat{b}_{\gamma} = -\hat{b}_{\gamma} \hat{b}_{\beta}$ , this reduces to

$$\begin{aligned} [\hat{\Psi}(t), \hat{H}(t)]_- &= \sum_{\alpha} \sum_{\beta} \sum_{\gamma} [\hat{b}_{\beta} \hat{b}_{\gamma} \hat{b}_{\alpha}^{\dagger} + \hat{b}_{\beta} \hat{b}_{\alpha}^{\dagger} \hat{b}_{\gamma}] h_{\alpha\beta} |\gamma\rangle \\ &= \sum_{\alpha} \sum_{\beta} \sum_{\gamma} b_{\beta} [\hat{b}_{\gamma} \hat{b}_{\alpha}^{\dagger} + \hat{b}_{\alpha}^{\dagger} \hat{b}_{\gamma}] h_{\alpha\beta} |\gamma\rangle \\ &= \sum_{\beta} \sum_{\gamma} \hat{b}_{\beta} h_{\alpha\beta} |\gamma\rangle \\ &= \sum_{\beta} \hat{b}_{\beta} \sum_{\gamma} |\gamma\rangle \langle \gamma | h | \beta \rangle \sum_{\beta} \hat{b}_{\beta} h | \beta \rangle h \sum_{\beta} \hat{b}_{\beta} | \beta \rangle \\ &= h \hat{\Psi}. \end{aligned} \quad (A3)$$

We note that a similar result was derived by Greiner *et al.* [22]. However, in their case the derivation required a thorough discussion of boundary terms. We also note that, if  $i \partial \hat{\Psi}(t) / \partial t = h \hat{\Psi}(t)$ , it also follows automatically that  $i \partial \hat{\Psi}(t) / \partial t = [\hat{\Psi}(t), \hat{H}(t)]_-$ . The equivalence between solution of the Dirac equation and the proper quantum field-theoretic treatment has also been established in the context of pair creation in heavy-ion collisions (see, e.g., Refs. [52,53]).

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