2015

Preservice Secondary Mathematics Teachers' Knowledge of Generalization and Justification on Geometric-Numerical Patterning Tasks

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PRESERVICE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE OF GENERALIZATION AND JUSTIFICATION ON GEOMETRIC-NUMERICAL PATTERNING TASKS

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238 Pages August 2015

Generalization is essential to mathematical thinking (Kaput, 1999), and justification is its inseparable twin (Lannin, 2005). If students will be expected to generalize and justify, then it is important to develop an understanding of teachers' thinking about these concepts. This study examined secondary preservice teachers understanding of generalization, justification, and the interaction between these constructs.

Data were collected from ten participants who solved three quadratic geometric-numerical patterning tasks administered during a single interview. Data reduction (Miles, Huberman, & Saldana, 2014) and constant comparative (Glaser & Straus, 1967) methodologies were used to analyze written transcripts of the interviews.

The results of this analysis indicated that participants developed or attempted to develop a variety of explicit, recursive, and hybrid rules that appealed to figural, numerical, and symbolic characteristics. The participants justified by verifying and explaining their generalizations through numerical, figural, and symbolic arguments.

Participants appeared to encounter the most success generalizing when appealing to figural characteristics and verifying their generalizations through a numerical lens.

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A Dissertation Submitted in Partial
Fulfillment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

ILLINOIS STATE UNIVERSITY

2015

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ACKNOWLEDGMENTS

I would first like to thank the chair of my committee, Dr. David Barker, for all his time and effort in helping me through the dissertation process. I would not be at this point without his patience and advice in completing this undertaking. I would also like to thank my committee members—Ken Clements, Beth Hatt, Tami Martin, and Jennifer Tobias—their time, suggestions, and guidance were invaluable in helping me with this endeavor.

I would also like to thank the students who participated in this study. Their willingness to share their time and thinking with me is much appreciated.

I would like to thank my parents, Bryon and Cyndee Kirwan, for valuing education. I would have never made it this far without their unconditional love and support.

I would like to thank my wife, Emily, for her endless patience, support, and love throughout my studies. You kept me sane and grounded when I felt like I was losing my mind and floating away. I am grateful to have been blessed with such a wonderful partner. This dissertation is dedicated to you.

J.V.K.

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CHAPTER I

THE PROBLEM AND ITS BACKGROUND

Generalization is the essence of doing mathematics. Lee (1996) stated, "algebra, and indeed all of mathematics is about generalizing patterns" (p. 103). Zazkis and Liljedahl (2002) agreed, stating, "patterns are the heart and soul of mathematics" (p. 379). Based upon this assumption, it is vital that all students experience and develop their ability to generalize during their study of mathematics. This vital component of mathematics implies the necessity for teachers to be aware of generalization and to be prepared to interpret and understand students' thinking about generalization. Mason (1996) commented that

> "generalization is the heartbeat of mathematics, and appears in many forms. If teachers are unaware of its presence, and are not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place." (p. 65).

Although it may be difficult for teachers to interpret and understand student generalization, they must be able to understand student thinking about generalization to be able to support and encourage mathematical growth in their students. Unfortunately, interpreting and understanding student thinking is difficult for many teachers (Maher & Davis, 1990). This implies that pre-service teachers need to be prepared to support student thinking about generalization.

Pre-service teachers' understanding of generalization alone is insufficient because justification is the inseparable twin of generalization (Lannin, 2005). Radford (1996) noted that justification is the process that supports generalization and Ellis (2007a) commented that engaging in justification influences a student's ability to generalize. Additionally, Lannin (2005) noted that having students justify can provide a medium for teachers to understand student generalizations. Teachers understanding and promotion of student justification can help them understand student generalization. Stated differently, teachers must also be able to interpret and understand how students justify their generalizations, in addition to how students make generalizations, because they are related phenomenon.

Purpose of the Study

What preservice mathematics teachers (PSTs) know about generalization and justification must be identified so that mathematics teacher educators can build upon it during teacher preparation. PSTs need to be able to interpret and understand their student's thinking about generalization and justification. The purpose of this study is to investigate the generalizations made by secondary PSTs, to explore the justifications provided for their generalizations, and to identify any relationships between generalization and justification.

Rationale for the Study

Teacher Knowledge

PSTs need to possess a deep understanding of the content they are to teach (Ball, 2003; Ball, Thames, & Phelps, 2008; Shulman, 1986, 1987; Stein, Baxter, & Leinhardt, 1990). Unfortunately, many PSTs do not possess this understanding (Bryan, 1999; Chinnappan, 2003; Davis, 2009; Even, 1993). Deficiencies in teachers' content knowledge have historically been addressed by requiring advanced content coursework (Ball, 1992; CBMS, 2012; Ferrini-Mundy & Findell, 2001), yet this approach has been found insufficient. Moreover, Nathan and Petrosino (2003) found this approach detrimental because PSTs viewed symbolic fluency as a precursor to being able to solve contextual problems, a view in contrast to students' actual problem solving abilities. PSTs knowledge matters, but is often not aligned with the mathematics needed for teaching.

Meaningful research regarding PST's content knowledge is largely an uninvestigated area (Kieran, 2007; Lewis, 2008; Lucas, 2006). Kieran (2007) commented that although the preparation of pre-service secondary teachers is a complex matter, "research that informs as to the nature of this complexity is still quite rare, especially with respect to the teaching of algebra" (p. 745). Identifying PSTs knowledge of generalization and justification is important because they are intertwined topics (Lannin, 2005) that can be used as an approach to teaching algebra (Lee, 1996). The Mathematics Education community needs to understand PSTs understanding of generalization and justification so that it can be built upon during teacher preparation coursework. This study investigated to what extent PSTs understand generalization and justification of quadratic relationships that arise from geometric-numerical (Radford, 1996) patterning contexts.

Generalization

PSTs knowledge of generalization should be studied for the following reasons. One, generalization is an essential component of mathematics. Kaput (1999) stated that "generalization and formalization are intrinsic to mathematical activity and thinking—they are what make it mathematical" (p. 137). Without generalization taking place, mathematical thinking is not occurring (Mason, 1996). Due to the centrality of generalization and its growing focus in research (Ellis, 2007a), it is important to understand PSTs thinking about generalization so it can be developed during teacher preparation to help support students' thinking about generalization.

Two, generalization is a core aspect of algebraic thinking (Kaput, 2008), and one method that can be used to help students transition from arithmetic to algebra (Lee, 1996; Blanton & Kaput, 2011). If PSTs are to be able to make use of this method to help students in this transition, a deeper understand of what PSTs know about generalization, and algebraic generalization in particular, is needed. This understanding can then be built upon during PSTs preparation to position them to help their future students' transition from arithmetic to algebra.

Three, the CCSS-M (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) indicated that generalization is a topic K–12 students should study. For example, the Standards for Mathematical Practice argue that students should "look for and make use of structure" (p. 8) and "look for and express regularity in repeated reasoning" (p. 8). If students will be expected to generalize, then PSTs need to possess this ability as well. Therefore, it is important to develop an understanding of PSTs generalization abilities so they can be further developed during their preparation coursework.

Ellis (2007a) stated that, "the ways in which students generalize will influence the tools that they can bring to bear when justifying their general statements" (p. 195). Thus, in order to make sure PSTs will be prepared to understand their students' generalization, it is important to research PSTs thinking about justification as well.

Justification

It is important to study PSTs knowledge of justification because students' justification abilities have been associated with their ability to reason algebraically (Blanton & Kaput, 2002). Ellis (2007a) echoed this argument stating that "engaging in acts of justification may be as likely to influence students' abilities to generalize as the other way around" (p. 169). Lannin (2005) noted that having students justify can provide a means for teachers to understand student generalizations, and what teachers know influences their classroom practice (Knuth, 2002). These statements taken together indicate that PSTs need to deeply understand justification, generalization, and the connections between them, to be able to help support their student's thinking in the classroom. Radford (1996) argued that "the logical base underlying generalization is that of justifying the conclusion" (p. 111). Unfortunately, few researchers have studied how pre-service teachers' knowledge of generalization and justification are associated (e.g., Richardson, Berenson, & Staley, 2009). This study seeks to gain an understanding of each construct and the interaction between them within the context of quadratic relationships.

The Context: Quadratic Relationships

Quadratic relationships were chosen as the context for this study for the following two reasons. First, researchers have noted the overabundance of studies focused on linear (e.g., Stump, 1999) and exponential (e.g., Presmeg & Nenduradu, 2005) relationships (Kieran, 2007; Vaiyavutjamai & Clements, 2006). Studies investigating quadratic relationships are limited (e.g., Li, 2011) and in need of further study. Moreover, only a subset of these studies on linear and exponential relationships have investigated the phenomenon of generalization, with even fewer investigating the generalizations given to the linear or exponential relationship under study (e.g., Healy & Hoyles, 1999; Lannin, 2003). Although researchers have identified some generalization strategies to linear and exponential relationships (e.g., guess-and-check, recursion, contextual), it may be the case that PSTs generalize quadratic relationships differently. Ellis (2004) commented that student's methods for generalization in one context may not transfer to other contexts. Investigating generalization in the context of quadratic relationships is a question largely uninvestigated.

Second, PSTs need a deep understanding of quadratic relationships because they are an integral part of the high school curriculum. Quadratic relationships are one of the "natural steps" in coming to understand polynomials, a central topic of algebra. Quadratic relationships also occur in geometry in the form of conic sections and area as well as in related fields such as statistics (e.g., quadratic regression curves) or physics (e.g., acceleration/velocity/position problems). Additionally, reform documents such as the *Common Core State Standards for Mathematics* (CCSS-M) (National Governors

Association Center for Best Practices, Council of Chief State School Officers, 2010) and NCTM's (2000) *Principles and Standards for School Mathematics* advocate the teaching of quadratic relationships. As future high school teachers (grades 9 – 12), PSTs will be expected to have a thorough understanding of quadratic relationships.

Statement of the Problem

In the rationale above, I argue that content knowledge is an essential component for teaching. Generalization is an important component of mathematical content knowledge, and its use is advocated for in reform documents. Moreover, generalization can be used to help students transition from arithmetic to algebra (Lee, 1996; Blanton & Kaput, 2011). Justification and generalization have been identified as related constructs (Lannin, 2005), but what secondary PSTs understand about generalization and justification for quadratic relationships is under researched. The purpose of this study was to provide a description of the generalizations and justifications given by PSTs when working with quadratic relationships. Additionally, this study sought to describe relationships between these generalizations and justifications.

Research Questions

The rationale and problem statement above prompted the following research questions. The participants in the study were secondary PSTs in their final year of teacher preparation.

1. What types of rules are given by preservice secondary teachers for quadratic patterning tasks presented in a geometric-numerical format? What patterns or relationships exist between the types of rules across tasks?

2. What types of justifications are given by preservice secondary teachers when solving quadratic patterning tasks presented in a geometric-numerical format? What patterns or relationships exist between the types of justifications across tasks?
3. What patterns or relationships exist between the types of rules and justifications given?

Rules Modeling Patterning Tasks

In the rationale and research questions above, quadratic relationships and quadratic patterning tasks are referred to—this requires some clarification. The use of this phrase is not meant to imply that *only* quadratic rules may be used to model the relationship present in the tasks. In fact, many different rules may be used to model the relationship, such as cubics, quartics, or piecewise functions. A quadratic is the smallest degree polynomial that may be used to model the relationship. The purpose in using the word "quadratic" was to attempt to describe a particular class of patterning tasks. Additionally, this name was utilized given that in many cases, the participants in this study did provide a quadratic rule. Thus, when the phrase "quadratic patterning tasks" is utilized in this dissertation, it is to refer to a particular class of patterning tasks, regardless of the rule utilized to model the relationship in the tasks.

Due to there being multiple rules that could be given to model the relationship in the patterning tasks, there is no single correct rule for each task. Rather, multiple correct rules may be given. It is important to note that what makes a rule correct depends upon what is meant by the word, correct. For example, a rule may be considered correct if it is one from a list of responses validated by an authority, such as a teacher or answer key. However, another example of a correct rule may be seen in the use of a particular set of assumptions, operating with those assumptions in a logical manner to arrive at a conclusion. Thus, if two different individuals made two different sets of assumptions about a task and operated with those assumptions in a logical manner, each may arrive at a different rule for the task, with each rule being correct. These two different facets of correctness illustrate the difficulty in trying to pinpoint what a correct rule to a task is. Thus, when participants provided rules to the tasks in this study, the correctness of those rules was not analyzed (see Chapter Three for additional details regarding attempts to analyze for correctness).

Theoretical Lens

In reading through the research literature, a variety of frameworks for generalization and justification are given. Although a variety of frameworks exist, such as those for linear or exponential relationships, I noticed that the literature did not contain one for the generalization and justification of *quadratic relationships with secondary PSTs*. Because researchers have wondered about the cross-relationship and cross-population applicability of existing frameworks (e.g., Ellis, 2004), I chose to not select one of these existing frameworks as a theoretical lens for this study. However, these frameworks influenced the study, especially in helping to situate it in terms of the existing literature (see Chapter Two, Review of Related Literature for a thorough discussion). Additionally, there were other theoretically influences that helped shape this study's design, which are further elaborated below.

Paradigm

This study was conceived in the interpretivist paradigm. This perspective assumes that all people do not respond to stimuli in the same way and so the responses given to stimuli are based upon each person's interpretation of the situation (Glesne, 2011; Miles, Huberman, & Saldana, 2014). Glesne (2011) commented that what matters "is how people interpret and make meaning of some object, event, action, perception, etc." (p. 8). An overarching goal of this study is to describe secondary PSTs generalizations and justifications with quadratic relationships. From the interpretivist perspective one would expect variation in secondary PSTs generalizations and justifications. It is this variation that I was interested in observing and characterizing.

The interpretivist paradigm was drawn upon for two reasons. One, with limited research on PSTs understanding of generalization and justification, I wanted to develop a description for the different types of generalization and justification, as well as any relationships between them. This paradigm allowed for variation in the participants' generalization and justification strategies because the paradigm assumes that not every participant will respond to the tasks in the same manner. Two, interaction with the participants during the interviews was unavoidable, implying that the understanding that was developed during each interview was situated in that context (Glesne, 2011). Variability across all of the interviews was inevitable and the interpretivist paradigm allows this variability to be captured in descriptions for the different types of generalization and justification.

Views on Knowledge

Teachers' knowledge has been a focus of research for decades. In this study, the type of teacher knowledge investigated was common content knowledge as described by Ball, Thames, and Phelps (2008). These researchers defined common content knowledge as "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399). Stated differently, the type of content knowledge examined was knowledge that would be expected to be held by any high school graduate, not any specialized forms of it (e.g., pedagogical content knowledge).

Common content knowledge was chosen instead of other types of knowledge (e.g., pedagogical content knowledge) because more studies are needed to identify what teachers know about the content they will be expected to teach. Specifically, what do teachers know about generalization and justification with regards to quadratic relationships? Using a generalization approach to teaching algebra has been identified to be a productive alternative to the traditional algorithmic, symbolically-focused approach (Blanton & Kaput, 2011; Lee, 1996). However, if teachers are going to be able to use this approach, then an understanding of what they know about the content underlying the types of tasks that students might be expected to solve is needed.

Views on Generalization

One tension that exists in the use of the word generalization is that it can mean product/object or process/activity (Yerushalmy, 1993). In this study, generalization was conceptualized as a product or object, in contrast to a process or activity. The generalizations that are described in this study are viewed as static objects. So when a participant develops a rule to a task, it is the rule that is the generalization as it is seen as the object/product. The process used to develop the rule is not viewed as the generalization, as the process is dynamic and changing in nature.

Another tension that exists in the study of generalization is whether generalization is conceptualized as internal or external. In this study, generalization was viewed as a phenomenon that is external to the individual. That is, generalizations exist outside of, and separate from, the individual. The generalizations that are described in this study are captured in the written rules or verbal statements made by the participants, pieces of evidence that are outside the mind of the individual. These written and verbal statements are not used as evidence to infer internal or cognitive generalizations, but are the generalizations themselves.

Views on Justification

A dichotomy exists in the study of justification. One area of research on justification is associated with deductive reasoning and work with general cases. In this vein, justification is often used as a synonym for proof (i.e., operating with general cases and/or statements to deduce information about other general cases and/or statements). The other area of justification research conceptualizes justification as the reasoning associated with inductive reasoning and specific cases. From this viewpoint, justification is used to describe arguments based upon specific cases that are then used to provide support for a general case or statement. In this study, the meaning for justification aligns more with the latter, associated with inductive reasoning and work with specific cases (see Definitions for and Meanings of Justification in the Literature Review for a more thorough discussion of these differences).

CHAPTER II

REVIEW OF RELATED LITERATURE

Review of Literature Chapter Structure

The literature reviewed in this chapter focuses around the constructs of generalization and justification. The review of literature on generalization begins with the presentation and discussion of definitions and characteristics of generalization, and then progresses into associated assumptions and research traditions. Next, the focus is narrowed to the different types of reasoning utilized when generalizing. After a review of the literature on generalization, I will shift to justification, beginning with the presentation of some definitions and meanings for justification. The section then continues with an overview of the traditions for researching justification, and then focuses on the types of reasoning utilized when justifying. The chapter concludes with a discussion of the relationship between generalization and justification.

Definitions of and Characteristics of Generalization

Generalization has been defined in multiple ways in the research literature. To illustrate, consider the definitions below.

1. "The process of applying a given argument in a broader context" (Harel & Tall, 1989, p. 38).

2. "Lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves, but rather on the patterns, procedures, structures, and the relations across and among them" (Kaput, 1999, p. 137).
3. "Engaging in at least one of three activities: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it is oriented, or (c) deriving broader results from particular cases" (Ellis, 2007b, p. 197).

These definitions provide a sense of what the construct of generalization is, but also point towards different characteristics and conceptualizations of generalization. These different characteristics and conceptualizations are discussed below.

One characteristic of generalization is that of extension. That is, increasing the boundaries for a given argument or reasoning. Ellis (2007b) utilized this exact root word, extension, in part b of her definition. Harel and Tall (1989) also appear to appeal to this characteristic in their phrasing, broader context. Generalization as extension takes an argument or reasoning and identifies additional cases in which that same argument or reasoning applies

A second characteristic associated with generalization is that of abstraction. That is, identifying properties or traits that are common or invariant among an entire class of objects. Ellis (2007b) appears to have appealed to this characteristic in part (a) of her definition. Kaput (1999) also appears to have appealed to this characteristic of generalization in describing that the focus is on the trends or commonalities among or between the cases. Generalization as abstraction synthesizes the commonalities present for a particular group of cases.

A third characteristic that is connected with the idea of generalization is that of process. Stated differently, generalization is an action or activity, not an object. Harel and Tall (1989) used this exact word, process, in their definition for generalization. Action can also be seen by the verb "lifting" used in Kaput's (1999) definition, as well as in the verbs "utilized" by Ellis (2007b) (i.e., identifying, extending, deriving). From this view, generalization is an activity one engages in.

Seemingly contrary to the third characteristic of generalization, a fourth characteristic is that it is itself an object. That is, generalization may also be viewed as the result of a process (i.e., a product). Kaput (1999) appeared to identify these generalization objects in his definition as patterns, procedures, structures, and relations. These objects can be seen in Ellis (2007b) definition as the commonality across cases (part a), the reasoning being extended (part b), and the derived results (part c). From this view, generalization is the result, or product, of one's actions.

The above discussion helps to identify commonalities and differences across the definitions and characteristics of generalization. It is worth noting that any one definition of generalization may contain a combination of these characteristics. For the purposes of this study, the definition of generalization that was utilized was the third definition (i.e., Ellis' 2007b definition) as provided above. That is, the definition for generalization I adopted in this study was "engaging in at least one of three activities: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it is oriented, or (c) deriving broader results from particular cases" (Ellis, 2007b, p. 197).

Assumptions about Generalization

A consequence of these different characteristics of generalization is that different assumptions may be made in the study of generalization. When studying generalization, some researchers have conceived it as an internal construct (e.g., cognitive) whereas others have conceived of it as external (e.g., empirical evidence). This tension between viewing generalization as internal versus external appears to be a consequence of studying student thinking (i.e., internal), yet needing empirical evidence (i.e., external) to support claims about student cognition. Tension between internal and external aspects of a construct have been documented in other areas as well, such as with representation. A brief discussion of assuming generalization as internal and external follows.

Generalization as an Internal Construct

One assumption about generalization that has emerged is that it is an internal construct. Phrased differently, generalization is an activity that occurs in one's mind. For example, Ellis (2007b) developed a taxonomy of actions one utilizes while generalizing. She referred to these actions as "generalizing actions" (p. 233) and defined them as "learners' mental acts as inferred through a person's activity and talk" (p. 233). This definition indicates that these acts of generalization are internal because they are identified as mental acts. Harel and Tall (1989) appeared to conceive generalization as an internal construct as well—their definitions for expansive, reconstructive, and disjunctive generalization are all described in terms of cognitive schema (i.e., internal structure). Although these definitions identify what these internal cognitions are, the definitions do not describe how one operates with them.

One manner in which the internal construct of generalization is understood to operate is through abstraction. Harel and Tall (1989) described abstraction as "when the subject focuses attention on the specific properties of a given object and then considers these properties in isolation from the original" (p. 39). Bills and Rowland (1999) referred to this as a structural generalization, and Dörfler (1991) called generalizations of this sort theoretical generalizations. Dubinsky and Lewin (1986) further extrapolated on one facet of Jean Piaget's reflective abstraction (e.g., Piaget, 2001), describing it as "a reflection of one or more structures onto a higher plane in which the structures function in greater generality" (p. 61). That is, in reflecting on the structures, the commonality across the structures becomes realized, allowing for greater generality. Ellis (2007b) identified this as the generalizing action of searching, describing it as when a student performs "the same repeated action in an attempt to determine if an element of similarity will emerge" (p. 238). Abstraction is one method for which generalization operates when assumed to be an internal construct.

Generalization as an External Construct

In contrast to the assumption of generalization as internal, a different assumption is that generalization is an external construct. Ellis (2007b) identified external aspects of generalization and referred to them as "reflection generalizations." She defined them as the verbal or written statements an individual makes as the mental processes are being carried out (Ellis, 2007b). Examples of this include stated general rules or methods (e.g., recursive, explicit), or commonalities for a group of cases (e.g., all cases possess characteristic or property X). In assuming that generalization is an external construct, the rules, commonalities, definitions, or other statements that are captured empirically are objects that are themselves understood to be the generalization.

Bills and Rowland (1999) and Dörfler (1991) also distinguished some generalizations as external and referred to them as empirical generalizations. Empirical generalizations are based upon identifying commonalities among a particular set of objects. A key feature of these external generalizations is that the generalization is not abstracted. This leaves the generalization tied specifically to the set of objects that the commonality was observed within. That is, that commonality is not extended (which would require abstraction) to encompass a larger set of objects.

Traditions for Researching Generalization

Despite the different characteristics of, and assumptions made about, generalization, two major traditions of generalization research have developed. One tradition focuses on generalization as a process (i.e., activity or action) and the other tradition focuses on generalization as a product (i.e., result of a process, or object). A discussion of each follows.

Generalization as a Process

Some researchers have interpreted generalization as a process and have investigated the processes associated with generalization (e.g., Radford, 1999, 2003; Zazkis, Liljedahl, & Chernoff, 2008). For example, Becker and Rivera (2006) investigated how sixth-grade students make and justify generalizations in algebra. The researchers identified several essential characteristics the participants used during this process. These characteristics included the ability to (a) compare figures to visually identify invariant relationships, (b) identify which components of the figures would be useful in developing a symbolic representation of the generalization, (c) utilizing multiplicative relationships over additive ones, and (d) being able to relate decomposed components of a visual figure to the symbolic generalization. These characteristics highlight some activities engaged in during the process of generalization.

Researchers working in this tradition sometimes appeared to assume generalization was an internal construct. For example, Ellis (2007b) identified three different types of actions carried out during the generalization process—relating (i.e., making associations between two or more elements), searching (i.e., iterating a single type of association to identify common structure among the elements), and extending (i.e., extrapolating the common structure to elements where an association was not originally viewed). She indicated that these activities described learners' mental actions, which implies that they are inherently internal. Lobato, Ellis, and Munoz (2003) also appeared to assume generalization was an internal construct. The authors stated that, "generalizing involves the extension of an existing mental structure to new objects and situations" (Lobato et al., 2003, p. 3). This statement implies that generalization must be an internal construct due to its work with mental structures. These two studies help provide evidence of the association between generalization as a process and the assumption of generalization as an internal construct.

Generalization as a Product

Other researchers have interpreted and studied generalization as the result or product of a process (e.g., Chua & Hoyles, 2010; Kirwan, 2013; Stacey, 1989). For example, Rivera and Becker (2008) investigated middle school students' ability to develop rules on linear patterning tasks. The researchers classified the students' rules from the patterning tasks' figures based upon whether they were constructive or deconstructive generalizations. A constructive generalization is one that is developed by perceiving a figure as composed of non-overlapping parts, where as a deconstructive generalization is one that perceives figures as composed of overlapping pieces. Thus, constructive generalizations add up quantities embodied in the non-overlapping parts in developing a rule, whereas deconstructive generalizations have to compensate for the over-counted parts in the development of a rule. The researchers sorting the students' rules in this manner treated them as products, the result of the generalization process.

Researchers working with this conceptualization appeared to assume generalization as an external construct. For example, Chua and Hoyles (2012) investigated whether including "jumps" in a sequence of cases (e.g., cases 1, 2, and 3 versus cases 1, 2, and 4) on pattern generalization tasks influenced secondary Singapore students' ability to develop rules for the task. The researchers used a six-point scale to measure the sophistication in the type of rule developed and the reasoning used to develop the rule as exhibited in the participant's work. Thus, this study measured only an external product produced by the participants (i.e., the participant's written work). This study helps to provide evidence of the association between generalization as a product and the assumption of generalization as an external construct.

Types of Reasoning Used to Generate Rules for Patterns

One area commonly studied within the generalization as product tradition are the rules used to describe a given pattern (e.g., English & Warren, 1995; Lee, 1996; Stacey, 1989). These rules are identified by the external statements, either written or verbal, made by the participants (Ellis, 2007b). Some researchers have argued for the importance of expressing generalizations symbolically (e.g., Kieran, 1989; Kinach, 2014), while other researchers have viewed non-symbolic or verbal descriptions (e.g., Zazkis & Liljedahl, 2002) as equally valid generalizations. The following synthesis of research did not assume that a particular representation (e.g., symbolic, verbal) was necessary to indicate a rule for a pattern. Additionally, research on pattern generalization is limited, and many of these studies are based upon linear relationships in the patterning tasks.

Recursive Reasoning

One common type of reasoning used in generalizing patterning tasks is the use of recursive reasoning (Becker & Rivera, 2006; Lannin, 2003; Mason, 1996; Townsend, Lannin, & Barker, 2009; Zazkis & Liljedahl, 2002). Recursive reasoning utilizes patterns that exist between successive cases to determine the next case in a sequence. Although not all researchers use this terminology when referring recursive reasoning, I saw this conception infused within descriptions for different types of generalizations. Thus, the descriptions that follow may be solely focused on recursive reasoning as described above, or they may indirectly appeal to this conception.

In one study, Becker and Rivera (2005) had 22 Grade 9 students work a linear patterning task during individual interviews. The researchers identified students using finite differences in tables in their working of this task. The authors described this approach as adding a constant to one value in a single variable the table to determine the subsequent value. Recursive reasoning also operates in this manner.

In another study, Stacey (1989) posed three linear generalization tasks to experienced and inexperienced secondary students, as well as primary students. One type of question included on these tasks was near generalization questions. She described these as a "question which can be solved by step-by-step drawing or counting" (p. 150). In order to make a drawing for a sequence of cases (i.e., step-by-step drawing or counting), one must identify a pattern that exists between successive cases. Without this pattern, the drawing or counting could not occur step-by-step to the desired case. To note, the depth of consciousness and understanding of the pattern being operated with may vary.

Although Stacey (1989) identified near generalization as a type of question on tasks, Kinach (2014) identified this as a way of reasoning and called it generalizing by analogy. She gave the example of "counting the bricks in a picture sequence of growing towers for the first four towers and then drawing the fifth tower" (Kinach, 2014, p. 433). In order to be able to draw the fifth tower (i.e., the subsequent case in the sequence of given cases), one must identify a pattern that exists between the first four given cases, so it can be utilized to develop the fifth tower (i.e., the subsequent case). Stacy's (1989) and Kinach's (2014) notions of near generalization and generalizing by analogy appear to incorporate aspects of recursive reasoning, but do not exactly match the description of recursive reasoning given.

Explicit Reasoning

Another common type of reasoning utilized in generalizing patterning tasks is explicit reasoning (Lannin, 2003; Mason, 1996; Townsend, Lannin, & Barker, 2009). Explicit reasoning is directly relating two (or more) co-varying quantities, often by a rule or formula. Although not all researchers use this terminology when referring to this way of reasoning, I saw this conception infused within descriptions for different types of generalizations. Thus, the descriptions that follow may be overtly focused on explicit reasoning as described above, or they may indirectly appeal to this conception.

Stacey's (1989) notion of far generalization appeared quite similar to explicit reasoning. In this study, she posed three linear patterning tasks to students, which included questions about far generalization. She described far generalization as "a question which goes beyond reasonable practical limits of such a step-by-step approach [near generalization]" (p. 150). Rather than determining a case that is near to a particular set of given cases, one must determine a case that is far away from the given cases. One cannot feasibly use recursive reasoning to determine a particular case that is substantially far into a sequence. Stacey (1989) commented that in order for students to determine a far generalization, they need general rules, or generalized forms of the number, that allow for determining any term in the sequence. Although far generalization is not the same as explicit reasoning described above, it appears to encourage explicit reasoning due to the necessity of determining cases that are substantially far into a given sequence. However, it should be noted that other ways of reasoning can be utilized to answer far generalization questions (e.g., recursive reasoning) because these questions appear to promote, but not require, explicit thinking.

Although Stacey (1989) identified far generalization as a type of question on tasks, Kinach (2014) identified this as a type of generalization called generalizing by extension. She gave the example of writing a formula that relates the size of a tower to the number of bricks needed to build that tower. Essentially, generalizing by extension identifies the direct relationship between the varying quantities that can be utilized for any case. That is, generalizing by extension models the general form of all cases. Kinach's (2014) notion of generalizing by extension appears to be closely related to the definition of explicit reasoning

Figural Reasoning

Although recursive and explicit reasoning are two of the most dominant types of reasoning identified in the literature, other types of reasoning are provided. One of these other ways is figural reasoning (Becker & Rivera, 2005; Chua & Hoyles, 2010, Mason, 1996; Rivera & Becker, 2003). Figural reasoning utilizes figures, diagrams, and other visuals to identify variant and invariant characteristics, properties, or structures in a set of objects. Although not all researchers use this terminology when referring to this way of reasoning, I saw this conception infused within descriptions for different types of generalizations. Thus, the descriptions that follow may be overtly focused on figural reasoning as described above, or they may indirectly appeal to this conception.

In a 2006 study, Becker and Rivera conducted clinical interviews with 29 Grade 6 students. The students were given a linear patterning task and asked to come up with a rule relating the number of tiles in a picture to that picture's number. The researchers found that some students utilized figural reasoning. Becker and Rivera (2006) called this figural generalization, describing it as a sequence of figural cues that posses "invariant structures and thus, are necessarily constructed in particular ways" (p. 466). The researchers concluded that developing rules from figural reasoning was challenging for students, but was influenced by the figural relationship identified.

In a 2008 study, Rivera and Becker further separated figural reasoning into two subcategories—constructive and deconstructive generalizations. Constructive generalizations occur when the figure is separated into non-overlapping pieces that can be counted and used to develop a rule. Deconstructive generalization is when the figure is viewed as being separated into overlapping pieces that can be counted and used to develop a rule after subtracting any overlapping parts. Chua and Hoyles (2010) further refined Rivera and Becker's (2008) constructive generalization, separating it into two types called additive construction generalization and non-additive constructive generalization. Additive constructive generalization is the same as Rivera and Becker's (2008) constructive generalization. Non-additive constructive generalization is "perceiving the given figure as part of a larger composite figure and then producing the rule by subtracting the sub-components from this composite figure" (Chua & Hoyles, 2010, p. 16). Chua and Hoyles (2010) also identified a third subcategory of figural reasoning called reconstructive generalization. Reconstructive generalization is similar to

Rivera and Becker's (2008) constructive generalization, except after the figure is decomposed into smaller pieces, the pieces are rearranged into new figures that can be counted and used to develop a rule.

The methods used to reason figurally sound oddly familiar to Piaget, Inhelder, and Szeminska's (1981) comments regarding the conservation of space, whether that space was length, area, or volume. Conservation of space is the understanding that a quantity remains unchanged regardless of the arrangement of any objects comprising that quantity. Piaget et al. (1981) noted that

> "when the child has learnt to perform concrete operations, whether these bear on logical 'groupings' or on the composition of parts, he automatically realizes that the parts are logically mobile, and the whole is therefore conserved because it corresponds to the (real or virtual) collection of its parts" (p. 327).

This realization that the whole remains unchanged regardless of the groupings of its parts appears to be the common thread between the different ways of reasoning figurally described in this section (e.g., deconstructive generalization, additive constructive generalization). That is, the different ways of operating figurally are all dependent on the original whole being composed of the sum of its parts. However, because the whole changes in each case, it appears that conservation of space is only part of the knowledge needed to reason figurally.

Figural counting.

Another manner of operating with figures appeared in the literature, which I refer to as figural counting (Becker & Rivera, 2005). Figural counting is constructing an image, figure, or other visual for a particular case and counting the desired attribute within it. Stacey (1989) identified this way of thinking in the primary and secondary students who worked linear tasks. She called this the Counting Method, defined as "counting from a drawing" (Stacey, 1989, p. 150). Lannin (2003) also recognized this type of thinking, which he described as "drawing a picture or constructing a model to represent the situation and counting the desired attribute" (p. 344). Figural counting develops a figure for a single case and counts the relevant characteristic.

Figural reasoning and figural counting are similar but distinct ways of thinking. Figural counting focuses on creating a figure for a *single case* to count the desired attribute. In contrast, figural reasoning utilizes a *set of cases* to identify variant and invariant characteristics useful for determining common structure from amongst the cases. That is, figural counting focuses on a single case whereas figural reasoning focuses on a set of cases. Although figural counting could lead to figural reasoning, the two are distinct ways of operating with figures.

Numerical Reasoning

Another type of reasoning utilized to develop rules on patterning tasks is numerical reasoning (Becker & Rivera, 2005, 2006; Healy & Hoyles, 1999; Rivera & Becker, 2003). Becker and Rivera (2006) defined numerical generalizers as students who draw upon numerical cues alone to establish their rule. Chua and Hoyles (2010) utilized this description but further elaborated it as drawing upon numerical cues from sequences of numbers, or tabulated ordered pairs in a T-table. Essentially, numerical reasoning is based upon cues from within the quantities involved, independent of any figure, symbolism, or other traits involved.

Numerical reasoning was one type of reasoning used across multiple populations on geometric-numerical patterning tasks. Rivera and Becker (2003) found that preservice elementary teachers utilized numerical reasoning when developing rules on linear tasks. Becker and Rivera (2005) also found that Grade 9 students utilized numerical reasoning more frequently compared to any other type of reasoning when developing rules for linear relationships. In contrast, Chua and Hoyles (2010) noted that preservice secondary teachers used numerical reasoning the least when developing rules on quadratic patterning tasks. Despite the change in population and type of relationship being worked with, numerical reasoning was utilized by all of these populations.

Pragmatic (Numerical + Figural) Reasoning

Another type of reasoning used to develop rules from patterns was pragmatic reasoning (Becker & Rivera, 2005, 2006; Chua & Hoyles, 2010). Pragmatic reasoning utilizes a combination of numerical and figural reasoning. Becker and Rivera (2006) noted that students who utilized pragmatic reasoning were fluent with both figural and numerical reasoning. These students were able to perceive both types of relationships and coordinate them flexibly when developing rules on patterning tasks. Chua and Hoyles (2010) found that preservice secondary mathematics teachers who were able to develop multiple rules on a quadratic pattern generalization task did so because they utilized pragmatic reasoning. Pragmatic reasoning is the hybrid use of both numerical and figural reasoning.

Proportional Reasoning

Another type of reasoning used when working with generalization tasks was proportional reasoning (Becker & Rivera, 2005; Rivera & Becker, 2003; Stacey, 1989). This type of reasoning was often associated with thinking about linear relationships and was used to determine particular cases in a pattern (Lannin, Barker, & Townsend, 2009). Proportional reasoning is the identification of a unit, and then scaling that unit by a factor. Depending upon the relationship and quantities involved, an adjustment may be necessary, such as adding or subtracting a constant. For example, imagine there are 35 objects associated with case 11. Using proportional reasoning, one might conclude that there are 70 objects associated with case 22. In terms of the definition, the unit constructed is 35 objects per 11 cases. This unit was then scaled by a factor of two, resulting in 70 objects per 22 cases. This may or may not be an appropriate way of scaling, depending upon a) if the relationship is linear and b) if the linear relationship $f(x) = mx + b$ has a zero or non-zero value for b.

Although not all researchers use the terminology of proportional reasoning when referring to this way of thinking, I saw this conception infused within descriptions for different types of generalizations. Thus, the descriptions that follow may be overtly focused on proportional reasoning as described above, or they may indirectly appeal to this conception. I have separated the following sections into different types of proportional reasoning, based upon what unit was identified to be scaled.

Rate-adjustment reasoning.

One of the most common ways of reasoning proportionally is rate-adjustment reasoning. Rate-adjustment reasoning identifies the rate of change as the unit, and then scales this unit by a factor. An adjustment to this scaled quantity may then be made, such as adding or subtracting a constant. The phrase “rate-adjustment” comes from Lannin (2003), in which he defined it as “using the constant rate of change as a multiplying factor. An adjustment is then made by adding or subtracting a constant to attain a particular value of the dependent variable” (p. 344).

Stacey (1989) also appeared to identify rate-adjustment proportional reasoning, though she did not use this phrase. She separated whether or not a constant was added as an adjustment into two methods—the Difference Method and the Linear Method. She described the Difference Method as scaling the rate of change by a factor, *without* adding or subtracting a constant. Stacey’s (1989) Linear Method also identified the rate of change as the unit and scaled by a factor, but then *did* add or subtract a constant from this product. Whether or not a constant was added to the scaled rate of change was the essential difference between these two ways of reasoning.

Rivera and Becker (2003) also appeared to identify rate-adjustment reasoning. Although not named this, Rivera and Becker (2003) found that elementary PSTs arranged co-varying quantities in a table of values, noted that there was a constant rate of change between successive values, and then scaled this constant difference by a factor. After scaling the rate of change, the PSTs then adjusted these products by adding a constant.

Whole-object reasoning.

Although rate-adjustment reasoning was one common way of reasoning, whole-object reasoning was also another common way. Whole-object reasoning is using a non-one multiple of the rate of change (or any multiple of the rate of change with a constant added) as your unit, and then scaling this unit by a factor. An adjustment to this scaled unit may then be made, such as adding or subtracting a constant or multiples of that constant. The phrase "rate-adjustment" comes from Lannin (2003), in which he defined it as "using a portion as a unit to construct a larger unit using multiples of the unit. This strategy may or may not require an adjustment for over- or undercounting" (p. 344). Lannin, Barker, and Townsend (2006) also identified this way of thinking, though the researchers noted that it could also be thought of as unitizing.

Stacey (1989) also appeared to identify whole-object proportional reasoning, referring to it as the Whole-Object Method. In her study, one of the tasks her participants worked was determining the number of matches needed to construct a ladder with a particular number of rungs. She described the Whole-Object Method as "taking a multiple of the number of matches required for smaller ladder" (p. 150). Symbolically, Stacey (1989) noted that this way of reasoning was equivalent to $M(mn) = m \times M(n)$. The essential issue with this way of thinking is that if the function $M(n)$ is linear with a non-zero constant added, then scaling by a factor of m will result in a miscounting of the number of objects for case mn, unless an adjustment is made for the multiples of the constant that were over-counted.

Guess-and-Check Reasoning

Another type of reasoning used to develop a rule for a task was guess-and-check reasoning (Becker & Rivera, 2005; Lannin, 2003; Rivera & Becker, 2003). Lannin (2003) described guess and check reasoning as "guessing a rule without regard to why the rule may work" (Lannin, 2003, p. 344). Becker & Rivera (2005) called this a trial-and-error approach, and argued that there were two subcategories of it. One trial-and-error approach used systematic selection of the numerical coefficients and terms in the rule being developed, whereas the other approach used unsystematic selection of the numerical coefficients and terms in the developing rule. Essentially, guess-and-check reasoning does not try to develop relationships between visual figures, numerical quantities, problem context, or other characteristics of the problem. Rather, guess-and-check reasoning seeks to identify a rule and then verify that the rule satisfies the parameters of the problem.

Chunking Reasoning

Another type of reasoning used to develop a rule for a task was chunking (Lannin, Barker, & Townsend, 2006). Lannin, Barker, and Townsend (2006) described someone using chunking as building "on a recursive pattern by building a unit onto known values of the desired attribute" (p. 6). The authors also provided an example which stated that "for or a rod of length 10 there are 42 stickers, so for a rod of length 15, I would take $42 + 5(4)$ because the number of stickers increases by 4 each time" (Lannin, Barker, & Townsend, 2006). In this example, one can observe two things. First, the identification of the rate of change (i.e., "the number of stickers increases by 4 each time") was noted, and then scaled by a factor of 5.   Second, this scaled unit (i.e., chunk) was added on to a different unit (i.e., chunk). Essentially, chunking takes two different chunks and combines to form a new chunk.

Contextual Reasoning

The final type of reasoning identified to develop a rule was contextual reasoning (Lannin, 2003). Lannin (2003) described contextual reasoning as “constructing a rule on the basis of a relationship that is determined by the problem situation” (p. 344). Although it is possible that the contextual information utilized is based upon the figures (i.e., figural reasoning), this may not necessarily be the case. Thus, contextual reasoning does not fit exactly under the umbrella of figural reasoning, though there may be overlap.

Contextual reasoning may overlap with other types of reasoning as well, such as explicit or recursive reasoning. Lannin (2003) indicated that a “contextual strategy is useful because it links the student’s rule to the situation and allows for the immediate calculation” (p. 345) of particular values. One of the trademarks of explicit reasoning is that it directly relates two or more co-varying quantities, often through a rule or formula. This direct relationship would allow for “immediate calculation” (Lannin, 2003, p. 345). Based upon this similarity, it appeared that there may be overlap between explicit reasoning and contextual reasoning.

Definitions of and Meanings for Justification

Multiple definitions for justification have been provided in the research literature. Consider the definitions below.

1. "*Proving* is the process employed by an individual (or a community) to remove doubts about the truth of an assertion" (Harel & Sowder, 2007, p. 808).
2. "Justification [is] an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning" (Staples, Bartlo, & Thanheiser, 2012, p. 448).
3. "Proof is just a convincing argument, as judged by competent judges" (Hersh, 1993, p. 398)
4. "Justification in the context of whole number computation is to provide a convincing argument for why carrying out a series of computations is a valid method for determining the answer of a given computation" (Lo, Grant, & Flowers, 2008, p. 6)

Immediately one may notice that the word "justification" was not always utilized, even though similar descriptions were provided. This was one challenge in determining the possible definitions of justification. Some researchers used the word "justification", others used "proof", and others situated themselves in terms of "argumentation." For purposes of this dissertation, the words justification and proof will be taken to be synonyms and will not be distinguished.

One meaning that can be seen in the first two definitions is that justification regards the *validity* (i.e., truth) of a statement. Harel and Sowder (2007) stated that proof removes any doubts about a statement's validity. Staples, Bartlo, and Thanheiser (2012) noted that justification demonstrates why a statement is or is not true. Essentially, justification (or proof) is about establishing the validity of a claim. That is, verifying a claim.

A second meaning that can be seen in the last two definitions provided above is that justification explains *why* a claim is correct. Both Hersh (1993) and Lo, Grant, and Flowers (2008) stated that justification (or proof) is a convincing argument. However, Lo et al. (2008) provided more detail, indicating that it addresses the why aspects of a claim. Essentially, justification (or proof) is a convincing statement that explains why the statement is true.

Hanna (2000) recognized both of these meanings, verification and explanation, as essential roles that proof (or justification) fulfills. So although there are two different meanings that can be made from these definitions, both are important when addressing justification. Thus, for this study, justification was defined as statements that verify and explain why an assertion is true (Hanna, 2000).

Traditions for Researching Justification

A consequence of these different definitions for justification is that the foci of research on justification have been different. These different foci have been broken into two major research traditions—the study of justification as associated with deductive reasoning, and the study of justification as associated with inductive reasoning (e.g., Bell, 1976). A discussion of each follows.

Justification as Associated with Deductive Reasoning

In deductive reasoning, justification is often a synonym for proof. Deductive reasoning has been defined as drawing a conclusion from a known set of information

(Baroody, Wilson, Kauchak, & Eggen, as cited by de Castro, 2003). The essential characteristic of justification, as associated with deductive reasoning, is that it does not attempt to make an inference. For example, if working with a specific case, no extension beyond that case is made (e.g., de Castro, 2004). Justification as associated with deductive reasoning does not try to extend the range in which the reasoning is oriented.

Staples et al. (2012) noted that "establishing a new result generally requires a rigorous deductive argument…that demonstrates the truth of a mathematical claim, that is, a proof (p. 448). Some studies conducted have identified justification to be that of deductive argumentation. For example, Knuth (2002) investigated experienced secondary teacher's conceptions of proof and found that "the majority of teachers stated, to varying degrees, that a proof is a logical or deductive argument that demonstrates the truth of a premise" (p. 71). He also found that the majority of teachers identified the role of proof in secondary school mathematics to be focused on the development of logical, deductive thinking skills. This is an example of one study where the meaning of proof (i.e., justification as associated with deductive reasoning) was made explicit to be that of deductive reasoning.

Other studies have been concerned with deductive argumentation itself. An example of this can be seen in the development of proof schemes (Harel & Sowder, 1998, 2007). A proof scheme "consists of what constitutes ascertaining [i.e., convincing yourself] and persuading [i.e., convincing others] for that person (or community)" (Harel & Sowder, 2007, p. 809). Harel and Sowder (1998; 2007) developed a taxonomy of these proof schemes composed of three major categories, one of which is the deductive (or analytical) proof scheme. The deductive proof scheme category is broken down into two subcategories—the axiomatic proof scheme and the transformational proof scheme. The axiomatic proof scheme is "when a person understands that at least in principle a mathematical justification must have started originally from undefined terms and axioms (facts, or statements accepted without proof)" (Harel & Sowder, 1998, p. 273). These undefined terms and axioms may have intuitive origins (e.g., $a + b = b + a$ where a, b are real numbers), but they do not have to (e.g., definition a group). The transformational proof scheme is similar, but what distinguishes it is that it involves the "transformations of images—perhaps expressed in verbal or written statements—by means of deduction" (Harel & Sowder, 1998, p. 258). The transformational proof scheme possesses three characteristics: generality, operational thought, and logical inference (Harel & Sowder, 2007). The generality characteristic indicates that the argument must apply to all objects in a particular class, not just some of them. The operational thought characteristic indicates that there are specific goals and sub-goals identified and progressed towards in the argumentation process. The logical inference characteristic indicates that the argument must utilize the rules of logical inference (e.g., definitions, axioms, theorems, corollaries). Harel and Sowder's (1998, 2007) work with the deductive proof scheme is another example of justification as associated with deductive reasoning.

Justification as Associated with Inductive Reasoning

Justification was also associated with inductive reasoning. One way inductive reasoning has been defined was as "generalizing knowledge from a finite sample of particular instances" (Rivera & Becker, 2003, p. 63). That is, reasoning based upon observing and/or operating with specific cases to explain why a conjecture must hold for a general case. The essential characteristic of justification as associated with inductive reasoning is that it *does* attempt to make an inference. If working with specific cases, an extension is made from the specific cases to a general case that encompasses all of the considered specific cases (and often other cases not originally considered).

Research on justification as associated with inductive reasoning often appeared in the literature as connected to another area of study (e.g., justification of developed generalizations). For example, Becker and Rivera (2007) investigated Grade 7 students' ability to justify the general rules (i.e., generalizations) they developed on a linear geometric-numerical patterning task. The researchers observed that some participants justified their rule by aligning their rules to figures presented in the task, whereas other participants aligned their rules to numerical values from the task to argue the appropriateness of their rule. Another example can be seen in the work of Richardson et al. (2009) where the researchers studied elementary preservice teachers' ability to justify their rules to linear geometric-numerical patterning tasks. The researchers found that in the beginning of the study the preservice teachers had difficulty justifying their developed rules, often being able to start their justifications, but unable to complete them. However, by the end of the teaching experiment the researchers concluded that the participants learned to justify their rules, noting that connecting the symbolic models to the given figures in the tasks was important to the improvement in justification. These examples help provide evidence of the relationship between studying justification associated with inductive reasoning and the study of generalization.

Distinguishing Justification as Associated with Deductive and Inductive Reasoning

The essential distinction between these two conceptions is that justification as associated with inductive reasoning utilizes reasoning that comes from a specific case to provide evidence for the general (i.e., Mason's (1996) "seeing a generality through the particular" (p. 65)). That is, an inference about the general is made based upon the specific. In contrast, justification as associated with deductive reasoning never attempts to make an inference. Rather, justification as associated with deductive reasoning operates by making deductions from a given set of information. For example, in a patterning generalization activity, a student may reference a specific case to justify why a conjectured general rule was true. This statement could constitute a justification as associated with inductive reasoning; however, such a statement would not be viewed as a justification associated with deductive reasoning because an inference from the specific to general case was made.

Types of Reasoning Used to Justify

Although research on justification as associated with inductive reasoning is limited, of the studies that exist, many were conducted on pattern generalization tasks. The literature that follows considered only studies that utilized justification as associated with inductive reasoning. Although some reviewed studies did not consider the context of pattern generalization, the majority of the following reviewed studies were situated within this context.

Using Examples to Justify

One of the most common types of reasoning drawn upon to justify is the use of one or more specific cases to illustrate the validity of the claim made (Balacheff, 1988; Becker & Rivera, 2007; Healy & Hoyles, 1999). Lannin (2003) referred to this as proof-by-example, and described it as providing examples of specific cases as support for one's answer (Lannin, 2005). Kirwan (2013) described this as providing examples for specific cases as support for the statement made. Harel and Sowder (1998, 2007) identified this type of reasoning in the inductive proof sub-scheme and described it as "evidence from examples (sometimes just one example) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, and so forth" (p. 809). Utilizing one or several examples to justify a statement was a common type of reasoning used to justify.

Not only was this type of reasoning common, but it also occurred in a variety of populations. Stacey (1989) found that both middle school and high school students utilized this type of reasoning, and it was the most dominant type of reasoning utilized by high school students to explain generalizations for patterning tasks. Harel and Sowder (2007) reviewed literature illustrating that students from multiple populations (e.g., middle school students, college students) utilized this type of reasoning. Utilizing one or more examples to justify a statement was a common type of reasoning used across multiple populations.

Radford (1996) observed that some students chose their examples purposefully. He observed that students sometimes appealed to specific cases in a sequence (e.g., 10^{th}, 50th, 100th) to argue why their generalization must work. Balacheff (1988) also saw that students sometimes were purposeful in their selection of which example(s) to draw upon when justifying. One common type of reasoning utilized to justify is the use of specific case(s) to argue the validity of or explain a made claim. Additionally, this type of reasoning was a common occurrence among students from varied populations.

Using a Generic Example to Justify

Another common way that students justified a claim was through the use of a generic example (Davydov, 1990; Lannin, 2005). That is, an example that described the situation for a general, non-specified, case (Lannin, 2005). Balacheff (1988) identified this type of reasoning and called it “the generic example” (p. 219) in his taxonomy of the different types of proof. He described it as “making explicit the reasons for the truth of an assertion by means of operations or transformations of an object that is not there in its own right, but as a characteristic representative of its class” (p. 219). Mason (1996) referred to this type of reasoning as “seeing the general in the particular” (p. 65). The essential characteristic of using a generic example to justify is that, although a specific example may be used, what is being attended to is not that specific case, but the characteristics of that case that are common to every other case as well.

Using Contextual Information to Justify

Another common type of reasoning used to justify one’s answer was through the use of contextualized information from the problem context (Becker & Rivera, 2003, 2007; Kirwan, 2013; Lannin, 2003; Townsend, Lannin, & Barker, 2009). Contextual reasoning utilizes information from a given problem or task, such as given figures and/or their characteristics, quantified aspects of the problem, or details of the situation, as part of the reasoning behind why an assertion is true. Many researchers have noted the importance of relating a rule to its associated figure (i.e., context) that it was developed from (e.g., Becker & Rivera, 2006; Healy & Hoyles, 1999; Radford, 2006). Richardson, Berenson, and Staley (2009) noted that students who were successful in justifying their generalizations often connected to the geometric models in each problem. Stacey (1989) found that only about 20% of the Grades 7 and 8 students surveyed from a suburban Australian high school were able to relate the rule developed to the figures given. Kinach (2014) suggested having students explain and justify why some rules do not work based upon figures can help them develop their justification abilities. One common way to reason contextually is by utilizing information given in the problem's context.

Another related way to use contextual information to justify is by relating components of a developed rule to quantified aspects of the problems context. For example, Rivera and Becker (2003) found that 69% of preservice elementary teachers cited constant first differences from a linear sequence of terms as the reasoning for where their rule, which was often recursive, was developed from. Lannin (2003) noted that it was important for students to explain each component of the rule they developed, and identify what it means in terms of quantified characteristics from the problem's context. For example, a student might argue that in a rod of blocks, every middle block has four sides, and there are always two fewer middle blocks than the total number of blocks in the rod to explain where their 4 and $b - 2$ factors come from in their rule. Richardson et al. (2009) found that some elementary preservice teachers struggled to explain the rate of change coefficient and/or y-intercept of the symbolically developed rule for a linear relationship. In this struggle, the participants attempted to explain these values by relating components of their rules back to tables with co-varying quantities (e.g., constant rate of change), or by quantifying different traits from the problem's context. The essential feature of this subtype of contextual reasoning is the relating of components of a developed rule to numerical aspects of a problem's context.

Appeal to an External Authority to Justify

Another type of reasoning utilized to justify a claim was through an appeal to an external authority, such as a teacher, textbook, or answer key (Simon & Blume, 1996). Harel and Sowder (1998; 2007) referred to this as an authoritarian proof scheme, subsumed under the external proof schemes subcategory. They noted that, "the underlying characteristic of this behavior is the view that mathematics is a collection of truths, with little or no concern and appreciation for the origin of the truths" (Harel & Sowder, 1998, p. 247). Harel and Sowder (1998) noted that students with this conception often focused on the *how* (i.e., instrumental (Skemp, 1976)) instead of the *why* (i.e., relational (Skemp, 1976)) aspects of the mathematics. In a teaching experiment conducted by Lannin (2005), he identified middle school student's justifications by utilizing a framework from the work of Simon and Blume (1996), with the second level of this framework being the appeal to an external authority. He described this type of reasoning as deferring to another individual or reference material to determine the correctness of a statement, and found that students tended not to use this type of justification to explain their generalizations on patterning tasks. The appeal to an external authority source to justify a statement is another type of reasoning used to justify.

Using a Non-Formalized Inductive Argument to Justify

One final type of reasoning utilized to justify a statement was through the use of a non-formalized inductive argument (Lannin, 2003; Rivera & Becker, 2003). For example, Lannin (2003) stated that a student might cite a particular case that satisfies their rule, and then describe how the rate of change allows them to determine the next subsequent case. Lannin (2003) found that middle students who attended to rates of change in linear patterning tasks sometimes drew upon this reasoning to justify the rule they developed. Rivera and Becker (2003) found that elementary PSTs frequently cited common first differences in the numerical values as to why the rules they developed for the tasks were correct, statements that were similar to those made by the middle school students in Lannin's (2003) study. It is important to note that in both of these studies, linear relationships provided the context in which the patterning tasks were situated, and the inductive arguments made did not contain any of the formalism of inductive proof. That is, they were unrefined, non-formalized inductive arguments.

A Relationship between Generalization and Justification

Although generalization and justification have been studied independently, few studies have considered the interaction between these constructs for geometric-numerical patterning tasks. Of these studies, I observed a common theme of relating a rule and justification to the figures or visuals associated with them. An unpacking of this theme follows

Relating a Rule to its Figure

A prominent theme present in the literature relating generalization and justification on geometric-numerical patterning tasks was the importance of relating a rule to the figures it emanated from (Richardson et al., 2009; Rivera & Becker, 2003). Lannin (2005) commented that

> "When justifying an algebraic model, an argument is viewed as acceptable when it connects the generalization to a general relation that exits in the problems context. This type of justification is often connected to a geometric scheme that is generated based on a visual conceptualization of the situation…This type of justification is valued because it explains rather than simply convinces [verifies], describing a relation that can be observed across all cases that exist in the situation" (p. 235).

In terms of patterning tasks, this implies that a rule must capture the general relationship identified in the task and the associated justification must address this general relationship. Moreover, Lannin (2005) noted the commonness of associating a justification with the figures or visuals it aligns with.

In his study, Lannin (2005) found that sixth-grade students who related their rules to figures or other visuals from the problem's context more frequently provided general and valid justifications. In contrast, Lannin (2005) observed that students who used empirical justification (i.e., proof-by-example) typically did not have a relationship between their generalization and a geometric model of it. This left the students unable to link their rule to the context from which it came. He concluded that problems that allow for relating figures and visuals to rules increase the likelihood of student success in developing a rule as well as justifying it.

However, in a 2006 case study of two sixth-grade students, Becker and Rivera found that developing a rule and justifying it was contingent upon more than one's ability to operate fluently with figures and visuals. Rather, the researchers concluded that it was also necessary to be flexible in ones use of variables. Becker and Rivera (2006) argued that, "the lack of competence in one aspect undermines the other in salient ways" (p. 471). Although relating a rule and justification to figures is a critical step, it is insufficient to develop appropriate rules and justifications alone.

In a follow up 2007 study, Becker and Rivera utilized pre- and post-interviews with eight, seventh-grade students and asked them to develop and justify a rule for a linear patterning task. The researchers observed that the participants were able to develop rules that were based upon numerical relationships (i.e., common differences) in the task. These developed rules were constructive generalizations (see the Figural Reasoning section in Generalization literature review) and were justified either by fitting them onto tables of values (i.e., proof-by-example) or by aligning them to figures. However, when the researchers presented the students with a deconstructive generalization (see the Figural Reasoning section in Generalization literature review), only one of the eight students could justifying it. The authors concluded that although an invariant property might be observed and established numerically (e.g., common differences), forcing an invariant property onto a figure in a way that does not corresponded to this invariance in the figures did not allow students to adequately justify their rules. Essentially, the type of rule to be justified (e.g., constructive versus deconstructive generalization) also plays a role in the development of an appropriate justification.

Summary of the Literature Review Chapter

This chapter began with a presentation of some definitions for generalization, followed by a discussion of some of the characteristics associated with the construct of generalization. Next, assumptions associated with generalization were presented, as well as traditions for researching it. The generalization section then focused on the different types of reasoning utilized to generalize. The literature review then progressed on to the construct of justification, beginning with the presentation of some definitions. Different meanings for justification were then identified, followed by a description of the different traditions for researching it. The justification section then narrowed to presenting the different types of reasoning associated with justifying a statement that was made. The chapter concluded with an identified theme between the constructs of generalization and justification.

CHAPTER III

RESEARCH DESIGN AND METHODOLOGY

Research Design

This study took a qualitative research perspective. Glesne (2011) noted that "qualitative researchers seek to make sense of actions" (p. 1). This study required making sense of the actions that PSTs made when solving generalization tasks and in justifying their generalizations. One aspect that separates qualitative research from quantitative is that qualitative research embraces the complexity and "messiness" of the phenomenon. In contrast, quantitative methods seek to reduce the phenomenon's complexity by identifying the essential variables to be measured. This study embraced complexity by developing rich descriptions of PSTs generalizing and justifying of quadratic relationships. Given the limited research on PSTs understanding of generalization and justification, it is important to first develop a rich, descriptive understanding of this phenomenon. That is, descriptive frameworks for how PSTs generalize and justify quadratic relationships are needed for further research.

This study drew upon elements of phenomenology and grounded theory. To be clear, this study was primarily a phenomenological study conceived within the qualitative research paradigm of interpretivism, but also drew upon aspects of grounded theory. Phenomenology is about finding the central shared experiences that individuals encounter for a particular phenomenon. For this study, the particular phenomenon was the generalization and justification of quadratic relationships while working geometric-numerical patterning tasks. Phenomenology was drawn upon because the primary goal of this study was to understand the phenomenon of how PSTs generalize and justify quadratic relationships. Aspects of grounded theory (e.g., the constant comparative method (Glaser & Strauss, 1967)) were drawn upon because of the limited research basis for PSTs generalization and justification strategies for quadratic relationships. Grounded theory is about developing theory (Strauss & Corbin, 1998), in contrast to testing a theory or hypothesis. Thus, grounded theory allowed for the development of the frameworks for the strategies used by PSTs for generalizing and justifying quadratic relationships.

Instrumentation and Development

There were two instruments designed for this study—a participant selection survey and an interview tasks instrument consisting of generalization and justification tasks. Details of each instrument are described below.

Participant Selection Survey

The participant selection survey (see Appendix A) was designed to determine which individuals would be willing to participate and to gather background information on the population the participants were sampled from (i.e., senior PSTs in their final semester prior to graduation). This instrument was designed by adapting the work of Ellerton and Clements (2011). Ellerton and Clements (2011) utilized a survey involving 16 "clever" (p. 387) questions composed of pairs of linear/quadratic equations and inequalities to measure pre-service middle school teachers' knowledge of linear and quadratic equations and inequalities, as well as their confidence in their solutions. What made these questions clever was that many of them were designed to elicit common manipulation errors (e.g., misapplying the square root method to solve $x^2 > 4$ as $x > \pm 2$), or required interpretation of the result of manipulation (e.g., manipulating $x^2 + 2 > 0$ into $x > \pm i\sqrt{2}$) when solving for all real-values of the unknown. A student solving $x^2 > 4$ would need to be aware that the square-root method applies only when solving quadratic equations. A student solving $x^2 + 2 > 0$ would need to know that the results of squaring any number is always greater than or equal to 0 (so increasing that result by 2 would yield a sum always be greater than zero), and that a true statement indicates that any value from the domain of the unknown (i.e., real-numbers) will satisfy the statement.

For my dissertation study, those questions that were originally for linear relationships were adapted to quadratic relationships. For example, the equation $\frac{1}{x} = 3$ was modified to be $\frac{1}{x^2} = 16$. The questions from Ellerton and Clements (2011) that were already for quadratic relationships were not modified—only the non-quadratic relationships were, with the hope of retaining the cleverness in each adapted question. My goal for making these adaptations was to develop a sense of the participants' ability to think non-algorithmically about quadratic relationships. Additionally, I wanted participants to indicate their level of confidence for each solution as well. This information was used to help inform me of which participants might be better suited to solve quadratic, geometric-numerical patterning tasks (see Appendix B). For example, participants who frequently answered questions correctly with a high-degree of confidence may be more likely to generate generalizations/justifications on patterning tasks and be able to unpack their thinking about them, versus participants who guessed correct answers.

From those 16 possible questions, six were included in the instrument used in my dissertation study—three quadratic equations and their associated inequalities. These six questions were selected because the equations and associated inequalities were accessible to the participants, yet their solutions were not immediately obvious. Because these six questions all used a symbolic representation, four other questions were added to the instrument. The seventh and eighth questions on the instrument presented quadratic relationships graphically and ask for an equation of the graph, or what real-numbered values of the variable satisfy a particular condition. These two questions were included to help assess each individual's ability to reason about quadratic relationships not presented in a symbolic format. The ninth and tenth questions on the instrument asked for individuals to write a rule for a geometric-numerical patterning task. These two questions were included to help determine which individuals might be able to generate rules to the generalization tasks utilized during the interviews. I also asked participants to indicate their degree of confidence in their solutions for each question. Once a copy of this instrument was drafted, it was then refined with the help of the committee.

Interview Tasks Instrument

The interview tasks instrument (see Appendix B) was designed to be used during the interviews with each participant. These tasks were designed by adapting questions from textbooks (i.e., Fulton & Lombard, 2001; Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998) and tasks (i.e., Kara, Eames, & Miller, 2014; NYC Department of Education,

2013) I had encountered as a teacher and student. The tasks were then piloted with a sophomore and a junior pre-service secondary mathematics teacher, and a graduate student in mathematics education. Piloting informed me that many of the tasks I originally designed required too much time to generate generalizations. Additionally, tasks utilizing rectangular or rectangular-like arrays of objects (i.e., the same number of objects per row for some number of rows) generated more generalizations than tasks whose objects did not have this arrangement. My committee also suggested administering the same set of tasks to all participants, instead of attempting to select one that would be at an appropriate level of cognitive demand for each participant. Based upon this feedback, I selected three tasks to be administered to all participants during each interview (see Appendix B). The fourth task (i.e., The Box Stacking task) was included to be administered as a confidence booster in case a participant struggled to solve the other tasks.

The Population and Sample

The participants in this study were selected from the population of 41 PSTs who were scheduled to graduate with a degree in secondary mathematics education from a mid-sized university in the Midwest. The PSTs had completed a minimum of 50 of the 53 required hours of mathematics content coursework which included calculus, discrete mathematics, geometry, linear and abstract algebra, probability and statistics, history of mathematics, computer programming or a technology in mathematics course, a capstone content course, and two mathematics specific methods courses. They also had completed 14 of their 24 required hours for professional education coursework, including issues in secondary education, instructional and evaluative methods in secondary education, and educational psychology. Additionally, these PSTs had completed four clinical experiences prior to their student teaching.

The sample for this study consisted of PSTs enrolled in a seminar for student teachers course who were willing to complete a sequence of tasks (see Appendix B) during an interview. These participants identified themselves on the participant selection survey (see Appendix A). Of the 41 students enrolled in the seminar for student teachers, 37 of them completed the participant selection survey (see Appendix A) during a 30-minute segment of a class in February 2014. This survey collected information regarding three characteristics: a) senior PSTs ability to solve quadratic equations and inequalities, b) how confident they were in their solutions to these quadratic equations and inequalities, and c) their ability to develop generalizations and justifications on a geometric-numerical patterning task for a quadratic relationship. These characteristics were included on the survey to provide a description of the population as well as to inform me which participants might be more adept at solving generalization tasks (i.e., characteristic c), and should be included in the sample.

However, of the 41 senior PSTs enrolled in this seminar, only ten consented to interviews (five males and five females). Because of this small number I opted to include all those consenting in the sample. These ten participants constituted the sample for this study. All ten participants were White, in their early 20s, and prepared in a traditional four-year secondary program. Three of the participants were student teaching in rural schools, two participants were student teaching in urban schools, and the remaining five were student teaching in suburban schools. Participant's GPAs were 2.80 or higher, both in mathematics coursework and overall GPA, as it was a requirement of the program. This was all of the demographic information that was collected.

Comparing the Sample to the Population

When considering the PSTs ability to solve quadratic equations and inequalities, the ten participants answered a larger proportion of the questions on the survey correct compared to the group of 37 students as a whole (on average, 0.73 of the questions answered correctly versus 0.65). Additionally, the ten participants had slightly less variation in the proportion of questions they answered correctly in comparison to the group of 37 students as a whole (average standard deviations of 0.40 versus 0.44). Overall, the ten participants answered slightly more questions about quadratic relationships correctly and they did this slightly more consistently than the group of 37 students as a whole.

To determine how confident the PSTs were in their solutions to the quadratic equations and inequalities on the participant selection survey, the students were asked to assign their confidence on a scale of 1 through 5 (i.e., I'm certain I'm wrong (1), I think I'm wrong (2), I've got a 50-50 chance of being right (3), I think I'm right (4), and I'm certain I'm right (5)). Based upon these assignments, the ten participants appeared to be more confident in their answers than the group of 37 students as a whole (on average, 4.74 versus 4.68). Additionally, the ten participants had less variation in their confidence than the group of 37 students as a whole (average standard deviation of 0.37 versus 0.42). Overall, the ten participants were slightly more confident in their answers and slightly more consistent in indicating this confidence than the group of 37 students as a whole, thought not by much.

When considering the number of generalizations made by the PSTs on a geometric-numerical patterning task for a quadratic relationship, the group of ten participants often provided more generalizations than the group of 37 students as a whole (on average, 1.9 generalizations made versus 1.46). Additionally, the ten participants had less variation in how many generalization statements they provided than the group of 37 students as a whole (average standard deviation of 0.3 versus 0.72). Overall, the ten participants typically made more generalizations and were more consistent in making more generalizations than the group of 37 students as a whole.

When considering the number of justifications made by the senior PSTs on a geometric-numerical patterning task for a quadratic relationship, the group of ten students appeared no different than the group of 37 students in the number of justifications provided, (on average, 0.70 justifications made versus 0.70). However, the group of ten participants had more variation in how often they provided justification statements compared to the group of 37 students as a whole (average standard deviation of 0.78 versus 0.73). Overall, the ten participants appeared to be similar to the group of 37 students with regards to the number of justification statements made, but were slightly less consistent in how often justification statements were provided.

Data Collection

Interviews

Once the ten study participants were identified, data was collected from a single audio and video recorded interview that lasted between one hour and 15 minutes and two hours. Interviews were conducted during a four-week period between the middle of March and the middle of April in 2014. Participants solved all three generalization tasks (see Appendix B) during each interview in the order of a) The Patio Tile task, b) The Happy Face Cutouts task, and c) The Star Sticker task. However, one participant was accidently administered the last two tasks in a reversed order. Interviews were conducted with participants in a one-on-one setting at their student teaching placements after the school day had ended, on campus after a seminar for student teachers session had concluded, or at another arranged time on campus (e.g., Saturday morning).

Interviews were conducted using a task-based interview structure (Goldin, 2000). More specifically, each participant solved tasks in accordance with the think-out-loud task protocol (see Appendix C), which was designed with the Newman Method in mind (Newman, 1977; White, Jaworski, Agudelo-Valderrama, & Gooya, 2013). Each participant began by reading the task's directions out loud (i.e., Reading). Following this, I asked the participant to explain what the directions were asking them to do in their own words (i.e., Comprehension). If there were any misunderstandings, additional clarification was provided. Next, the participant worked to generate a rule for the task (i.e., Transformation, Process Skill, Encoding). Each participant verbalized their thinking out loud as they worked. If more than 30 seconds passed without the participant making a written or verbal statement, I asked them to verbalize what they were thinking.

Once a participant concluded their work towards generating and justifying a rule, I debriefed them using the post-task interview protocol (see Appendix D). More specifically, I would first ask the participant to tell me what their rule meant in their own words. Next, I asked what influenced them in developing a rule to this task. If the participant made any comments related to the given figures or numerical quantities, I would press for more details. I then asked them to rate how confident they were in their rule on a scale of one to five. Following these questions about the rule, I would then ask the participant to explain why their rule was correct. I followed this question by asking what influenced them in providing that explanation for why their rule was correct. I then concluded this string of questions by asking the participant to rate how confident they were in why their rule was correct on a scale of one to five.

The rule-generation (i.e., Transformation, Process Skill, Encoding) and debriefing questions cycle was iterated for each task until participants were unable to work the task in any other ways. Once a participant indicated this, I collected the written work on the current task from the participant, gave the participant the next task in the sequence, and began the rule-generation and debriefing cycle again. Once all tasks were completed the interview concluded.

Data Preparation

Once interviews had been conducted and recorded, they were transcribed. Transcription was done from both the audio and video recordings to allow for the inclusion of not only the words uttered by the participants, but also brief descriptions of any gestures that were captured in the video. To further aid in the accuracy of the transcripts, the participant's written work was reviewed when it was not clear from the audio and video recordings what the participant was communicating.

Unit of Analysis

Once each interview had been transcribed, data reduction methodology (Miles, Huberman, & Saldana, 2014) was employed. In the first step, I identified the generalization episodes within each transcript. I defined a generalization episode as a portion or portions of the transcript focused on a single approach for searching and using a productive pattern that led to, or could have led to, a developed rule. These generalization episodes (GEs) served as the unit of analysis in this study. Once I had identified the GEs within a transcript, the transcript was check-coded by another researcher. Any disagreements in the identification of GEs were discussed until consensus was reached. The total number of generalization episodes identified was 77.

Data Analysis

Following the identification of the generalization episodes (i.e., units of analysis), a second layer of data reduction methodology (Miles, Huberman, & Saldana, 2014) was employed. Generalization episodes were read to identify the rules and justifications. A rule was defined as a statement made that describes a relationship for a non-specified (i.e., general) case. A justification was defined as a statement made that indicates whether or not a previous statement (e.g., rule, pattern) is reasonable. These two components were captured on a summary sheet for each generalization episode (see

Appendix E). On the summary sheet there was a section where I could place any notes or summarize the rule/justification, excerpts from the episode that I viewed as evidence of the rule/justification, as well as what the rule/justification appeared to focus around.

The summary sheets created a concise summary of the rule and justifications associated with each GE (i.e., unit of analysis). Once the GE summaries were created, they were check-coded by another researcher. Any disagreements in the rule or justifications were discussed until consensus was reached.

Research Questions One and Two

Recall that research questions one and two sought to identify the types of rules/justifications on the tasks, as well as search for relationships between these different types. To identify the rules/justifications elicited by the tasks, constant comparative methodology was employed first (Glaser & Straus, 1967) to develop categories for different sortings of the GE summaries. With regards to the types of rules (i.e., research question one), the GE summaries were first sorted into major categories based upon different characteristics of the rules, (e.g., a rule's construction category, the characteristics a rule appealed to). Subcategories within each major category were then looked for, to further distinguishing the types of rules within each major category. Descriptions for each of these categories and subcategories were developed and checked with another researcher. Any disagreements between categorizations and/or their descriptions were discussed and revised until consensus was reached. The same process was repeated to categorize the GE summaries according to the type of justification (research question two).

Once these categorizations and their descriptions were developed for each task, they were explored for relationships between them across tasks. A variety of comparisons were used to help make the data speak, regarding the types of rules or justifications. For research question one and two, the comparisons of the categorizations for rules and justifications were explored separately. These comparisons included a) separating the data by each major category, b) separating the data by each participant, c) looking for similarities and differences within and between descriptions for major categories and their subcategories, d) looking for dominant trends (e.g., common visualizations referenced, common explanations given) within and between each major category, e) looking for common errors or potential misconceptions made, and f) considering cases that I thought were intriguing and wanted to investigate further. The examples below help illustrate how this analysis unfolded.

After categorizing and describing the types of rules/justifications developed on each task, these descriptions were then compared (see Appendix F). This side-by-side-by-side comparison of the different rules allowed for similarities (e.g., developed explicit rules were nearly always symbolized) and differences (e.g., rules from task's one and three often appealed to figural characteristics whereas rules from task two often appealed to numerical) to be identified. This example helps illustrate how a comparison between subcategory descriptions unfolded. Additionally, the information provided in this example emerged from analyzing the data. It was provided in this example to illustrate how the analysis unfolded in making comparisons between different categories.

Another example can be seen by aggregating counts across the three tasks based upon the different characteristics the data were categorized by (e.g., construction category). After categorizing the rule's construction category for all GEs across the task, these counts were compared by each subcategory (i.e., developed rules, attempted rules, not attempted rules) and task (see Table 29). Which construction category was most common? Was it always most common regardless of task? Questions of this nature were able to be asked and explored by synthesizing across the findings from each task. Again, the information provided in this example emerged from analyzing the data. It was provided in this example to illustrate how the analysis unfolded in making comparisons between different categories.

An inadequate attempt to analyze rules and justifications for correctness.

I also attempted to analyze the rules and justifications given for correctness. However, this quickly became problematic in the difficulty to identify exactly what was meant by correct. For example, was a participant's rule checked against a particular rule, or list of acceptable rules, that were deemed correct to determine correctness (i.e., an authoritarian view)? What if two different participants read and interpreted a task slightly differently, each making a different set of assumptions, and then operated in a logical manner to determined two different reasonable conclusions that followed from their different sets of assumptions—are both correct? Neither? What if a participant is operating in a logical manner, but makes a slight error in the construction of their rule or justification? Is the resulting rule/justification correct or incorrect? Essentially, determining the correctness of a rule or justification became complex quickly. This complexity ultimately resulted in a decision not to analyze a rule's or justification's correctness. A further description of this attempt ensues below.

I began the attempt to analyze if a rule was correct based upon if it was the same or algebraically equivalent to a rule I specified (e.g., $x^2 + 3x + 4$ and $(x + 1)(x + 2) + 2$). When a rule satisfied this condition, it was identified as correct. When a rule did not satisfy this condition, it was identified as incorrect. However, I soon realized that rules that were not symbolized did not work with this definition for correctness. I did not want to translate a participant's verbal rule to symbols as the participant may not have considered this different form as appropriately capturing their rule. Thus, even if I reconsidered an equivalent form of the participant's rule symbolically from their verbal description (or vice versa), I had to infer that participants would view these different representations as equivalent. A primary concern was that I could not find evidence that the participants would make this inference. Stated differently, a participant made a choice in giving their rule to the task, and I wanted to make sure to not make inferences beyond the rule provided.

Another issue that arose in determining if a rule or justification was correct was accounting for rules/justifications that were almost correct, but not quite. For example, I had participants develop rules that coordinated co-varying quantities that were off by one count. That is, if the rule $x^2 + 3x + 4$ (or equivalent) coordinated these quantities, the rule $(x - 1)^2 + 3(x - 1) + 4$ (or equivalent) may have been given. Essentially, classifying a rule as correct or incorrect didn't allow room for rules that were close to correct. Although I attempted to account for partially correct rules/justifications, I observed that accounting for the diversity in the partially correct rules became too numerous and complex to utilize efficiently with my focus in this study on the different types of rules and justifications given. Succinctly, trying to capture the depth of variation in the spectrum of correctness became too large to address as part of my research questions. Thus, I chose not to address correctness as part of the generalizations and justifications that the participants gave.

Research Question Three

Recall that research question three sought to identify patterns and relationships between the rules and justifications. To do so, the frameworks that emerged from the categorizations in research questions one and two were drawn upon because they identified each GE with a type of rule and at least one type of justification. This association between the type of rule and the type of justification created pairs of codes that were associated with each GE. If more than one type of justification was utilized in a GE, the type of rule was held fixed, and it was paired with each of the different types of justification, creating several, unique pairs of codes for the type of rule and associated justification.

These pairs of codes were placed into their corresponding cells in two-way tables, with the rows and columns based upon the three major categories from the rules framework (i.e., construction category, kind of rule, characteristics appealed to) and the one major category from the justifications framework (i.e., justification categories). Again, it is important to note that the rule’s and justification’s frameworks emerged from the data in addressing research questions one and two. The purpose in providing them here is to give additional clarity to how the data were organized in different tables to identify relationships between the rules and justifications for research question three.

Once the pairs of codes were placed within the tables and aggregated across all three tasks, I looked for trends. I did this by making comparisons between pairs of codes from different major categories (e.g., how did explicit rules associated with verification compare to recursive rules associated with verification) (see Table 51). However, on some occasions these pairs of codes were stratified further, such as by another major category (e.g., how did *developed* explicit rules associated with verification compare to *developed* recursive rules associated with verification) (see Table 52), or by the task the pairs of codes originated from (e.g., how did explicit rules associated with verification compare on task's one, two, and three) (see Appendix H). I also made note of any observations that I noted as interesting, such as pairs of codes that occurred more frequently compared to other pairs of codes. Again, the purpose in providing these examples was to illustrate how the analysis unfolded in searching for patterns and relationships between the rules and justifications for research question three.

CHAPTER IV

ANALYSIS OF THE DATA AND RESULTS

Data were analyzed as described in the proceeding chapter. The results of this analysis are presented in this chapter, which is organized by research question and task. For research questions one and two, the results are presented for the three quadratic tasks, followed by a synthesis across these tasks, culminating in the development of a rules framework (research question one) and a justifications framework (research question two). For the section associated with research question three, a comparison between these two frameworks is given, identifying commonalities, differences, and observations between the two frameworks.

Research Question One

Research question one investigated the types of rules given by the participants on the quadratic geometric-numerical patterning tasks. Specifically, what types of rules are given for quadratic patterning tasks presented in a geometric-numerical format? What patterns or relationships exist between the types of rules across tasks? The complete collection of generalization episode (GE) summaries was separated according to interview task (e.g., The Patio Tile Task). The GE summaries that were associated with each task were then categorized based upon the rule associated with them. The presentation that follows is first broken down by individual task and then followed by a comparison between and synthesis across the three quadratic tasks.

Task One: The Patio Tile Task

Once GE summaries associated with the Patio Tile task (see Figure 1) were identified, they were sorted into three major groups based upon the extent to which the rule was developed—developed rules, attempted rules, and no attempted rule. A discussion of each of these major groups follows, with the developed and attempted rules further separated as developed/attempted explicit and recursive.

You are laying circular stones for a patio you are building. You take a picture of the patio each day to capture your progress, which is shown below. Determine and write a rule that relates the **day number** to the **total number of stones laid** in your patio for that day. Please explain why your rule is correct. Solve this task in as many ways as you can.

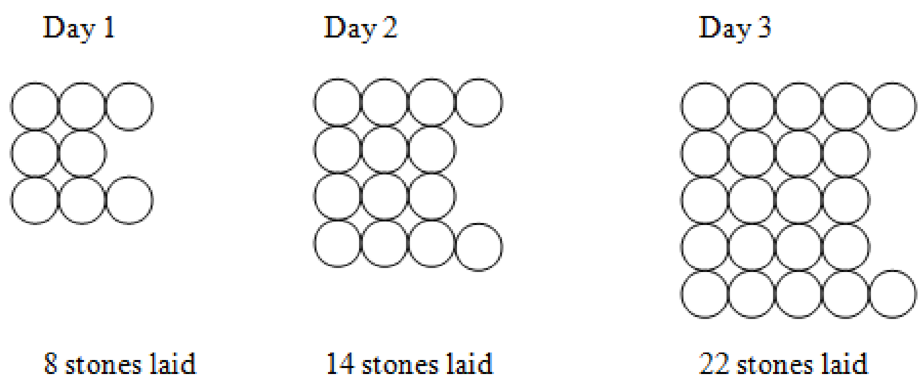

Figure 1. The Patio Tile Task Administered to Participants.

Developed rules – explicit.

A GE summary was categorized as a developed rule when the participant concluded that he/she had finished the task and was satisfied with the rule. A GE summary was categorized as an explicit rule when the rule directly related two co-varying quantities. There were a total of 14 explicit rules developed, from which I identified seven different types. These types of explicit rules ranged in frequency from four to one, and were provided by six of the ten participants. The table below presents the distribution of these different types of explicit rules along with their descriptions.

Table 1

Frequencies and Descriptions for Explicit Rules Developed

(Explicit) Rule Developed/Description	**Frequency**
Count stones in the top and bottom row, then add stones counted in the middle rectangular array. Symbolized as $2(d+2)+d(d+1)$	4
Count stones assuming there is a full square array, then remove stones not present in right column. Symbolized as $(d+2)^2-d$	4
Count stones in rectangular array formed by full left columns, then add remaining two stones in right column. Symbolized as $(d+1)(d+2)+2$	2
Count stones in the middle rows, then add in stones in the top and bottom rows, assuming the 6 stones from day 1 are always there. Symbolized as $d^2+d+6+2(d-1)$	1
Count stones in square array in upper-left corner of figure, then count *n* stones below square array, then count the remaining three stones.	1
Simplify $(d+2)^2-d$ to standard form.	1
Model part of the stones laid with the exponential 2^{d+1}, and then correct for the number of stones not counted with a quadratic that passes through the points (1,4), (2,6), and (3,6). Partially symbolized as 2^{d+1}	1

Table 1 presents evidence that there were three explicit rules whose frequencies were greater than one. The first of these rules was described as counting the stones in the top and bottom row of the array, and then adding the stones in the middle rectangular array. Figure 2 below visually illustrates this counting for the day two figure.

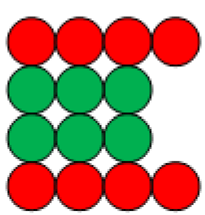

Figure 2. Counting Stones in the Top and Bottom Rows, as well as the Middle Rectangular Array.

Four of the ten participants developed this rule, with each participant fully symbolizing the coordination of the two co-varying quantities. The only differences in the symbolization of the rule were in the use of different letters to represent the varying day number. The following excerpt from Eli helps to illustrate how this rule appeared in the data.

> "So I'm going to start with those outside ones [top and bottom rows] which would be 2 times the day plus 2 (writes $2(n + 2)$). (points at day 1 figure) So the day number, adding 2 onto that (points at length of top row), and multiply that by 2 (points at top and bottom rows). (pointing to day 2 figure) The day number, adding 2 onto that, multiplying that by 2. (points at day 3 figure) The day number, adding 2 onto that, multiplying it by 2. That takes care of those outside ones [rows] and then with those inside [rows of stones forming] rectangles…It goes from 2 [stones for day 1], to 6 [stones for day 2], to 12 [stones for day 3]. (writes $n(n + 1)$) I mean, so my first thing was to do the day number times the day number plus 1. So for that first one [day 1], that would be 1 times 2, so that would be 2 [stones] there. For that second one [day 2], it would be 2 times 3. That would be 6. And for that third one [day 3], it would be 3 times 4. So yeah, that would give you 12. And each of those would evaluate out. (circles $2(n + 2+n(n+1)$)"

As illustrated by this excerpt, Eli first began by counting the number of stones in the top and bottom rows. He recognized that each figure always contained these two rows, as illustrated by the leading factor of two in the term $2(n + 2)$. He also recognized that the length of the top and bottom rows varied, captured in the $n + 2$ factor of the first term in the rule. Next, Eli proceeded to count the stones between the top and bottom rows that formed a rectangular array. He identified the length of each row as being one more than the day number, and the number of middle rows as always being equivalent to the day number. He captured this in the second factor of his rule, symbolized as $n(n + 1)$. Thus, counting the stones in the top and bottom rows, as well as those in the middle rectangular array resulted in the rule $2(n + 2) + n(n + 1)$.

The second rule with frequency greater than one was described as counting the stones assuming there is a full square array, and then removing the stones not present in right column of this square array. Figure 3 below illustrates this counting visually.

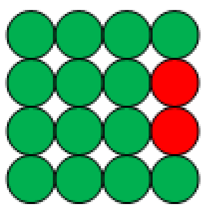

Figure 3. Counting the Stones Assuming there is a Full Square Array, and then Removing Stones Missing in the Right Column.

Four of the ten participants developed this rule, with each participant fully symbolizing their rule. The only differences in the symbolization of the rule were in the use of different letters to represent the variables. The following excerpt from Brooke helps to illustrate how this rule appeared in her solving of this task.

> "So first I'm going to look at it [case 1 figure] like a square (traces pencil around day 1 figure). So I have a 3x3 [square] on day 1. And then on day 2 I have a 4x4 [square], (writes 3x3 and 4x4 under day 1 and 2 figure respectively) and, 5x5 (writes 5x5 under day 3 figure). And then this one [figure] is missing one [tile]

(points to day 1 figure), and 2, and 3 (points to 3 missing tiles along right side of day 3 figure)… I'm going to represent the day by D. So then I have $(D + 2)^2 - D$ (writes $(D + 2)^2 - D$)."

In this excerpt, Brooke first began by looking at the figure as a square array of stones. She identified the dimensions of this array, writing them as an un-simplified product, and then noted the number of stones missing in each array. She then used this information to write a generalized rule to describe this relationship.

The third and final rule with frequency greater than one was described as counting the stones in the rectangular array formed by the left columns, and then adding in the two remaining stones in the right-most column. Figure 4 below illustrates this counting visually.

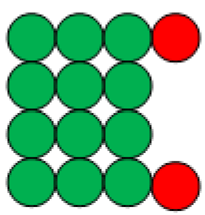

Figure 4. The Stones in the Left-Hand Columns, and the Remaining Two Stones in the Right-Hand Column.

Two of the ten participants developed this rule, with each participant fully symbolizing their rule. The only differences in the symbolization of the rule were in the use of different letters to represent the variables. The following excerpt from Frank helps to illustrate how this rule appeared in his solving of this task.

"(points at day 1) so this is, I have 2 rows of 3 [stones] and then plus 2 (writes $2 \cdot 3 + 2$ above day 1 figure). Here [day 2] I have 3 rows of 4 [stones] (writes

$3 \cdot 4 + 2$). And here [day 3] I have 4 rows of 5 [stones] and 2 (writes $4 \cdot 5 + 2$)… since this is day 1 (points to day 1 rule and numerical statement) this is really day plus 1 times day plus 2 plus 2 (writes $(d + 1)(d + 2) + 2$). And [on day 2], 3 is that day number plus 1, times the day number plus 2 gives me 4, plus the 2 stones at the end (writes $(d + 1)(d + 2) + 2$). Here [on day 3], I have the day number plus 1 to give me 4, plus the day number plus 2, giving me 5, plus the 2 additional stones on the end [of the figure] (writes $(d + 1)(d + 2) + 2$)…the number of stones laid, which is s is equal to the day number plus 1, plus the day number plus 2, plus 2 (writes $(d + 1)(d + 2) + 2$)."

In this excerpt, Frank began by describing the array formed by the columns on the left, followed by adding the two stones in the right-hand column. After writing a numerical sentence for each case, he then related these quantities to the day number, developing a general rule that would describe each day.

Developed rules – recursive.

However, not all developed rules were explicit, some were recursive. A GE summary was categorized as a recursive rule when the rule began with a given case and then iteratively built upon that case to determine subsequent cases in the sequence. There were a total of six recursive rules developed being of three different types. These types ranged in frequency from one to three, and were provided by three of the ten participants in this study. The table below presents the distribution of these different types along with their descriptions and frequencies.

Table 2

Frequencies and Descriptions for Recursive Rules Developed

(Recursive) Rule Developed/Description	Frequency
Add a stone to the end of the top row and each middle row. Then add a new bottom row that has as many stones as the new top row. Most often symbolized as (e.g., $d_1 = 8; d_n = d_{n-1} + 2n + 2$)	3
Identify the current number of stones laid on day *n*. Determine how many stones were added to day *n-1* to give the number of stones laid on day *n*, and increase that number of stones added by 2, then add that quantity to the number of stones laid on day *n*. Most often symbolized as (e.g., $d_n = d_{n-1} + 6 + 2^{n-2}$)	2
Add a stone to the end of the top and bottom rows, as well as each middle row. Then add a new middle row. Not symbolized.	1

The most common type of rule was described as adding a stone to the end of the top row, as well as each middle row in the figure. Then, a new row of stones was added across the bottom of the figure that contained as many stones as the new top row. Figure 5 below illustrates this counting visually.

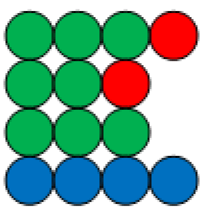

Figure 5. Adding a Stone to the End of the Top and each Middle Row, Followed by Adding a New Row of Stones across the Bottom.

In Figure 5 above, the green stones represent the stones that were laid on day one. The red stones illustrate the stones added to the end of the top and each middle row. The blue stones represent the new row of stones added across the bottom of the figure that contain as many stones as the new top row (i.e., green stones plus the new red stone).

Three of the ten participants developed this rule. The following excerpt from Dane helps to illustrate how this rule appeared in his solving of this task.

> "What I'm seeing is from here (points at day 1 figure) is I had my original, my original 8 stones. And then, now I'm adding one here [to the end of the top row], I'm adding one here [to the end of the middle row], and now instead of adding this one [stone] here [to the end of the bottom row], I'm really adding a bottom row of 4 [stones] here to get to that [day 2 figure]. So I'm adding one [stone] on the top [row], one [stone] on [the end of] the next row, and then my next row [original bottom row of day 1 figure] is fine, and then 4 [stones are added] here [to form a new bottom row 4 stones long]…then from here [on day 2], you do the same thing, I'm just introducing one more [stone] on the first row, one more [stone], now two more [stones] on each of the next two consecutive rows (draws one stone on the end of each row), and then I would be adding 1,2,3,4,5 [stones] now to the bottom row.… so I would say that I would add a stone to the top row, and then add one stone each to the middle rows, and then add a row on the bottom that matches the [new] top row [of stones]."

In this excerpt, Dane first describes how to convert the figure from day one into the figure from day two by adding a stone to the end of the top row and the middle row, followed by adding a new row of four stones across the bottom of the figure. He then used this same relationship to convert the figure from day two into the figure from day three. Dane then synthesized his rule as adding a stone to the end of the top and each middle row, followed by adding a row of stones across the bottom that contains as many stones as the new top row.

The second most common type of rule was described as identifying the rate of change in the number of stones laid between the current day (i.e., case *n*) and the previous day (i.e., case $n - 1$), increasing that rate of change by two, and then adding that value to the current number of stones laid on day *n*. However, rather than counting different components of the figures as in the previously described rules, what was counted in this rule was the quantity of stones laid each day as an entire conglomerate. To help illustrate this rule, consider the excerpt from Dane below.

> "to get from one day to the next day…I look at how many stones were added the day before, and then I'm adding 2 to that [amount] to project what my next day will look like, what the amount of stones will be on my next day [how many stones should be added to the current day to determine the total number of stones on the next day]…on day 1 I have 8 stones and then I'm adding 6 stones to get to day 2. And then from day 2 [to day 3], instead of adding 6 [stones], I'm adding 2 more to that 6, so I'm adding 8 [stones]. Then, to get from day 3 to day 4…instead of 8 I'm adding 2 more than 8, which is going to be the 10 [stones]."

In this excerpt, Dane began by describing how his rule worked for the general case. Next, he explained how the rate of change was being increased by two each time by comparing the rates of change of six, eight, and ten between day's one and two, two and three, and three and four respectively.

Attempted rules – explicit.

A GE summary was categorized as an attempted rule when a rule was conjectured, but the participant never concluded that he/she had finished the task. Stated differently, the participant began developing their rule but never completed it to his/her satisfaction. There were a total of nine rules attempted—four attempts at developing explicit rules made by three of the ten participants, and five attempts at developing recursive rules made by four of the ten participants. These attempted rules are separated into two tables below; the first is focused on attempted explicit rules and includes a description of each type and their associated frequency.

Table 3

Frequencies and Descriptions for Explicit Rules Attempted

Attempted (Explicit) Rules/Description	**Frequency**
Identified that the relationship is quadratic. The participant then tried to use the Vertex Form of a quadratic to symbolize this relationship, but was unable to adequately develop the appropriate symbolism to describe this relationship.	1
Identified that the relationship was linear, and that the goal was to directly relate the total number of stones laid each day with the day number. However, the participant was not able to determine a linear relationship that satisfied all three given cases.	1
Identified that the goal was to develop a direct (i.e., explicit) exponential rule that over-counts the number of stones laid on a given day, and then remove the over-counted amount. The participant was able to develop the exponential portion of the rule, but became stuck in trying to correct the over-counting.	1
Identified that the relationship should be an explicit rule. The participant then conjectured linear, quadratic, and exponential rules but was never satisfied with them.	1

In three of the four cases presented in Table 3, the participants overtly acknowledged that they were attempting to develop an explicit rule. Additionally, some of the participants stated the type of relationship (e.g., quadratic, linear, exponential) they believed the rule needed to capture. Each type of attempt occurred only once, and all attempts never resulted in the development of an explicit rule, regardless of any assumptions about the type of relationship. To help illustrate what an attempted rule looked like, consider the excerpt from Ian below, an excerpt that was captured in the last description in Table 3.

> "So I have 8,14,22 (writes 8,14,22). This is 1, this is 2, and this is 3 (writes 1 below 8, 2 below 14, 3 below 22). It's 4 (writes 4, pauses, then erases 4). 6 (writes 6n+2; draws arrow from this to 1, 2, and 3; writes $6n + (n^2)$ below this, then erases n^2; points at 6n and 1; replaces n^2 with 2n, then erases 2n; points at 1 and 8)…I'm trying to come up with an explicit rule… but because the numbers [rates of change] go up by different [values], it makes it difficult to come up with the proper numbers to use [for the rule]. So now, if I take 6 times 1, I have to add plus 2 (writes 6+2). If I use 2, 12 plus 2 (writes 12 + 2). If I [use 3], 18 plus 4 (writes 18+4). Right now I'm just thinking about multiples of 6 and how I can relate them. (points at 32 stones laid for day 4; counts number of missing stones in right column for what would be the day 4 figure) 1,2,3,4 (counts tiles in bottom row of what would be day 4 figure; adds 32 to list of 8,14,22), this would be 24 plus 8. That's 1,2,3,4 (writes 1,2,3,4 next to 6+2, 12+2, 18+4, 24+8 respectively; adjusts rule to $6n + (2^{n-1})$). Yeah, I don't think I can come up with another one [rule]."

In this excerpt, Ian first indicated that he was treating 1, 2, and 3 as corresponding with 8, 14, and 22 respectively. Next, he began conjecturing different rules, beginning first with a linear rule ($6n + 2$), and then adjusting it to a quadratic rule ($6n + n^2$). He then acknowledged that he was attempting to develop an explicit rule, but stated that because the rate of change was not constant it was difficult to determine which numbers to use when symbolizing the rule. Ian then attempted to use multiples of six to write different numerical decompositions of 8, 14, 22, and then 32, which was followed by the conjecture of a new rule ($6n + (2^{n-1})$). He then concluded that he was stuck and could not develop a rule for this task.

Attempted rules – recursive.

As previously noted, not all attempts to develop a rule were explicit. The table below presents the different types of recursive attempts and their associated frequencies.

Table 4

Frequencies and Descriptions for Recursive Rules Attempted

Attempted (Recursive) Rules/Description	Frequency
Identifies the recursive pattern +6,+8,… and attempts to develop a recursive rule based upon this pattern, but the participant was not able to develop the appropriate symbolism to capture this pattern as a recursive rule.	3
Identifies the recursive pattern +6,+8,… but confounds this recursive pattern, which builds off of the number of stones laid the previous day, with the symbolism associated with explicit rules and is unable to capture this recursive pattern as a recursive rule symbolically (i.e., no use of subscripts to denote terms in a sequence).	2

The most common type of attempted, but not developed, recursive rule was the identification of the recursive pattern +6,+8,... between the number of stones laid on days one and two, and days two and three, but not being able to capture this pattern as a rule symbolically. Three of the ten different participants attempted to develop this kind of recursive rule. To illustrate what an attempt to develop this kind of rule looked like, consider the excerpt from Hailey.

> "I feel like there's still a way to write it where it's like, so there's 8 and then you added 6 to get 14 (writes $8 + 6 \rightarrow 14$) and then you added 8 to get 22 (writes $14 + 8 \rightarrow 22$)…I keep thinking there's those functions where you have to take what you did before and you have to put it into the next line to find out what you get that time [recursive rule]. And then you have to take that and put it into the next line. And you have to figure out how you…I'm having difficulty setting it [recursive rule] up…If you just did f1 is equal to 8 (writes $f_1 = 8$), then, err, f0 is equal to 8. And then every time, err, f n was equal to 8, I don't know…So if you knew that one of them was 8 and 0 was, um, let's try this. So f1 is 8, plus f of n minus 1, so then what was f of 0, equals the number of tiles (writes $8 + f_0 = tiles$). I don't know if you set it [rule] off of 0 or if it's 2…I can't think of another way to do it [task]."

In the excerpt above, Hailey first identified that she wanted to use the recursive pattern of +6,+8,… and add to the previous case to determine the subsequent case. Next, she noted that she was having difficulty in capturing the rule symbolically. She then continued to symbolize this rule, but became stuck or confused multiple times, ultimately concluding that she could not develop a recursive rule to the task.

One characteristic that was common to both kinds of attempts to develop a recursive rule was the participants' difficulty in developing the appropriate symbolism to capture the rule. This difficulty was encountered not only by Hailey as illustrated in the excerpt above, but by all four of the ten participants who attempted to develop recursive rules. Participants who attempted to develop recursive rules frequently encountered difficulty in symbolizing their rule.

No rule attempted.

The final major category of rules was that of not developing and not attempting to develop a rule. A GE summary was categorized as no rule developed when an attempt was not made to develop a rule. That is, participants searched for a pattern or characteristic from the problem that might lead to a rule, but were not able to determine or act upon a pattern or characteristic and, thus, did not attempt to develop a rule. The table below presents the distribution of these different types of no rule attempted along with their description and associated frequencies.

Table 5

Frequencies and Descriptions for No Rule Attempted

No Rule Attempted/Description	Frequency
Searching for useful information or problem characteristics to develop a direct rule by investigating the rates of change	2
Searching for useful information or characteristics of patterns when assuming the relationship is linear or exponential	2
Searching for useful information to develop a direct rule by rearranging the stones in the figures into columns based upon values in the Fibbonacci sequence	1

Similar to the developed and attempted rules, one common type was to identify the rates of change in the number of stones laid between days one and two, and days two and three. Two of the ten participants were identified in this sub-category. To help illustrate what this looked like, consider the excerpt from Gina below.

"So what I would do is make a table…And I would determine the slope. So thinking of these as ordered pairs, one ordered pair would be (1,8), and (2,14). Slope is y2 minus y1 over x2 minus x1. So using this as x1 and y1 (points at ordered pair (1,8)), and this is x2 and y2 (points at ordered pair (2,14)), 14 minus 8 over 2 minus 1 is equal to 6 over 1, so your slope is 6 (writes $\frac{14-8}{2-1} = \frac{6}{1} = 6$). I don't think that's right though because this one doesn't increase by 6 (points at ordered pair (3,22) in table), so that's not right because it's not a constant slope [rate of change]. Hmm…"

In the excerpt above, Gina first stated that she wanted to organize the information in a table and then considers the slope (i.e., rates of change). Next, she determined the slope between the points (1,8) and (2,14), and then noted that the rate of change between these two points, six, was different than the rate of change between the points (2,14) and (3,22), which was eight. After noting that these rates of change were not the same, she then appeared to be unsure of what to do next, as indicated by her "Hmm…" statement.

The other equally common category from the no rule attempted category was described as searching for useful information, or characteristics of patterns, when assuming the relationship was linear or exponential. Two of the ten participants were identified as belonging to this category. To help illustrate, consider the excerpt from Ashley below.

"If our number of days are our input in some kind of function to get to our output, I was trying to see if it would be some multiplication. So if this is 8 times (points to arrow between 1 and 8 in f(1)) and 7 times (points to arrow between 2 and 14 in f(2)) and 6 times (points to arrow between 3 and 22 in f(3)) that there'd be some kind of relationship there. But it doesn't work for all of them [cases]…now I'm going to see if there some kind of…some exponential or some powers. Like 2^3 (writes 2^3 next to the 8 in f(1)). (points to f(2)), but 2 doesn't evenly go into 14 with powers of 2. These [problems] always give me trouble."

In this excerpt, Ashley first noted that she was looking to determine a factor she could multiply the day number by to produce the number of stones for that day. After hypothesizing that this factor is reduced by one for each day, she commented that this trend did not continue across all days—it failed on the third day. Next, she attempted to rewrite the number of stones as a power of two. After doing this for the eight stones laid on the first day, she commented that the number of stones laid on day two (14) could not be rewritten as a power of two. Ashley then commented that determining rules for patterning tasks was a challenge.

The distribution of the types of rules.

The number of GE summaries associated with each of the three major categories were counted. The largest number of GE summaries was associated with the development of a rule (59%), with attempted rules totaling less than half of this amount (26%). Even including the no rule developed category, rules were developed to this task more often than not. The distribution of the GE summaries is shown in the table below.

Table 6

Distribution of the Major Types of Rules from Task 1 (The Patio Tile Task)

Major Type of Rule	**Frequency (Relative Frequency)**
Developed rules	20 (59%)
Attempted rules	9 (26%)
No rule attempted	5 (15%)
Total	**34 (100%)**

Observations from Task One: The Patio Tile Task

After identifying the different types of rules developed, attempted, or not attempted, I explored the data for additional information. The results are presented below.

Comparing developed and attempted explicit and recursive rules.

After categorizing and developing descriptions for the different types of rules for task one, the Patio Tile task, I noticed that both explicit and recursive rules were both developed and attempted. This information was organized in the table below.

Table 7

Explicit and Recursive Rules, both Developed and Attempted

	Developed	**Attempted**	**Totals**
Explicit Rule	14	6	**20**
Recursive Rule	6	3	**9**
Totals	**20**	**9**	**29**

Looking at Table 7 one can observe several things. One, explicit rules were more than twice as common as recursive rules (20:9). A similar pattern was also observed when you looked at developed (14:6) and attempted rules (6:3) separately. Two, rules were developed 69% of the time (20/29). A similar pattern was observed when you looked at explicit rules, where they were developed in 70% (14/20) of all of the explicit cases, and recursive rules where they were developed in 67% (6/9) of all recursive cases. Drawing across this information, the participants appeared to more frequently appeal to explicit rules, and demonstrated the ability to develop a rule, regardless of the strategy used. I also separated the data by participant to search for trends.

Separating the types of rules by participant.

I separated the different types of rules by participant for the Patio Tile task, which is presented in the table below.

Table 8

Separation of the Types of Rules on the Patio Tile Task by Participant

	Rule Type			
Participant	Developed	Attempted	Not Attempted	**Totals**
Ashley	-----------------------	1	1	**2**
Brooke	3	------------------------	-----------------------	**3**
Claire	2	------------------------	-----------------------	**2**
Dane	3	------------------------	-----------------------	**3**
Eli	4	------------------------	1	**5**
Frank	3	1	2	**6**
Gina	-----------------------	2	1	**3**
Hailey	1	3	-----------------------	**4**
Ian	2	1	-----------------------	**3**
Jack	2	1	-----------------------	**3**
Totals	**20**	**9**	**5**	**34**

Looking at Table 8, three participants (Brooke, Claire, and Dane) always developed rules, two participants (Ashley and Gina) never developed rules, and five participants (Eli, Frank, Hailey, Ian, and Jack) developed a rule some of the time. Considering the participants who sometimes did and did not develop rules, three of these five participants tended towards developing rules more frequently than they did not (4:1, 2:1, and 2:1), one of the participants tended towards not developing rules more frequently than developing them (1:3), and one participant had an even split between the number of rules developed and the number of rules not developed (3:3). Overall, six of the ten participants developed or trended towards developing rules to this task, three of the ten participants did not develop or trended towards not developing rules to this task, and only one participant was an even split between developing and not developing rules to this task. These results appear to present evidence of the variability in which students develop rules.

Separating the explicit or recursive rules developed or attempted by participant.

I also analyzed the explicit and recursive rules that were attempted and/or developed by each participant. The frequencies of each participant's explicit or recursive rules are presented in Table 9.

Table 9

Separating the Explicit and Recursive Rules Developed or Attempted by Participant

	Explicit or Recursive Rule Developed or Attempted		
Participant	Explicit	Recursive	**Totals**
Ashley	------------------------------------	1	**1**
Brooke	3	-----------------------------------	**3**
Claire	1	1	**2**
Dane	------------------------------------	3	**3**
Eli	4	-----------------------------------	**4**
Frank	4	-----------------------------------	**4**
Gina	2	-----------------------------------	**2**
Hailey	3	1	**4**
Ian	1	2	**3**
Jack	2	1	**3**
Totals	**20**	**9**	**29**

Looking at Table 9, one can observe that only one participant, Claire, had an even balance (1:1) between explicit and recursive rules. The remaining nine participants more frequently developed or attempted either an explicit or recursive rule. Of the remaining nine participants, six of them trended towards developing explicit rules, with four of these six *only* developing or attempting explicit rules. The remaining two, of these six participants, only developed or attempted a single recursive rule. The theme across these six participants was that they drew upon explicit rules exclusively, or nearly exclusively on this task.

The remaining three participants trended towards developing recursive rules, with two of these three *only* developing or attempting recursive rules. The other participant, of these three, developed or attempted an explicit rule once. The theme across these three participants was that they drew upon recursive rules exclusively, or nearly exclusively on this task. Overall, the data in Table 9 present evidence that indicate participants trended towards developing or attempting a particular style of rule (i.e., explicit rules or recursive rules) versus a more balanced combination of rules.

Task Two: The Happy-Face Cutouts Task

Once GE summaries associated with the Happy-Face Cutouts Task (see Figure 6) were identified, they were sorted into three major groups based upon the extent to which a rule was (or was not) developed— developed rules, attempted rules, and no rule attempted. A discussion of each of these major groups follows, with the developed and attempted rules further separated as developed/attempted explicit and recursive.

To help positively reinforce good homework completion habits, a teacher puts happy-face cutouts on her classroom's bulletin board in the arrangement shown below. The teacher adds more cutouts to the board for each week all of the students complete all homework assignments. Determine and <u>write a rule</u> that relates the **number of weeks** the students have completed all homework assignments to the **total number of cutouts** required to construct that pattern. Please <u>explain</u> why your rule is correct. Solve this task in as many ways as you can.

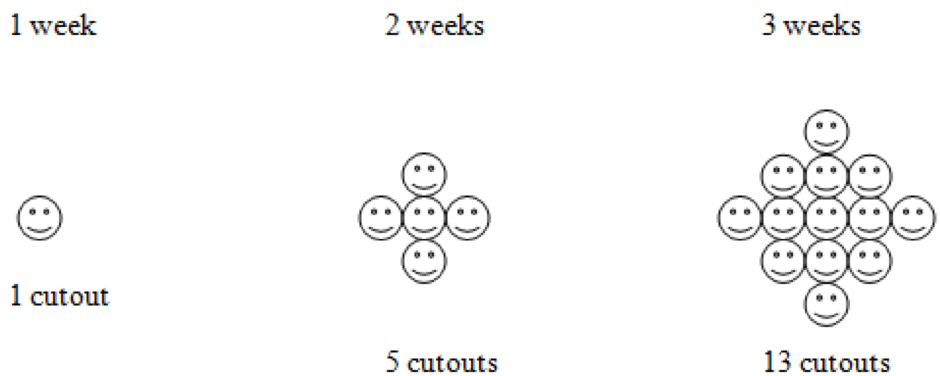

Figure 6. The Happy Face Cutouts Task Administered to Participants.

Developed rules – explicit.

A GE summary was categorized as a developed rule when a rule was developed and completed to the satisfaction of the participant. There were a total of eight rules developed—five of these were developed explicit rules made by four of the ten participants and three of these were developed recursive rules made by three of the ten participants. These developed rules are separated into two tables below. The results presented in the first table consist of a description of the different types of developed explicit rules along with their associated frequencies.

Table 10

Frequencies and Descriptions for Explicit Rules Developed

(Explicit) Rule Developed/Description	Frequency
Count the number of cutouts in the array by squaring the week number and adding the remaining portion, realizing that the remaining portion is a perfect square of one less than the week number. Always symbolized as $w^2 + (w-1)^2$	4
Count the number of cutouts in the array by adding one less than the week number of multiples of four to the initial one cutout from week one to determine the number of cutouts in the array for that week. Symbolized as $1 + 4(w-1)$	1

Table 10 shows that there was only one explicit rule developed with frequency greater than one. This rule was described as finding the number of cutouts for a particular week by squaring the week number and adding the remaining portion, realizing that the remaining portion is the square of one less than the week number. Four of the ten participants developed this rule, with each participant symbolizing the rule as $w^2 + (w-1)^2$. The only difference in the symbolizations of the rule was in the letter used to denote the changing week number, and the order of the two squared terms. The following excerpt from Ashley helps illustrate how this rule appeared in her solving of this task.

> "So, if I related 2 to 5 (for the number of cutouts in week 2) and 3 to 13 (for the number of cutouts in week 3) and 4 to 25 (for the number of cutouts in week 4), I wonder (writes $2^2 + 1$ above week 2; $3^2 + 4$ above week 3; $4^2 + 9$ above week 4). So trying to relate the number of weeks to the cutouts, so (rewrites 4 as 2^2 in week 3 expression; rewrites 9 as 3^2 in week 4 expression; rewrites 1 and 1^2 in

week 2 expression). So let's say that it's [the rule is] weeks squared plus weeks minus 1 squared (writes $w^2 + (w-1)^2$)."

In this excerpt, Ashley began by relating the week number to the associated number of cutouts. She then rewrote the number of cutouts for that week as the square of that week number, plus another quantity whose sum would give the number of the cutouts for that week. Next, she rewrote this additional quantity being added as the square of one less than the week number. Ashley then stated her rule, $w^2 + (w-1)^2$.

Developed rules – recursive.

As previously noted, not all developed rules were explicit. The table below presents descriptions of the different types of recursive rules developed with their associated frequencies.

Table 11

Frequencies and Descriptions for Recursive Rules Developed

(Recursive) Rule Developed/Description	Frequency
Given the number of cutouts in an array for a particular week, add on four times the week number associated with that array to determine the number of cutouts for the subsequent week. Symbolized in one of the two cases as $c_1 = 1, c_n = c_{n-1} + 4(n-1)$	2
Assuming the pattern is a square array with an extra cutout above/below the middle column and to the left/right of the middle row, to determine the number of cutouts for a week (i.e., week *n*), one takes the number of cutouts on the previous week (i.e., week $n-1$) and increases this amount by 2^n. Symbolized as $a_{n-1} + 2^n$	1

Table 11 demonstrates that there was only one recursive rule developed whose frequency was greater than one. This type of rule occurred twice, each time by a different participant. This type of developed recursive rule was described as taking the number of happy-face cutouts for a particular week and then adding on four times that week number to determine the number of cutouts for the next week. The following excerpt from Claire helps to illustrate how this rule appeared in the participants' solving of this task.

> "So this is like one of the recursive things again where I need to be adding, like, I start off with the number I had the week before and then I have to add to it…each week we're adding 4, err, 4 then 8, so…I mean, I think you still start off with 1 [cutout] and then you have to add 4 [cutouts], and then 8 [cutouts]…So, you start off with your 1 [cutout] and then you add 4 (writes $1 + 4$). So you have 1 (writes 1 above $1 + 4$) and then you have 4 again (writes $1 + 4 + 4$). But you don't add 4 you add 8 (adjusts previous statement of $1 + 4 + 4$ to $1 + [4] + (4 + 4)$). So you'll always have 1, and then to that you have to add 4. You add 4 on week 2, which is…so, like, 4 times the week number minus 1 (writes $1 + 4(w - 1)$)…So w represents the number of weeks but then I think it would be helpful if I had one [variable] that represents the number of cutouts. I'll call that [variable] c. So here's c on week 2 (writes c_2) and then w 2, or just w. It's not necessary [to write the subscript on the variable w]. Umm, and then the number of cutouts on day [week] 3 is 1 plus (writes $c_3 = 1 +$)…and then you have to use…no, you have to use c 2, plus 4 times, w minus 1 (writes $c_3 = c_2 + 4(w - 1)$; rewrites rule for general case of $c_n = c_{n-1} + 4(n - 1)$)."

In this excerpt, Claire began by first indicating that she thought she would need to use a recursive rule to capture this relationship. She then identified the change in the number of cutouts between successive weeks as +4, +8, followed by writing the number of cutouts for each week in terms of the cutout from week one and then adding multiples of 4. In writing this she realized that the number of multiples of four is changing between successive weeks, and indicated that the number of factors of four that would need to be added was one less than the week number. Claire then stated that she needed two variables—one variable to indicate the number of cutouts and one variable that identified the week number. She then rewrote the number of cutouts in week three based upon the number of cutouts from week two, and then stated her recursive rule for this relationship.

Attempted rules – explicit.

A GE summary was categorized as an attempted rule when a rule was conjectured, but the participant never concluded that he/she had finished the task. Stated differently, the participant began developing their rule but never completed it to his/her satisfaction. There were a total of eight rules attempted—five attempts at developing explicit rules made by five of the ten participants, and three attempts at developing recursive rules made by three of the ten participants. These attempted rules are separated into two tables below. The first table is focused on attempted explicit rules and includes a description of the different types of attempts and their associated frequencies.

Table 12

Frequencies and Descriptions for Explicit Rules Attempted

(Explicit) Rule Attempted/Description	Frequency
Conjecture an explicit rule, and then adjust this rule based upon whether it accurately relates the week number with its associated number of cutouts.	3
Attempts to count the number of cutouts in the middle column and row (the "cross"), and then adds this to the number of cutouts not on this middle column and row (not on the "cross"). Able to count the cutouts on the cross but becomes stuck when trying to count cutouts not on the cross.	1
Assumes the pattern is a square array with an extra cutouts above/below the middle column and to the left/right of the middle row, and attempts to count the number of cutouts in the square array and then add on the four cutouts on the top/bottom and left/right of the middle column/row. Able to count all portions except the number of cutouts in the square array.	1

The most common type of attempted explicit rule was described as conjecturing a rule and then adjusting the conjectured rule based upon whether or not it appropriately related the week number to the number of cutouts associated with that week. Three of the ten participants attempted to develop a rule of this subtype. This rule was always symbolized by participants, but never in a way that captured the direct relationship between the week number and the number of cutouts for all three weeks at once (e.g., $n^{n-1} + n^2 - 1, 1 + 4(x-1)^2$). To help illustrate what an attempted explicit rule of this type looked like, consider the excerpt from Jack below.

> "If I start off with x [the number of weeks] (writes x). So for week 1, x is 1. For week 2, x would be 2, plus 3, not 4, that was my problem—I was thinking in terms of week 1. So for week 3, that's 3 plus 10. 2 plus 3 [for the number of cutouts in week 2], 3 plus 10 [for the number of cutouts in week 3]. Unless, what if we just start with 1 (erases x and writes 1), and then for each week it's 4 times x minus 1 (adjusts to 1 + 4(x-1)). So for week 1 (writes week 1), that would give

me 1 (writes 1 next to "week 1"), which is what I need. For week 2 (writes week 2), that would give me 1 times 4, plus 1 is 5 (writes 5 next to "week 2"). For week 3 (writes week 3), that would give me 2 times 4 is 8, but that's still 9. (adjusts rule to $1 + 4(x - 1)^2$) If that's squared, for week 3, that would be 2 times 2 is 4. No, that wouldn't work (adjusts rule back to 1 + 4(x-1)). I think this one [task] stumped me."

In this excerpt, Jack started by conjecturing the rule *x*. Next, he checked this rule with weeks one, two, and three, noting that he would be adding three and then ten to the week number for weeks two and three respectively. He then adjusted his rule to $1 + 4(x - 1)$. He then checked his rule with weeks one, two, and three, commenting that it satisfied weeks one and two, but not week three. He then adjusted his rule to $1 + 4(x - 1)^2$ and checked it for week three, commenting that it would not work. He then concluded that he was unable to further develop his rule and complete the task.

Attempted rules – recursive.

As previously noted, not all attempts to develop a rule were explicit. The table below presents the different types of attempted recursive rules with their associated frequencies.

Table 13

Frequencies and Descriptions for Recursive Rules Attempted

(Recursive) Rule Attempted/Description	Frequency
Recognizes the recursive pattern of +4, +8, … in the change in the number of cutouts between successive weeks, but confounds this recursive pattern, which builds off of the number of cutouts from the previous week, with trying to write an explicit rule and cannot develop the symbolism to capture this pattern.	2
Counts the number of cutouts in the array and then adds a "border" of cutouts around the outside of the array, counting the number of cutouts in this border, with the sum of these two quantities being the number of cutouts in the subsequent array. Does not conclude that this rule is correct or developed though.	1

The most common type of attempted recursive rule was described as identifying the recursive pattern of +4, +8,… in the number of cutouts between successive weeks and confounding this recursive pattern with an explicit rule's symbolization to determine the number of cutouts for a particular week. Two the ten participants attempted to develop a rule of this type. The two participants always symbolized it as an explicit rule, despite that they identified the rule as building off of the number of cutouts from the previous week. To illustrate what an attempt of this type looked like in the context of solving this task, consider the following excerpt from Ashley below.

> "In this one we're adding, (draws an arrow between number of cutouts in week 1 and 2) we're adding 4 here. (draws an arrow between number of cutouts in week 2 and 3) And this time we're adding 8…Okay, so then it looks like we're adding 4 more [cutouts] every time (points at the arrows between the number of cutouts for weeks 1,2, and 3) to the number of smiley faces [cutouts]…Week 4 would be 1,2,3,4,5,6,7,8,9,10,11,12 more…So this is $4^1 + 1$ (writes below 5 cutouts for

week 2), this is 8 plus the previous [amount] (writes $4^2 + 5$ below 13 cutouts for week 3). This is 16 (writes $4^3 + 9$), hmmm (writes 4^0 below 1 cutout on week 1) hmm…I don't know [how to do] this [task]."

In this excerpt, Ashley first identified the change in the number of cutouts between successive weeks. She then indicated that the rate of change is increasing by four every time. She the utilized this pattern to extend the relationship to the fourth week, commenting that it would have 12 more cutouts than the previous week's cutouts. Following this, she rewrote the number of cutouts as an exponential with a base of four, plus whatever amount needed to be added to produce the number of cutouts associated with the week number. Ashley then stated that she did not know how to do this task and was not able to finish developing a rule to it.

No rule attempted.

The final major category associated with the Happy-Face Cutouts Task was that of no rule attempted. A GE summary was categorized as no rule attempted when no attempt was made to develop a rule. That is, participants searched for a pattern or characteristic from the problem that might lead to a rule, but were not able to determine or act upon a pattern or characteristic, so they did not attempt to develop a rule. The table below presents the distribution of these different types of no rule attempted along with their description and associated frequencies.

Table 14

Frequencies and Descriptions for No Rule Attempted

No Rule Attempted/Description	Frequency
Searches for useful information by identifying the change of +4, +8, … between successive cases in a single variable	2
Searches for useful information by rewriting the number of cutouts as a sum of the square of a week number and what amount remains	1

Both types appeared to consider only numerical aspects of the task. The most common type of rule not attempted was described as identifying the recursive pattern of +4, +8, … in the number of happy-face cutouts between consecutive weeks. Two of the ten participants identified this pattern, but were not able to act upon it to attempt a rule. To help illustrate what this looked like in the context of working this task, consider the excerpt from Brooke below.

> "Okay, so this one [week 2 figure] added 1,2,3,4 on [to week 1 figure to produce week 2 figure] (writes +4 beside week 2 figure). And so [the week 3 figure] added 1,2,3,4,5,6,7,8 [on to week 2 figure to produce week 3 figure] (writes +8 beside week 3 figure). And so, maybe [extend] that pattern for next time [week 4], and then add 1,2,3,4,5,6,7,8…And then you would add, umm, hmm, I'm not sure. "

In this excerpt, Brooke first identified how many cutouts would be added to the cutouts from week one to give the number of cutouts for week two. She then repeated this, identifying the number of cutouts to be added to the cutouts from week two to produce the number of cutouts for week three. She then commented that she wanted to extend this pattern to week four, but stated that she was not sure how to extend it.

The distribution of the types of rules.

The number of GE summaries associated with each of the three major categories were counted. Unlike task one, there was an even split between the number of developed rules (42%) and the number of attempted rules (42%). However, similar to task one, the number of cases where no rule was attempted remained low. The distribution of the GE summaries associated with task two is shown in the table below.

Table 15

Distribution of the Types of Rules from Task Two (The Happy-Face Cutouts Task)

Major Type of Rule	**Frequency (Relative Frequency)**
Developed rules	8 (42%)
Attempted rules	8 (42%)
No rule attempted	3 (16%)
Total	**19 (100%)**

Observations from Task Two: The Happy-Face Cutouts Task

After identifying the different types of rules developed, attempted, or not attempted, I explored the data for additional information. The results are presented below.

Comparing developed and attempted explicit and recursive rules.

After categorizing and developing descriptions for the different types of rules for the Happy-Face Cutouts task, I noticed that both explicit and recursive rules were both developed and attempted. These developments and attempts were then separated and organized in the table below.

Table 16

Explicit and Recursive Rules, both Developed and Attempted

	Developed	**Attempted**	**Totals**
Explicit Rule	5	5	**10**
Recursive Rule	3	3	**6**
Totals	**8**	**8**	**16**

Looking at Table 16, one can observe several things. One, explicit rules were developed or attempted in 62.5% (10/16) of the of the GE summaries for this task, with recursive rules accounting for the remaining 37.5% (6/16) of the GE summaries for this task. This distribution was also observed when you looked at developed (5/8) and attempted rules (5/8) individually. Comparing explicit to recursive rules, explicit rules were 67% more common as compared to recursive rules (10:6). Two, participants developed rules in 50% (8/16) of the GE summaries for this task, with the reaming 50% (8/16) attempted. This distribution was also observed if you looked at explicit (5/10) or recursive rules individually. There was an even split between developed and attempted rules (8:8). Synthesizing across these statements, explicit rules appeared to be the preferred type of rule used by participants, but there was an even split between rules that were developed and those that were only attempted. I also separated the data by participant.

Separating the types of rules by participant.

I also separated the different types of rules captured in the three major rule categories by participant for the second task (the Happy-Face Cutouts task), which is presented in the table below.

Table 17

Separation of the Major Types of Rules on the Happy-Face Cutouts Task by Participant

	Rule Type			
Participant	Developed	Attempted	Not Attempted	**Totals**
Ashley	1	1	-----------------------	**2**
Brooke	-----------------------	1	1	**2**
Claire	3	-----------------------	-----------------------	**3**
Dane	1	-----------------------	-----------------------	**1**
Eli	1	2	-----------------------	**3**
Frank	1	-----------------------	-----------------------	**1**
Gina	-----------------------	1	-----------------------	**1**
Hailey	-----------------------	-----------------------	2	**2**
Ian	1	1	-----------------------	**2**
Jack	-----------------------	2	-----------------------	**2**
Totals	**8**	**8**	**3**	**19**

Looking at this table (Table 17), three participants (Claire, Dane, and Frank) always developed rules, four participants (Brooke, Gina, Hailey, and Jack) never attempted to developed rules, and three participants (Ashley, Eli, and Ian) sometimes did and did not develop rules. Considering the participants who sometimes did and did not develop rules, zero of these three participant trended towards developing rules, one of these three trended towards not developing rules (1:2), and two of these three were an even balance between sometimes developing and sometimes not developing a rule to this task (1:1). Synthesizing, three of the ten participants developed rules, five of the ten participants did not develop or trended towards not developing rules, and two of the ten participants were an even split between developing and not developing rules for this task. As a whole, participants appeared to not develop or trend towards not developing rules to this task (3:5).

Separating the explicit or recursive rules developed or attempted by participant.

The explicit or recursive rules that were developed or attempted were also separated by participant. The frequencies of each participant's explicit or recursive rules developed or attempted are presented in Table 18 below.

Table 18

Separating the Explicit and Recursive Rules Developed or Attempted by Participant

	Explicit or Recursive Rule Developed or Attempted		
Participant	Explicit	Recursive	**Totals**
Ashley	1	1	**2**
Brooke	1	-----------------------------------	**1**
Claire	2	1	**3**
Dane	------------------------------------	1	**1**
Eli	2	1	**3**
Frank	1	-----------------------------------	**1**
Gina	1	-----------------------------------	**1**
Hailey	------------------------------------	-----------------------------------	**0**
Ian	1	1	**2**
Jack	1	1	**2**
Totals	**10**	**6**	**16**

Looking at this table (Table 18), one can observe that participants developed or attempted to develop explicit or recursive rules with frequencies ranging from zero to three, with about one and a half rules developed or attempted to be developed per participant on average. Eight of the ten participants developed or attempted to develop explicit rules and six of the ten participants developed or attempted to develop recursive rules. Stated differently, nearly all participants developed or attempted to develop at least one explicit rule to the task, whereas just over half of the participants developed or attempted to develop one or more recursive rules to the task.

Five of the ten participants more frequently developed or attempted to develop explicit rules with three of those fire participants only developing explicit rules. One of the ten participants only developed recursive rules, and this participant developed only one rule. Three of the ten participants had an even balance between developing or attempting to develop explicit and recursive rules, with all three participants each developing or attempting to develop one explicit and recursive rule. Additionally, one of the ten participants never developed or attempted to develop either an explicit or recursive rule. Overall, it was most common for participants to develop or attempt to develop a particular type of rule (i.e., explicit rules or recursive rules).

Task Three: The Star Sticker Task

Once GE summaries associated with the Star Sticker task (see Figure 7) were identified, they were then sorted into three major groups based upon whether the rule that was (or was not) developed—developed rules, attempted rules, and no rule attempted. A discussion of each of these major groups follows, with the developed rules further separated as explicit, recursive, and hybrid, and the attempted rules separated as explicit and recursive.

Sophia has been collecting star-shaped stickers. At the end of each week, she arranges all of the stickers she has collected, forming the pattern shown below. Determine and <u>write a rule</u> that relates the **number of weeks** Sophia has been collecting stickers to the **total number of star stickers** she has collected for that week. Please <u>explain</u> why your rule is correct. Solve this task in as many ways as you can.

Figure 7. The Star Sticker Task Administered to Participants.

Developed rules – explicit.

A developed rule occurred when a rule was developed and completed to the satisfaction of the participant. There were a total of 15 developed rules—ten were developed explicit rules made by eight of the ten participants, four were developed recursive rules made by three of the ten participants, and one was a developed hybrid explicit/recursive rule that was made by one of the ten participants. These developed rules are separated into three tables below. The results presented first are the descriptions of developed explicit rules along with their associated frequencies.

Table 19

Frequencies and Descriptions for Explicit Rules Developed

(Explicit) Rule Developed/Description	Frequency
Determine the number of stars in the array by multiply the dimensions of the array. The number of rows and columns is directly related to the week number. Always symbolized as $w \times (2w - 1)$ or $w \times (w + w - 1)$	6
Count the number of stars in the left columns that for a *n* x *n* square, and then count the number of stars in the rectangular array formed by the remaining columns. Always symbolized as $w^2 + w(w - 1)$	3
Count the number of stars as if the rectangular array was a full square where the height is a great as the width, and then subtract the overcounted (i.e., missing) rows at the top of the array. Always symbolized as $(2w - 1)^2 - (w - 1)(2w - 1)$	1

Table 19 above presents evidence that there were two explicit rules developed whose frequencies were greater than one. The first rule was described as relating the dimensions of the width and the height of the array to the week number, and then using the product of these dimensions to determine the number of stickers in the array. Six of the ten participants developed this rule, with each participant able to fully symbolize their rule. A minor difference in the symbolization of the rule occurred in the second factor—some participants counted the number of stickers in the width of the array as $2w - 1$ and some counted them as $w + w - 1$. Additionally, some participants utilized different letters to represent the varying week number (e.g., x, w, n). The following excerpt from Claire helps to illustrate how this rule appeared in her solving of this task.

> "maybe I should think about the number of sides [dimensions of each figure]. So here [week 2 figure] it's a 3 by 2, here [week 3 figure] it's a 5 by 3, so is it [rule] the number of weeks, times the number of weeks plus something (writes $w(w +$ $)$? Here [week 2] it's the number of weeks plus 1, here [week 3] it's the number

of weeks plus 2. (points at week 2) So 1,2,3, so 1, the week number minus 1…To get the number of columns you do the week number and then you're going to add 1 less than the week number as well…(writes $w(w + w - 1)$)"

In this excerpt, Claire first considered the dimensions of the array. She then indicated the dimensions of each array of stars for week's 2 and 3 and found that the number of rows was the same as the number of weeks. Next, she counted the number of columns, first as the number of weeks plus something, and then refined this to the number of weeks plus one less than the number of weeks. Claire then symbolized her rule as $w(w + w - 1)$.

The second rule whose frequency was greater than one was described as determining the number of star stickers that are in the $n \times n$ square formed by the first n columns and then counting the remaining stars in the rectangular array formed by the remaining columns. Figure 8 below illustrates this counting visually.

Figure 8. Counting Star Stickers in the Square and the Remaining Rectangular Array.

Three of the ten participants developed this rule, with each participant fully symbolizing their rule. The only difference in the rule's symbolization was the use of different letters to represent the varying week number. The following excerpt from Eli helps to illustrate how this rule appeared in his solving of this task.

"I'm just going to notice that that's a 1 by 1 [square] (circles 1x1 square in week 1 figure), that's a 2 by 2 (circles 2x2 square on left side of week 2 figure), that's a 3 by 3 (circles 3x3 square on left side of week 3 figure; writes n^2). Then, you've got 0 [stickers] left here [week 1], 2 [stickers] left on that one [week 2], 6 [stickers] left on that one [week 3]…this is like a 1 by 2 [rectangle left in week 2 figure], this is a 2 by 3 [rectangle left in week 3 figure]. So then, maybe that's a 0 by 1 [rectangle left in week 1 figure]. So that's going to be 1, err, 0 I guess. So that n squared, you're going to add an n times n minus 1 (adjust rule to $n^2 + n(n-1)$)."

In this excerpt, Eli first identified a square array of stars in the figure with the dimensions equal to the week number, symbolizing this as n^2. He then considered the stars that were unaccounted for by this square. Eli gave the dimensions of these rectangles for weeks one, two, and three. He then adjusted his rule to $n^2 + n(n-1)$ to account for these remaining rectangular arrays of stars.

Developed rules – recursive.

As previously noted, not all developed rules were explicit. The table below presents descriptions of the different types of recursive rules developed with their associated frequencies.

Table 20

Frequencies and Descriptions for Recursive Rules Developed

(Recursive) Rule Developed/Description	Frequency
Add two columns of stars to the right of the array where each column height is as great as any other column, and then add a row of stars across the top of all columns. This rule was never symbolized.	2
Add a row of stars across the top of all columns, and then add two columns of stars to the right of the array whose height is a great as any other column. This rule was symbolized once as $a_n = a_{n-1} + 3(n-1) + n$	2

Both types of rules occurred twice, given by two different participants in each case. The first type of recursive rule developed was described as adding two columns of star stickers to the right-end of the array, then adding a row of star stickers across the top of all of the columns. Figure 9 below illustrates this counting visually.

Figure 9. Adding Two Columns of Star Stickers to the Right-End, Followed by Adding a New Row of Stars Across the Top of all Columns.

In Figure 9 above, the green stars represent the rectangular array of stars from week two. The red stars illustrate the two columns added to the right-end of the week two figure. The blue stars represent the new row of stars added across the top of all of the columns. Two of the ten participants developed this rule. The following excerpt from Ian helps to illustrate how this rule appeared in the participants' solving of this task.

> "I think that all you do is add two columns to the end [of the figure], and a row on the top, that could be a rule. So to go from here (points at week 1 figure), you add two columns on the right, or two starts on the right of this [given figure], and add one row of stars [across the top], if you're seeing it visually."

In this excerpt, Ian was looking at the figures from weeks one, two, and three and commented that to convert week one's array of stars into the subsequent week's array you add two columns of stars to the end of the figure, and then a row of stars across the top of all of the columns. He then explained his rule in converting the week one figure into the week two figure by adding two columns of stars and then a row of stars across the tops of the columns.

The other equally common developed recursive rule was similar to the rule presented above. However, what distinguished it was the reversal of the order in which the row and columns were added. This rule added a row of stars across the top of the array and then added two columns to the right of the figure whose height was equal to the original array with the newly added row. Figure 10 below illustrates this counting visually.

Figure 10. Adding a New Row of Stars Across the Top of the Columns, Followed by Adding Two Columns of Star Stickers to the Right-End.

In Figure 10 above, the green stars represent the rectangular array of stars from week two. The blue stars represent the row of stars added across the top of the columns from week two. The red stars illustrate the two new columns of stars added to the right-end of the modified array of stickers. Two of the ten participants developed this rule. The following excerpt from Dane helps to illustrate how this rule appeared in his solving of this task.

> "So I'm kind of seeing it as increasing it 1 here (draws new row across top of week 1 figure), and then I'm taking this and then just introducing another (draws 2 more columns with the same height as amended week 1 figure [2 stars high])…So adding this one [row across the top of figure] and then adding these 2

rows [columns] to match that row [new height of figure]…And then to get from week 2 to week 3, so now I'm adding (draws row across top of week 2 figure), I'm adding these 3 [stars] up here, and then I'll want to add 2 more rows [columns] of those 3."

In this excerpt, Dane began by converting the array of stars from week one by drawing a row of stars across the top of the array, and then adding two columns of stars to the end of the amended array. Next, he stated the rule could then be used to convert one week's array of stars into the subsequent week's array of star stickers. Following this statement, he illustrated his rule by converting the array of stars in week two to the array of star stickers in week three via this rule.

Developed rules – hybrid explicit/recursive.

The third subcategory of rules that were developed, which I have termed a hybrid explicit/recursive rule, only occurred in one case by a single participant. It is described in the table below.

Table 21

Frequencies and Descriptions for Hybrid Explicit/Recursive Rules Developed

(Hybrid Explicit/Recursive) Rule Developed/Description	Frequency
Determine the number of stars in the array by multiplying the dimensions of the array. The number of rows is directly related to the week number, whereas the number of columns is based upon how many columns were in the preceding week. Always symbolized as $w \times (c_{w-1} + 2)$	1

Although this rule only occurred once and utilized the dimensions of the rectangular array to count the number of stickers, what made it unique was how it counted them. The number of rows (i.e., the first factor in the symbolization of the rule) was directly (i.e., explicitly) related to the week number. The number of columns (i.e., the second factor in the symbolization of the rule) was based upon the number of columns from the preceding week (i.e., recursive), which was then increased by two to determine the number of columns of stickers in the array for the current week (i.e., $c_{w-1} + 2$). The following excerpt from Ian helps to illustrate how this rule appeared in his solving of this task.

> "The way I did this one [came up with this rule] was I kind of figure out the dimensions of each one [figure]. Like how this [figure 1] is just a 1 by 1, kind of like a matrix. Like this [figure 2] is a 2 by 3 and that [figure 3] is a 3 by 5. I realize that every number in the first one is the same as the week (point at leading digit of 1,2,3 in 1x1, 2x3, 3x5)…So then I realized that this was a 1 (points at second 1 in 1x1 expression), so in every single one of these (points at week 2 figure), they add 2 columns (points at 2 columns drawn on week 2 figure to convert to week 3 figure). So you add 2 [to the number of columns each time]. 1 plus 2 is 3, 3 plus 2 is 5. So you take your previous one, your previous week [number] (points at second 1 in 1x1) and you add 2. Like this number (points at second 1 in 1x1 and 3 in 2x3), and add 2…And then if you came up with another one it would be a 4 by 7 (writes 4x7). So this number stays the same (points at 4 in 4x7) as this one (points at 4 in week number), and then you add 2 from here (points at 5 in 3x5) to get that (points at 7 in 4x7). So the n_{-1} means this number [number of columns in the array from the week before] (points to 5 in 3x5). And then to find the number of stickers you just multiply this number [number of

rows] (underlines n factor in rule) times this number [number of columns] (underlines $n_{-1} + 2$ factor in rule)."

In this excerpt, Ian unpacked his thinking after stating his rule. He first commented that he identified the dimensions of the array of stars for each week's rectangular shape. Next, he noted that the number of rows in the array is the same as the week number, and that the number of columns in each array increased by two from the previous week. He then illustrated his rule by extending it to the fourth week, noting that the dimensions of the array of stars would be four-by-seven (rows by columns) where the four rows is the same as the week number and that the seven columns is two more than the week before.

Attempted rules – recursive.

An attempted rule occurred when a rule was conjectured, but the participant never concluded that he/she had completed the task. Stated differently, the participant began their rule but never completed the rule to his/her satisfaction. There were a total of six rules attempted but never developed—four attempts were recursive rules made by three of the ten participants, and two attempts were explicit rules made by two of the ten participants. These attempted rules are separated into two tables below. The first table provides descriptions of the attempted recursive rules along with their associated frequencies.

Table 22

Frequencies and Descriptions for Recursive Rules Attempted

(Recursive) Rule Attempted/Description	Frequency
Attempts to develop a recursive rule based upon the figures by adding a border of stars along the left, top, and right sides of the array, but cannot develop the appropriate symbolism to capture counting the given cases.	2
Attempts to develop a recursive rule based upon the figures by adding two columns to the right of the array and a row across the top of all of the columns, but cannot develop the appropriate symbolism to capture this pattern for the given cases. Symbolized in one case as $5w - 4$	2

The first type of attempted recursive rule was the addition of a "border" along the left, top, and right sides of the figure. The participants were not able to capture this rule symbolically. Figure 11 below illustrates this counting visually.

Figure 11. Adding a Border of Stars Along the Left, Top, and Right Side of the Figure.

Two of the ten participants attempted to develop recursive rules of this subtype. To illustrate what an attempt of this subtype looked like, consider the following excerpt from Gina below.

> "I think she just added a row (traces pencil along left column, across top row, and down right column), all the way around. That's what she did here (traces pencil up the left of the left column, above top row, and to the right of the right column in week 1 figure). So this one [week 4 figure], I'm just going to write them [stars] as x's. This is what three weeks was (sketches week 3 figure). So then 4 weeks, you need a row here (adds column of 3 stars to the left of sketched week 3 figure),

and a row here (adds column of 3 stars to the right of sketched week 3 figure), and a row on top (sketches row of stars across top of all columns). So this is 1,2,3,4 times 1,2,3,4,5,6,7, so 28 [star stickers]. So this is increasing by 13 [star stickers from week 3 to week 4] (writes +13 between number of star stickers laid on week's 3 and 4). So, the difference in the increase [second difference] is 4 [star stickers]. So, this is 3 times 2 (writes 3(2) below week 2 figure). No, that's not right (erases 3(2))."

In this excerpt, Gina first identified the pattern of adding a column, row, and column of stickers along the left, top, and right sides of the previous array of stickers. She noted that this could be done to convert the week one array of stickers into the week two array of stickers. Gina then used this visual pattern to extend the array of star stickers from week three to week four, and then counted the number of star stickers present in the array. She then attempted to symbolize this counting, but decided her symbolization was incorrect after attempting to do so.

The other attempted recursive rule was adding two columns of star stickers to the right of the array and then a row of stars across the top of all of the columns. This attempted rule was not captured symbolically. Two of the ten participants attempted this type of recursive rule. This description of an attempted recursive rule was almost the same as the description of the recursively developed rule presented in Table 20 (see Figure 9). The difference was whether the attempt culminated in the developed rule or if the attempt did not culminate in a rule. It is worth noting that this is an example of an attempted rule that could be transitioned into a developed rule with further work.

One characteristic that was common to both of these attempted recursive rules was the participants' difficulty in developing appropriate symbolism for the rule. This difficulty was encountered by all three of the ten participants who attempted recursive rules. In both types of attempted recursive rules, the participants always encountered difficulty in symbolizing their rule.

A second characteristic that was common to both attempted recursive rules was the operation upon the figures. In both attempted recursive rules the participants worked to convert one figure into the subsequent figure in sequence, either by adding two columns and a row or by placing a border around the left, top, and right sides of the figure. This attention to the figure can be seen not only in the descriptions provided, but also in Gina's excerpt above.

Attempted rules – explicit.

As previously noted, not all attempts to develop a rule were recursive. The table below presents the different attempted explicit rules with their associated frequencies.

Table 23

Frequencies and Descriptions for Explicit Rules Attempted

(Explicit) Rule Attempted/Description	Frequency
Rearrange the rectangular arrays of stars into triangular arras of stars (i.e., triangular numbers—a sequence of columns where each column has one more star than the preceding column), but is unable to coordinate the week number and the number of stars when trying to count them.	1
Conjectures a quadratic, then linear rule, unsatisfied with each conjecture after it is made.	1

Each attempted explicit rule occurred only once. To illustrate what this looked like, consider the excerpt from Gina below, an excerpt that was captured in the last description in Table 23 above.

> "Let's try a quadratic. So if this is 2 squared plus 4 (writes $2^2 + 4$ above week 2 figure), err, no, 2 squared plus 2 (adjusts to $2^2 + 2$), 2 to the third, plus 7 (writes $2^3 + 7$ above week 3 figure), 2 to the fourth…so it would be 45 [star stickers in week 5]. So that would be 2 to the fifth plus… No, it's not going to be 2 to the (erases $2^2 + 2$ and $2^3 + 7$)…I don't think it is [quadratic]…4 times 2 is 8, minus 2 (writes 4(2)-2 above week 2 figure). 4 times 3 is (writes 4(3)+3 above week 3 figure). 4 times 4 is 16, plus (writes 4(4)+12 above week 4 figure), no, that's not it (erases 4(4)+12 and (4(3)+3). Umm… I don't know."

In this excerpt, Gina first considered the rule to be quadratic, searching for common structure in the symbolization of the number of stickers for each week. However, after extending to weeks four and five she did not determine a common symbolic structure, and states that the rule isn't quadratic. She then proceeded to symbolize each week's stickers as a multiple of four, with an adjustment made to this multiple of four to account for any over or undercounting. Gina then stated that this was not correct and concluded that she could not develop another rule for this task.

No rule attempted.

The final category was that of no rule attempted. A GE summary was categorized as no rule attempted when an attempt was not made to develop a rule. That is, participants searched for a pattern or characteristic from the problem useful for developing a rule, but were not able to determine or act upon a pattern or characteristic to develop a rule. The table below presents a description of each of the no rule attempted categories and associated frequency.

Table 24

Frequencies and Descriptions for No Rule Attempted

No Rule Attempted/Description	Frequency
Searching for useful information by identifying the changes of +5, +9, … between successive cases for the number of star stickers in each array.	1
Searching for useful information using the slope formula, concluding the relationship is not linear.	1
Searching for useful information by determining the ratio between the number of star stickers and the week number.	1

Unlike the descriptions of the developed and attempted rules, the rules that were not attempted appeared to consider only numerical aspects of the task (e.g., numerical recursive pattern of +5, +9, …, determining slopes using the slope formula). To help illustrate what consideration of numerical aspects of the task looked like, consider the excerpt from Hailey below.

> "So this one has that 1 and a 1 thing again (writes (1,1) below week 1 figure; writes (2,6) below week 2 figure; writes (3,15) below week 3 figure). I guess I'll try to find the slope here (writes $\frac{6-1}{2-1} = \frac{5}{1}$ between week 1 and 2 figures; writes $\frac{15-6}{3-2} = \frac{9}{1}$ between week 2 and 3 figures). So nope, it's not linear."

In this excerpt, Hailey began by identifying the week number and the associated number of stickers in that week's rectangular array. Next, she commented that she should determine the slope between the points (1,1) and (2,6), as well as (2,6), and (3,15). After doing so, she concluded that the relationship was not linear. This excerpt illustrates what a no rule attempted looked like, as well as what consideration of the numerical aspects of the task looked like.

The distribution of the types of rules.

The number of GE summaries associated with each of the three major categories was counted. The largest number of GE summaries was associated with the development of a rule (62.5%), with attempted rules accounting for less than half of this amount (25%). Even including the no rule developed category, rules were developed for this task more often than not. The percentage of no rule attempted on task three was similar to that of task one and two. The distribution of the GE summaries is shown in the table below.

Table 25

Distribution of the Types of Rules from Task Three (The Star Sticker Task)

Major Type of Rule	**Frequency (Relative Frequency)**
Developed rules	15 (62.5%)
Attempted rules	6 (25.0%)
No rule attempted	3 (12.5%)
Total	**24 (100%)**

Observations from Task Three: The Star Sticker Task

After identifying the different types of rules developed, attempted, or not attempted, I explored the data for additional information. The results are presented below.

Comparing developed and attempted explicit, recursive, and hybrid rules.

After categorizing and describing the different types of rules for the Star Sticker task, I noticed that explicit, recursive, and hybrid (explicit/recursive) rules were both developed and attempted. These developments and attempts were then separated and organized in the table below.

Table 26

Explicit, Recursive, and Hybrid Rules, both Developed and Attempted

	Developed	Attempted	Totals
Explicit Rule	10	2	**12**
Recursive Rule	4	4	**8**
Hybrid Rule	1	---------------------------	**1**
Totals	**15**	**6**	**21**

Looking at Table 26, one can observe several things. One, explicit rules were developed or attempted in 57% (12/21) of the of the GE summaries for this task, with recursive rules accounting for 38% (8/21), and the one hybrid rule accounting for only 5% (1/21) of the rules developed or attempted on this task. However, this distribution was different for developed and attempted rules. For instance, when looking only at developed rules, explicit rules were in 67% (10/15) of the GE summaries. Yet, if we focus only on the attempted rules, only 33% (2/6) of these attempts utilized explicit rules. Two, participants developed rules in 71% (15/21) of the GE summaries, while they only attempted rules in 29% (6/21) of the GE summaries for this task. Comparing the developed to the attempted rules, rules were developed two and a half times more frequently than attempted (15:6). However, this distribution was different for recursive and explicit rules. When looking at only explicit rules, 83% (10/12) were developed as compared to recursive rules, where only 50% of them (4/8) were developed. Synthesizing across these statements, explicit rules that were developed appeared to be the preferred type of rule by participants.

Separating the types of rules by participant.

I also separated the different types of rules by participant for the Star Sticker task. These results are presented in the table below.

Table 27

Separation of the Types of Rules on the Star Sticker Task by Participant

	Rule Type			
Participant	Developed	Attempted	Not Attempted	**Totals**
Ashley	1	------------------------	------------------------	**1**
Brooke	2	------------------------	------------------------	**2**
Claire	1	2	------------------------	**3**
Dane	2	------------------------	------------------------	**2**
Eli	3	------------------------	------------------------	**3**
Frank	1	1	1	**3**
Gina	------------------------	2	------------------------	**2**
Hailey	1	------------------------	2	**3**
Ian	3	1	------------------------	**4**
Jack	1	------------------------	------------------------	**1**
Totals	**15**	**6**	**3**	**24**

Looking at Table 27, five participants (Ashley, Brooke, Dane, Eli, and Jack) always developed rules, one participant (Gina) never developed rules, and four participants (Claire, Frank, Hailey, and Ian) developed a rule some of the time. Considering the participants who sometimes did and did not develop rules, one of these four participants trended towards developing rules (3:1), and three of these four trended towards not developing rules (1:2, 1:2, 1:2). Synthesizing, six of the ten participants developed or trended towards developing rules and four of the ten participants trended towards not developing rules for this task. As a whole, participants appeared to develop or trended towards developing rules to this task, but only slightly (6:4).

Separating the explicit, recursive, and hybrid rules developed or attempted by participant.

I analyzed the explicit, recursive, and hybrid rules that were attempted or developed by each participant. The frequencies of each participant's explicit, recursive, and hybrid rules are presented in Table 28 below.

Table 28

Separating the Explicit, Recursive, and Hybrid Rules by Participant

Explicit, Recursive, or Hybrid Rule Developed or Attempted				
Participant	Explicit	Recursive	Hybrid	**Totals**
Ashley	1	-----------	-----------	**1**
Brooke	1	1	-----------	**2**
Claire	1	2	-----------	**3**
Dane	-----------	2	-----------	**2**
Eli	3	-----------	-----------	**3**
Frank	2	-----------	-----------	**2**
Gina	1	1	-----------	**2**
Hailey	1	------------	-----------	**1**
Ian	1	2	1	**4**
Jack	1	-----------	-----------	**1**
Totals	**12**	**8**	**1**	**21**

Looking at Table 28, one can observe that participants developed or attempted explicit, recursive, or hybrid rules with frequencies ranging from one to four, with an average of just more than two rules developed or attempted per participant. Nine of the ten participants developed or attempted explicit rules, five of the ten participants developed or attempted recursive rules, and only one participant developed a hybrid rule. Stated differently, nearly every participant developed or attempted at least one explicit rule, half of the participants developed or attempted one or more recursive rules, and only one of the participants developed a hybrid rule.

Five of the ten participants developed or attempted explicit rules more frequently, with all five of these participants *only* developing explicit rules. Three of the ten participants more frequently developed or attempted recursive rules, with one of those three *only* developing recursive rules. Two of the ten participants had an even balance between developing or attempting explicit and recursive rules, with both participants each developing or attempting one explicit and one recursive rule. Overall, participants trended towards developing or attempting a particular style of rule (i.e., explicit rules or recursive rules) versus a more balanced combination of rules.

Comparing and Synthesizing across the Types of Rules for the Quadratic Tasks

After analyzing the three quadratic tasks individually, I constructed tables to look for trends across the tasks (e.g., what was the most common type of developed rule), and compared the descriptions for the different categories across these tasks (see Appendix G). The account below begins with a potential misconception observed across all three tasks, continuing with the tables and their frequencies, and then progresses to a comparison and synthesis of the categories.

A potential misconception—confounding a recursive pattern with writing an explicit rule.

After categorizing and describing the different types of rules and their subcategories, I noticed that one of the attempts to develop a recursive rule contained a potential misconception—confounding a recursive pattern with writing an explicit rule (e.g., see Tables 4, 13). This stood out to me because I remembered multiple participants identifying a recursive pattern and attempting to capture it using an explicit rule. After reviewing the GE summaries and their categorizations, I observed that five of the ten participants might have possessed this misconception. That is, identifying a recursive pattern and attempting to describe it in the form of an explicit rule. To help illustrate what confounding a recursive pattern with an explicit rule looked like, consider the excerpt from Jack from task two below.

> "To get from 1 [cutout] to 5 [cutouts], for week 2 (writes 2), it's like we're adding 4 (writes x+4 below 2). And then to get from week 2 to 3 (writes 3), that's adding 8 [cutouts] from the original x (writes x+8 below 3). (points at where outside border of cutouts would be on week 3 figure to produce week 4 figure) Then that's adding 12 [cutouts], assuming the pattern stays the same to get to week 4…So if I were to relate this back to week 1 (points at x+8 below 3, then erases it and writes x+2*4), it would be 2 times 4. So at week 1, week 1 is just going to be 1 [cutout], then we're adding 4 [cutouts to it], and then we're adding twice 4, which is [adding] 8. And then we would be adding 3 times 4 [cutouts] for the next week, which would be 12 [cutouts]. So assuming there was a week 4, I'm guessing it would be x plus 3 times 4 (writes x+3*4). But how do I come up with a rule that describes that [pattern] is the question. This 4 (points at 4 in x+4, x+2*4, and x+3*4), x minus 1, times 4. No, that can't be right…So x plus 4 times [the quantity] x minus 1 (writes x + 4(x-1)). So for week 3, that would be 3 plus 4 times [the quantity] 3 minus 1 (writes 3 + 4(3-1)). That's 11 though, that's not right. Unless that's not supposed to be x (points at leading x term in x + 4(x-1)).

If that's [a constant] 1, no. That's not right either…So that's not the pattern. I'm just going to scrap all this and try to start over."

In this excerpt, Jack identified the rate of change of +4, +8 between weeks one and two, and weeks two and three. He then extended this pattern to +12 between weeks three and four. He then identified all of these rates of change as a multiple of four. Jack then worked to develop an explicit rule for this task. He first conjectured $4(x-1)$, then $x+4(x-1)$, followed by $1+4(x-1)$. After each conjecture he decided that the stated rule was not correct. Jack then decided that the pattern of +4, +8, +12,… was not the pattern and decided to return to searching for another useful characteristic or pattern in the task.

Although some participants were not able to work though this confusion, as in the case of Jack above, some were able to recognize the error being made. Of the five participants who confounded a recursive pattern with writing an explicit rule, two of them were able to identify and overcome their error. To help illustrate what it looked like for a participant to identify and overcome confounding a recursive pattern with writing an explicit rule, consider the excerpt from Claire from task one below.

"This is going to be the same [rule] as before (rewrites $8 + 1 + d + 2 + (d - 1)$ as $2d + 10$)…But day 3 doesn't work [doesn't check out with the rule]. So on day 3…ohhhhh, it's like recursive…I gave this number from the day before (circles 8 in rule), I was just counting it as 8 but it's actually the number of stones you have the day before."

In this excerpt, Claire had identified the number of stones laid on day one (8), and then had accounted for the other stones being added onto the day one figure to produce the day two figure ($1 + d + 2 + (d - 1)$), a recursive relationship. She then simplified this expression to $2d + 10$, not realizing that this simplification had treated the eight, the number of stones laid the previous day, as a constant eight. She then checked this explicitly symbolized rule against the day three case, stating that her rule did not work. She then returned to her un-simplified expression ($8 + 1 + d + 2 + (d - 1)$) and realized her error—the eight stones she had added with the one, two, and negative one were not the same type of object, even though they were symbolized the same way. Stated differently, although Claire was able to describe the rate of change between successive cases explicitly ($1 + d + 2 + (d - 1)$), she did not initially recognize that her rule needed to account for combining this rate of change with different cases in the sequence.

Comparing and synthesizing rules by task and categorization.

A table considering whether a rule was developed, attempted, or not attempted for the different tasks is presented below.

Table 29

Comparison of the Types of Rules across Tasks

	Task 1	**Task 2**	**Task 3**	**Totals**
Developed Rule	20	8	15	**43**
Attempted Rule	9	8	6	**23**
Not Attempted	5	3	3	**11**
Totals	**34**	**19**	**24**	**77**

Looking across all three of the quadratic tasks, rules were developed in 56% (43/77) of the GE summaries, rules were attempted in 30% (23/77) of the summaries, and rules were not attempted in 14% (11/77) of the summaries for these tasks. Developed rules were slightly less than twice as common as attempted rules (43:23), and just under four times as common as rules not attempted (43:11).

When considering rules developed versus attempted for individual tasks, there were roughly twice as many rules developed versus attempted on tasks one (20:9) and task three (15:6). However, there were an even split between developed and attempted on task two (8:8). The aggregation of these counts across the tasks resulted in just under twice as many rules developed versus attempted overall (43:23).

Looking across all three of the quadratic tasks, there was little variation in the number of rules attempted (9:8:6) or not attempted (5:3:3). However, there was more variation in the number of rules that were developed (20:8:15). Stated differently, the group of participants was less consistent in the number of rules developed when separated by task. Consider Table 30 below, which further decomposes the rules that were developed, attempted, or not attempted.

Table 30

A Finer-Grained Comparison of the Types of Rules across Tasks

	Task 1	**Task 2**	**Task 3**	**Totals**
Explicit (Developed)	14	5	10	**29**
Recursive (Developed)	6	3	4	**13**
Hybrid (Developed)	------------	------------	1	**1**
Explicit (Attempted)	4	5	2	**11**
Recursive (Attempted)	5	3	4	**12**
Not Attempted	5	3	3	**11**
Totals	**34**	**19**	**24**	**77**

Looking across all three quadratic tasks, there were 43 cases of developed rules, 23 cases of attempted rules, and 11 cases of not attempted rules. Of the 43 developed rules, 67% (29/43) were developed explicit rules, 30% (13/43) were developed recursive rules, and 2% (1/43) was a developed hybrid explicit/recursive rule. That is, developed explicit rules were the most common, developed recursive rules were a little less than half as common as explicit developed rules (13:29), and hybrid explicit/recursive rules were rare. Of the 23 cases of attempted rules, 48% (11/23) explicit attempted rules and 52% (12/23) were recursive attempted rules. In contrast to the skew towards developed explicit rules, there was nearly an even balance between attempted explicit versus attempted recursive rules (11:12).

Looking across all three tasks, there was little variation in the number of recursive (6:3:4) or hybrid (0:0:1) rules that were developed, explicit (4:5:2) or recursive (5:3:4) attempted rules, or rules not attempted (5:3:3). However, the most variation occurred in the number of developed explicit rules when looking across the three quadratic tasks (14:5:10). Stated differently, the group of participants was less consistent in the number of developed explicit rules depending upon the task, whereas the group of participants was fairly consistent in all other rules developed, attempted, or not attempted regardless of the task. Additionally, developed or attempted explicit rules were the most common type of rule developed or attempted across the three tasks. With the exception of task two, developed explicit rules were more than twice as common as any other type of rule developed or attempted for the quadratic tasks.

The GE summaries were also reorganized based upon whether the rule developed or attempted was an explicit or a recursive rule. Unlike Table 30, these counts did not include GEs where a rule was not attempted, as well as the one hybrid rule developed. This is why there are only 65 GEs accounted for in Table 31, versus the 77 GEs accounted for in Table 30 above.

Table 31

Explicit/Recursive Rules Developed or Attempted Separated by Task

	Task 1	**Task 2**	**Task 3**	**Totals**
Explicit Rule	18	10	12	**40**
Recursive Rule	11	6	8	**25**
Totals	**29**	**16**	**20**	**65**

Looking across the three quadratic tasks, developed or attempted explicit rules occurred in about 62% (40/65) of all of the cases, leaving developed or attempted recursive rules accounting for the other 38% (25/65) approximately. Comparing explicit to recursive rules developed or attempted, explicit rules were 60% more common versus recursive rules (40:25) across all three tasks. When considering each task individually, there were always more developed or attempted explicit versus recursive rules (18:11, 10:6, 12:8). Explicit rules ranged from being 50% more common as on task three (12:8) to 67% more common as on task two (10:6). Developed or attempted explicit rules occurred most frequently whether considering a task individually or the aggregation across all three tasks.

Each task accounted for between 25% (16/65) and 45% (29/65) of the developed or attempted explicit or recursive rules, with task two containing the fewest cases (16) and task one containing the most (29). Explicit rules on task one were 50% more common compared to task three (18:12), and 80% more common compared to task two (18:10). Recursive rules on task one were also approximately 40% more common than task three (11:8) and 80% more common compared to task two (11:6). Overall, task one generated the most developed or attempted rules, followed by task three, then task two.

However, there were 25% more developed or attempted rules on task three compared to task two (20:16) total. Developed or attempted explicit rules were 20% more common on task three versus two (12:10), with a difference of only two counts. Similarly, developed or attempted recursive rules were 33% more common on task three versus two (8:6), with a difference of two counts. That is, tasks two and three had nearly the same number of explicit and recursive rules developed or attempted on them. So although task one generated the most developed or attempted rules, the number of developed or attempted rules generated on task three was only slightly more than task two.

The GE summaries were also reorganized based upon whether the rule developed or attempted was explicit, recursive, or hybrid. Unlike Table 30, these counts did not include GEs identified as rule not attempted. This is why there are only 66 GEs accounted for in Table 32.

Table 32

Explicit/Recursive/Hybrid Rules Compared to Developed/Attempted

	Explicit Rule	**Recursive Rule**	**Hybrid Rule**	**Totals**
Developed	29	13	1	**43**
Attempted	13	10	--------------------	**23**
Totals	**42**	**23**	**1**	**66**

Looking across the three quadratic tasks, explicit rules occurred in 64% (42/66) of the GE summaries and recursive rules occurred in 35% (23/66). For developed rules, explicit were more than twice as common as recursive (29:13). In contrast for attempted rules, explicit were only 30% more common than recursive (13:10). Overall, explicit rules were 83% more common compared to recursive rules (42:23).

Looking across the three quadratic tasks, participants developed rules in 65% (43/66) of the GE summaries for this task and attempted rules in 35% (23/66). Explicit rules were developed more than twice as often as they were attempted (29:13). In contrast, recursive rules were developed only 30% more often than they were attempted (13:10). Overall, developed rules were 87% more common compared to attempted rules (43:23).

Comparing and synthesizing rules by participant.

GE summaries across all three tasks were reorganized based upon the participant and whether a rule was developed, attempted, or not attempted. This information is presented in Table 33 below.

Table 33

Developed/Attempted/Not Attempted Rules Separated by Participant

	Developed Rule	**Attempted Rule**	**Not Attempted**	**Totals**
Ashley	2	2	1	**5**
Brooke	5	1	1	**7**
Claire	6	2	----------------------	**8**
Dane	6	----------------------	----------------------	**6**
Eli	8	2	1	**11**
Frank	5	2	3	**10**
Gina	---------------------	5	1	**6**
Hailey	2	3	4	**9**
Ian	6	3	----------------------	**9**
Jack	3	3	----------------------	**6**
Totals	**43**	**23**	**11**	**77**

Aggregating across all three tasks, only one of the ten participants (Dane) always developed a rule, and only one of the ten participants (Gina) never developed a rule. The remaining eight of the ten participants sometimes did and sometimes did not develop rules to the three quadratic tasks.

Of the eight participants who sometimes did and sometimes did not develop rules, four of them (Brooke, Claire, Eli, and Ian) trended towards developing rules (5:2, 6:2, 8:3, and 6:3 respectively), two of them (Ashley and Hailey) trended towards not developing rules (2:3 and 2:7 respectively), and the remaining two (Frank and Jack) had an even balance of cases between developing and not developing rules (5:5 and 3:3 respectively). Synthesizing, five of the ten participants developed or trended towards developing rules, three of the ten participants trended towards not developing rules, and two of the ten participants had and even balance between developing and not developing rules. Overall, it was more common for participants to develop or trend towards developing rules on the tasks.

The GE summaries were also sorted based upon the participant and whether the rule, developed or attempted, was explicit, recursive, or hybrid. These counts did not include GEs where a rule was not attempted. This is why there are only 66 GEs accounted for in this table, versus the 77 GEs in other tables.

Table 34

Explicit/Recursive/Hybrid Rules Separated by Participant

	Explicit Rule	**Recursive Rule**	**Hybrid Rule**	**Totals**
Ashley	2	2	--------------------	**4**
Brooke	5	1	--------------------	**6**
Claire	4	4	--------------------	**8**
Dane	---------------------	6	--------------------	**6**
Eli	9	1	--------------------	**10**
Frank	7	----------------------	--------------------	**7**
Gina	4	1	--------------------	**5**
Hailey	4	1	--------------------	**5**
Ian	3	5	1	**9**
Jack	4	2	--------------------	**6**
Totals	**42**	**23**	**1**	**66**

Looking across the three tasks, participants developed or attempted rules with frequencies ranging from four to ten, with an average of more than six and a half rules developed or attempted per participant. Nine of the ten participants developed or attempted explicit rules, with a total of 42 developed or attempted explicit rules. Nine of the ten participants developed or attempted recursive rules, with a total of 23 developed or attempted recursive rules. One of the ten participants developed a hybrid rule, occurring only once during the three quadratic tasks (see task three). Additionally, developed or attempted explicit rules were a little less than twice as common as recursive rules. Synthesizing, nearly every participant developed or attempted at least two explicit and one recursive rule when looking across all three tasks.

Looking across the three tasks, six of the ten participants more frequently developed or attempted explicit rules with one of these six participants *only* developing or attempting explicit rules. Two of the ten participants more frequently developed or attempted recursive rules with *only* one of these two developing or attempting recursive rules. Two of the ten participants had an even balance between explicit and recursive rules. Overall, participants trended towards developing or attempting explicit rules when generalizing on the three quadratic tasks. However, this distribution of explicit, recursive, and split was slightly different when considered on a per task basis.

When considering the participant's explicit, recursive, or hybrid rules on a per task basis (see Tables 9, 18, and 28), participants often gave the same kinds of rules, regardless of task. Six of the ten participants (Brooke, Eli, Frank Gina, Hailey, and Jack) trended towards or exclusively gave explicit rules on each task. Of these six participants, one (Frank) gave only explicit rules on each task, and four (Brooke, Eli, Gina, and Hailey) gave only explicit rules on each task with the exception of one case. Additionally, one participant (Hailey) had a single task where she did not give any rules. The remaining one of these six participants (Jack) gave only explicit rules on task three (1:0), trended towards giving explicit rules on task one (2:1), and was split in giving explicit and recursive rules on task two (1:1). One of the ten participants (Dane) exclusively gave recursive rules on each task. The remaining three of the ten participants (Ashley, Claire, and Ian) were flexible in their choice of providing explicit, recursive, or hybrid rules. Two of these three participants (Ashley and Claire) each had one task where they only or dominantly gave explicit rules, one task where they only or dominantly gave recursive rules, and one task where they gave an even balance of explicit and recursive rules. The remaining one of these three participants (Ian) provided at least one explicit and recursive rule for each of the three tasks, as well as provided a hybrid rule for the third task. As a whole, participants tended to give the same kind of rule independent of the task, with the majority being explicit rules.

Comparing and synthesizing the category descriptions.

The developed explicit rules from tasks one and three often appealed to figural aspects, such as decomposing a figure into smaller components and counting the number of objects in each piece. In contrast, developed explicit rules on task two seemed to appeal to numerical aspects, such as identifying numerals that were perfect squares. Thus, appealing to figural or numerical aspects was used to develop explicit rules. Additionally, independent of the task, developed explicit rules were nearly always symbolized, with one exception occurring in task one.

The developed recursive rules in tasks one and two sometimes appealed to figural aspects (e.g., adding stones to different locations within a figure to produce the subsequent figure) or numerical aspects (e.g., increasing a given amount of cutouts by four times the week number). In contrast, the developed recursive rules in task three always appealed to figural aspects. Additionally, five of the seven types of recursive rules developed were often or always symbolized.

The attempted explicit rules in task two and three sometimes appealed to figural aspects (e.g., attending to the cutouts on the “cross”) or numerical aspects (e.g., adjusting a rule based upon whether or not it accurately related the week number and associated number of cutouts). In contrast, the attempted explicit rules described for task one, as well as some from task three, appeared to focus on the type of relationship (e.g., linear, quadratic, exponential). Additionally, descriptions for attempted explicit rules appeared to indicate the struggle to identify whether the relationship was quadratic or not. This was observed in the conjecturing of different linear and exponential rules, as well as incorporating non-quadratic components into the rules (e.g., over counting with an exponential term and then adjusting for this over counting). In contrast to the developed explicit rules, which appeared to focus primarily on figural or numerical aspects of the task, the focus in the attempted explicit rules appeared to be primarily on the type of relationship (e.g., linear, quadratic, exponential). It is important to note that some attempted explicit rules did consider figural or numerical aspects of the task, but that this focus was secondary to the type of relationship.

The attempted recursive rules in task one always appealed to numerical aspects (e.g., numerical change of +6, +8,…). In contrast, the attempted recursive rules in task three always appealed to figural aspects (e.g., adding a "border" of stars around the sides and top of a given figure). The attempted recursive rules in task two were evenly split between attending to numerical and figural aspects of the task. Additionally, only one of the six attempted recursive rules was symbolized, though this symbolization did not fit the pattern identified by the participant. For the other five of six attempted recursive rules, the participants encountered difficulties in attempting to symbolize the rule, such as with the use of subscripts.

For rules not attempted, participants identified rates of change between consecutive cases in all three tasks. In tasks one and three, rules not attempted also considered the relationship to not be quadratic (e.g., linear, exponential) before searching for additional information based upon this assumption. In the case of task three, one conclusion that was reached was that the assumption of the relationship being linear (i.e., non-quadratic) was inaccurate. Additionally, in task one, rearrangement of the objects in the figures was performed. For rules not attempted on task two, rewriting the numerals based upon the co-vary quantity occurred. In summary, rules not attempted appeared to identify features from the task (e.g., numerical and figural foci), as well as rates of change and the type of relationship.

A Synthesized Framework for the Rules on Quadratic Tasks

Considering the rules identified on the three quadratic tasks presented above, a framework for the different rules on quadratic geometric-numerical patterning tasks is now presented and described. This framework contains three dimensions—the rule's state of development, the kind of rule (to be) developed, and the characteristics the rule appeals to. The following three sections unpack and describe each of these three dimensions.

Dimension one: the rule's construction category.

The first dimension of the framework identifies the degree to which the rule was developed, for which there are three different categories. One category is for a rule *not attempted*. That is, searching through the task for a useful pattern or trend, but not observing a pattern or not attempting to develop a rule from the observed trend or pattern.

It is important to note that if a rule is not attempted, then the following two dimensions of the framework do not apply. Another category is for an *attempted* rule. That is, searching for a useful pattern or trend to develop a rule, identifying one, and acting upon this pattern or trend to create a rule that captures it. However, an attempted rule is not finished to the satisfaction of the individual. The third category is for a *developed* rule. That is, a rule that is identified as completed in a finalized form for the given task. What distinguishes this second category from the third one is that an attempted rule is not completed, whereas a developed rule is one that is completed. Additionally, a caveat about these categories is that although they may appear to be hierarchical, it is currently unclear if this is the case.

Dimension two: the kind of rule (to be) developed.

The second dimension of the framework identifies the kind of developed or attempted rule. One category is that of an *explicit rule*. An explicit rule directly relates two co-varying quantities from the task. The other category is that of a developed or attempted *recursive rule*. A recursive rule considers a particular case in a sequence and builds upon this case to determine the subsequent case(s) in the sequence. The last category is that of a developed or attempted *hybrid rule*. A hybrid rule contains some components that are explicit rules, and some that are recursive rules (see Table 21). Recall, a hybrid rule occurred in only one of the 77 GEs associated with the quadratic tasks, whereas explicit and recursive rules were more common. Thus, hybrid rules appear to be rarely developed or attempted.

A key observation with this dimension is that the developed or attempted rule does not make any assumption or implication about the representation of the rule. A verbal description is viewed no differently than a symbolic description for the different kinds of rules. Rather, the focus is on directly relating two variables (i.e., explicit rule), building upon a case to determine a subsequent case within a single variable (i.e., recursive rule), or a combination of these two (i.e., hybrid rule).

Dimension three: the characteristics the rule appealed to.

The third dimension of the framework identifies the characteristics that the rule appeals to. One category is that of a rule appealing to *figural* characteristics. That is, appealing to figures, pictures, or other visual images. Another category is that of appealing to *numerical* characteristics. That is, appealing to the quantities, numerals, or their decompositions. The final category is that of appealing to the *symbolic* format in which it is presented. That is, focusing on the mathematical symbols used to represent a mathematical object or relationship.

It is important to note that participants occasionally created tables of values and/or graphs in their process of developing a rule. Although these may have been influential to the participants thinking, this study sought to describe the resulting end product/object of the generalization process—the rule that was developed. With this narrowed focus, the participant's rules that were developed, attempted, or not attempted, and their associated descriptions capturing them were analyzed, not these potentially influential factors. Thus, the potentially influential factors were not included in this framework, as they were part of the participant's generalization process, and not in the objects produced by the participants as a result of this process. Additionally, participants sometimes appealed to both figural and numerical aspects of the tasks during their generalization process, however the rule that was attempted or developed only appealed to one of these aspects. Just as with the participant's tables of values and/or graph use, only the factors appealed to in the rules were captured in the associated descriptions.

Research Question One Summary

A taxonomy and description for the different types of rules given on the three quadratic geometric-numerical patterning tasks was presented, described, and illustrated with excerpts from transcripts. Additionally, trends within and between the rules and participants were investigated. Following this, a comparison and synthesis across all three tasks was presented and described, noting any trends within the descriptions and frequencies of the rules. Lastly, a framework for the rules on quadratic geometric-numerical patterning tasks was given and described.

Research Question Two

Research question two investigated the justifications given by the participants on the quadratic geometric-numerical patterning tasks. Specifically, what types of justifications are given when solving quadratic patterning tasks presented in a geometric-numerical format? What patterns or relationships exist between the types of justifications across tasks? The complete collection of generalization episode (GE) summaries was separated by interview task (e.g., the Patio Tile task). The GE summaries associated with each task were then categorized based upon the justification(s) associated with them. The presentation that follows is broken down by individual task, followed by a comparison between and synthesis across the three quadratics tasks.

Task One: The Patio Tile Task

Once GE summaries associated with the Patio Tile task were completed, they were sorted into three major groups based upon the justification that was (or was not) given—justification as verification, justification as explanation, and no justification given. A distribution of these different major groups is presented in Table 35 below.

Table 35

Distribution of the Types of Justifications from Task One (The Patio Tile Task)

Major Type of Justification	**Frequency (Relative Frequency)**
Justification as verification	17 (42.5%)
Justification as explanation	13 (32.5%)
No justification given	10 (25.0%)
Total	**40 (100%)**

Although there were 34 GEs sorted, six of those GEs utilized multiple justification types. The six GEs associated with multiple types of justification resulted in a total of 40 justification codes applied to the GEs. Based upon these codes, the data indicated that a justification was provided in 71% of the GEs (24/34). A discussion of each major justification category ensues below.

Justification as verification.

A GE summary was categorized as justification as verification when the justification verified or validated the conjectured rule. There were two types of justification as verification, one occurred in 15 GEs and the other in two, resulting in 17 instances of verification. The table below presents the distribution of these two types of verification along with their associated description and frequencies.

Table 36

Frequencies and Descriptions for Justification as Verification

Justification as Verification/Description	**Frequency**
Determining that a conjectured rule is true or false by substituting given cases in that result in a true or false statement (Verification as a Numerical Check)	15
Symbolically manipulating a conjectured rule such that it matches a previous rule assumed to be true (Verification as an Algebraic Check)	2

Table 36 presents the two types of verifications. The most common type was verification as a numerical check, described as substituting in values for particular cases and using the truth of this statement to determine the validity of the conjectured rule. All ten of participants utilized this reasoning with four of the ten participants using it in multiple GEs. This type of verification was used exclusively to validate conjectured rules, or a part of a conjectured rule. The following excerpt from Jack illustrates how this justification appeared in the participants' solving of this task.

“So what if the rule is…x squared, plus x (adjusts rule to $x^2 + x$). From day 1 to day 2, because for day 1 that would be x squared is 1, plus 1 gives me 2 here. For day 2 that gives me 2 squared, is 4, plus 2, is 6. For day 3, 3 squared is 9, plus 3 is 12. Okay, so that takes care of the middle part [rows] for the rule.”

In this excerpt, Jack had developed a rule to count the number of stones in the middle rows of the figure (i.e., between the top and bottom rows) based off of the day number. Jack then substituted in the day number and checked whether the resulting computation provided the appropriate number of stones for the middle rows in each day’s figure.

The other type of verification given was described as symbolically manipulating a conjectured rule so that it matched the form of another rule that was assumed to be true (i.e., verification as an algebraic check). Only one of the ten participants utilized this reasoning, but did so in two separate GEs. The excerpt below from Ian illustrates how this verification appeared in his solving of the task.

“If I use any of these other ways [rules], umm, I could come up with an infinite amount of rules that would describe it [pattern]…so if I multiplied out $(n + 2)^2 - n$, so $n^2 + 4n + 4 - n$. So you start with this [and transform it into] $n^2 + 3n + 4$. I mean, that’s standard form for a quadratic equation…I think all of these rules (points at previous 3 rules), I could find this rule (points to $(n + 2)^2 - n$) using this (points to $n^2 + 3n + 4$), and I could find this rule (points to $(n + 1)^2 + 3 + n$)…my rules are all interrelated. If I distribute this out, n plus 1 squared…n squared plus 2n plus 1, you’re still getting $n^2 + 3n + 4$.”

In this excerpt above, Ian commented that if he was to use any of his previously developed rules, he could come up with a multitude of rules for this task. He then extrapolated on this, indicating that if he were to symbolically manipulate one rule, he could transform it into a new rule. He then noted that his rules were all related to one another. Throughout this excerpt, Ian argued the interrelationships between all of the rules by symbolically manipulating the form of one rule into another.

Justification as explanation.

A GE summary was categorized as justification as explanation when the justification explained and provided additional details or insights for a conjectured rule. There were a total of 13 instances where justification as explanation occurred. These instances were subdivided in two subcategories. The table below presents the distribution of the two types of explanations along with their associated description and frequencies.

Table 37

Frequencies and Descriptions for Justification as Explanation

Justification as Explanation/Description	Frequency
Providing insight into why a statement cannot be made or is false	8
Providing insight into why a statement can be made or is true	5

Table 37 presents frequency counts and descriptions for the two types of explanations. The most common was described as explaining why a conjectured rule was false or could not be made. Six of the ten participants drew upon this reasoning with two of these six participants using it in multiple GEs. The following excerpt from Gina illustrates how this explanation appeared in her solving of this task.

> "I would determine the slope. So thinking of these as ordered pairs, one ordered pair would be (1,8), and (2,14)…so your slope is 6 (writes $\frac{14-8}{2-1}=\frac{6}{1}=6$). I don't think that's right though because this one doesn't increase by 6 (points at ordered pair (3,22) in table)…this is obviously not linear because these are not all increasing by the same number (points at change in number of stones laid between day's 1 and 2, 2 and 3, 3 and 4, and 4 and 5)"

In this excerpt, Gina first used ordered pairs to coordinate the day number and the number of stones laid that day. She identified that she wanted to determine the slope between successive pairs of points. However, after determining the slope between the points (1,8) and (2,14), she noted that this rate of change was different than that rate of change between (2,14) and (3,22). Gina then extrapolated this to other pairs of points and concluded that the relationship cannot be linear because the rate of change is not constant between all the pairs of points.

In contrast, the other type of explanation explained why a statement was true or could be made. Four of the ten participants used this reasoning with one of the four utilizing it in multiple GEs. The following excerpt from Gina illustrates how this type of explanation appeared in her solving of this task.

> "There would be 22 plus 10 [stones laid] (draws line between 22 stones laid and where the number of stones laid would be on day 4 and writes +10 above it) because this is increasing by 6 (points at change in stones laid from day 1 to day 2), this is increasing by 8 (points at change in stones laid from day 2 to day 3), so 6, 8, 10, just increasing by 2 every time."

In this excerpt, Gina claimed there would be 32 stones laid on day four in this sequence. She then explained why, noting that the rate of change in the number of stones laid between days one and two is +6, and that the rate of change in the number of stones laid between days two and three is +8. She then extended this pattern to +10, arguing that the rate of change in the number of stones laid between two subsequent days is increasing by two each day.

Observations from Task One: The Patio Tile Task

After identifying the different types of justification, I explored the data for additional information. The results are presented below.

Separating the types of justifications by participant.

After categorizing and developing descriptions for the types of justifications on the Patio Tile task, I separated and analyzed them by participant, which is presented in the table below.

Table 38

Separation of the Types of Justifications on the Patio Tile Task by Participant

	Justification Type					
Participant	Verification as a Numerical Check	Verification as an Algebraic Check	Explanation For or Why	Explanation Against or Why Not	No Justification Given	**Totals**
Ashley	1	-------------	-------------	-------------	1	**2**
Brooke	3	-------------	-------------	1	-------------	**4**
Claire	2	-------------	-------------	1	-------------	**3**
Dane	2	-------------	1	-------------	1	**5**
Eli	1	2	-------------	-------------	2	**5**
Frank	1	-------------	2	1	3	**7**
Gina	1	-------------	1	2	-------------	**4**
Hailey	2	-------------	-------------	1	1	**4**
Ian	1	-------------	-------------	-------------	2	**3**
Jack	1	-------------	1	2	-------------	**4**
Totals	**15**	**2**	**5**	**8**	**10**	**40**

Looking at Table 38, we can observe that every participant utilized verification as a numerical check at least once. Additionally, seven of the ten participants utilized verification more frequently than explanation, assuming a justification was provided. If we consider the participants who utilized explanation more than verification (Frank, Gina, and Jack), we can observe that they each only utilized verification in one case, and utilized explanation in all others. This data indicates that participants appeared to trend towards using verification or explanation instead of a balanced combination of them, with verification being the dominant justification given.

Explanation of why not sometimes used to adjust a conjectured rule.

Participants sometimes used explanations as to why a statement was false to adjust their conjectured rules. To help illustrate what this adjustment looked like, consider the excerpt from Brooke given below.

"But look at these ones together that have the full rows (points to the top and bottom rows of tiles of day 1 figure), and put them together…and then add in these extra ones (points to middle row of tiles in day 1 figure)…So I'm thinking this [rule] is…$(D+1)(D+2)+(D+1)$…So then 2 plus 1, times 2, no (points at top and bottom rows of tiles in day 1 figure). So this is always going to be 2. (erases $(D+1)$ factor from first term in rule) So this [top and bottom rows of pattern] is going to be 2, so this is 2 (rewrites rule for pattern as $(D+1)(D+2)+(D+1)D$)…So here [case 2] I have $(2+1)(2+2)+(2+1)$. So 3 times 4, plus 3…(writes $12+3$). Okay, that [rule] doesn't work because this one [case 2] grows too this way [vertically]. So you have to have times D (writes $(D+1)(D+2)+(D+1)D$). So then 2 plus 1, times 2…no (points at top and bottom rows of tiles in day 1 figure) So this is always going to be 2. (erases $(D+1)$ factor from first term in rule) So this [top and bottom rows of pattern] is going to be 2, so this is 2 (rewrites rule for pattern as $2(D+2)+(D+1)D$)."

In this excerpt, Brooke first conjectured her explicit rule—count the number of stones in the top and bottom rows, determine the number of stones in the rectangular array formed by the middle rows (i.e., rows of stones between the top and bottom rows), and then relate this counting to the day number. She then symbolized her rule as $(D+1)(D+2)+(D+1)$. Next, she considered her rule in the context of day two and realized that the first term in her rule ($(D+1)(D+2)$) incorrectly counted the number of stones in the top and bottom rows. Although she identified the leading term of $D+1$ as incorrect and that it should be a factor of two instead, she incorrectly adjusted the symbolization of her rule. Next, she reconsidered her rule again in the context of day two. In doing so, she realized that the second term in her rule $(D + 1)$ was incorrect because the number of middle rows, the number of stones in each row, changed for each case. She noted that the number of middle rows were always the same as the day number, and adjusted the symbolization of the second term in her rule to be $(D + 1)D$. In considering her rule a third time in the context of day two, she realized that she did not change the symbolization of the first term in her rule, and changes the $D + 1$ factor to a 2, leaving her with a final rule of $2(D + 2) + (D + 1)$. Throughout this excerpt, Brooke repeatedly used why her rule was incorrect to make adjustments so her rule would correctly count the number of stones for each day.

Task Two: The Happy-Face Cutouts Task

Once GE summaries associated with the Happy-Face Cutouts task were identified, they were sorted into three major groups based upon the justifications that were (or were not) given—justification as verification, justification as explanation, and no justification given. A distribution of these three major groups is presented in Table 39 below.

Table 39

Distribution of the Types of Justifications from Task Two (The Happy-Face Cutouts Task)

Major Type of Justification	**Frequency (Relative Frequency)**
Justification as verification	8 (42%)
Justification as explanation	3 (16%)
No justification given	8 (42%)
Total	**19 (100%)**

There were 19 GEs from this task and none utilized justification in multiple ways. That is, no GEs were associated with multiple types of justification (e.g., justification as verification and justification as explanation). The data indicated that justification was provided in 58% of the GEs (11/19). A discussion of each major type of justification ensues below.

Justification as verification.

A GE summary was categorized as justification as verification when the justification verified or validated the conjectured rule. There were a total of eight instances of verification, all of the same type. The table below presents this type of verification along with the associated description and frequency.

Table 40

Frequencies and Descriptions for Justification as Verification

Justification as Verification/Description	Frequency
Determining that a conjectured rule is true or false by numerically substituting given cases in that result in a true or false statement (Verification as a Numerical Check)	8

Table 40 presents this type of verification, described as substituting in numerical values from particular cases and using the truth of this statement to determine the validity of the conjectured rule. Five of the ten participants utilized verification, with three of the five participants using it in multiple GEs. This type of verification was used exclusively to validate conjectured rules, or a part of a conjectured rule. The only variation that occurred within this type of verification was that participants substituted values into different cases to determine if their rule was correct. Some participants utilized only one of the given cases, some checked all three, and others utilized cases extrapolated from the pattern. The following excerpt from Ashley illustrates how verification appeared in her solving of this task.

> "So let's say that it's [the rule is] weeks squared plus weeks minus 1 squared (writes $w^2 + (w-1)^2$). Now let's see if it technically works out. So 5 squared is 25, plus 16 is 41 (checking the number of cutouts for week 5). So I think this should be my rule (circles $w^2 + (w-1)^2$)."

In this excerpt, Ashley started by stating her conjectured rule. She then checked her rule by substituting the week number into her rule to verify that she obtained the correct number of cutouts for that week. She then concluded that she believed her rule to be correct.

Justification as explanation.

A GE summary was categorized as justification as explanation when the justification explained and provided additional details or insights for the conjectured rule. There were a total of three instances of explanation, all of the same type. The table below presents this type of explanation along with its associated description and frequency.

Table 41

Frequencies and Descriptions for Justification as Explanation

Justification as Explanation/Description	Frequency
Providing insight into why a statement can be made or is true	3

Table 41 presents the frequency count and description for the one type of explanation given on task two. This explanation was described as explaining why a conjectured rule was true or could be made. Three of the ten participants drew upon explanation with none of these participants using it in multiple GEs. Of these three participants, two of them provided insight into why a statement can be made or is true by explaining why a symbolic rule fit or agreed with a pattern or trend observed in the task's pictures or images. That is, the explanations argued why a rule fit the figures given in the task. The following excerpt from Brooke illustrates how this subtype of explanation appeared in her solving of this task.

> "I was looking at, like, this cross (points to week 2 vertical center column and horizontal middle row) and this cross (points to week 3 vertical center column and horizontal middle row)…So I have my center one [cutout] and then I'm adding one [cutout] on each end of the cross (looking at week 2 figure in making this statement). So I have 1, and then I'm adding 4 times, I guess I'm just adding (writes $1 + 4$). You need 4 [times] n minus 1 (writes $1 + 4(n - 1)$) because the next one [week 3 figure] is 1 (points to center cutout), plus I'm adding 2 (points to 2 cutouts on cross above, below, left, and right). So 4 times $n - 1$ which is 2. So that takes care of the cross."

In this excerpt, Brooke began by identifying the "cross" in the week two and week three figures (see Figure 12 below).

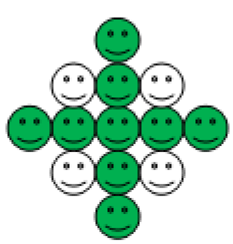

Figure 12. The "Cross" Referred to in the Figure.

Next, she identified that there will always be a center cutout in each cross, and that a cutout is added to the top, bottom, left, and right columns/rows that emanated from the center cutout. She then used this visual information to quantify and symbolize her counting as $1 + 4(n - 1)$. Brooke then related her symbolization to the figure, noting that the one corresponds to the center cutout, and that the $n - 1$ factor corresponds to the number of cutouts in the column/row other than the center cutout accounted for by the one. She then concluded that her symbolization accounted for the counting of the cutouts that fall along the cross.

Observations from Task Two: The Happy-Face Cutouts Task

After identifying the different justifications, I explored the data for additional information. The results are presented below.

Separating the types of justifications by participant.

After categorizing and developing descriptions for the types of justifications on the Happy-Face Cutouts task, I separated and analyzed them by participant, which is presented in the table below.

Table 42

Separation of the Types of Justifications on the Happy-Face Cutouts Task by Participant

	Justification Type			
Participant	Verification as a Numerical Check	Explanation For or Why	No Justification Given	**Totals**
Ashley	1	------------------------	1	**2**
Brooke	------------------------	1	1	**2**
Claire	2	1	----------------------	**3**
Dane	------------------------	------------------------	1	**1**
Eli	2	------------------------	1	**3**
Frank	1	------------------------	----------------------	**1**
Gina	------------------------	------------------------	1	**1**
Hailey	------------------------	------------------------	2	**2**
Ian	------------------------	1	1	**2**
Jack	2	------------------------	----------------------	**2**
Totals	**8**	**3**	**8**	**19**

Looking at Table 42, one can see that every participant provided at least one justification but no more than three. Assuming a justification was provided, five of the ten participants used verification in at least one case, with three of these five participants utilizing verification in two cases. This made verification the most common type of justification provided by the participants (eight total cases). In contrast, explanation was not as common (three cases), with three of the ten participants utilizing it. Overall, verification was the more common justification utilized, occurring nearly three times as frequently compared to explanation (8:3).

In comparing verification to explanation, five of the ten participants only provided or more frequently provided verifications, and two of the ten participants only provided or more frequently provided explanations. Overall, participants trended towards using justification as verification compared to justification as explanation (5:2).

Seven of the ten participants had at least one case where no justification was provided, with one of these seven participants having two cases where no justification was provided. Comparing these to the most common justification provided (i.e., justification as verification), there were more participants who had at least one case where no justification was provided (7/10) compared to participants who had at least one case of justification as verification (5/10). As a whole on this task, more participants did not provide a justification (seven participants), compared to participants who used justification as verification (five participants) or justification as explanation (three participants).

Task Three: The Star Sticker Task

Once GE summaries associated with the Star Sticker task were completed they were sorted into three categories of justification—justification as verification, justification as explanation, and no justification. A distribution of these different major groups is presented in Table 43 below.

Table 43

Distribution of the Types of Justifications from Task Three (The Star Sticker Task)

Major Type of Justification	Frequency (Relative Frequency)
Justification as verification	9 (33%)
Justification as explanation	8 (30%)
No justification given	10 (37%)
Total	**27 (100%)**

Although there were 22 GEs sorted, five of those GEs utilized justification in multiple ways. The five GEs associated with multiple justifications resulted in a total of 27 justification codes applied to the GEs. Based upon these codes, the data indicated that a justification was provided in 55% of the GEs (12/22). A discussion of each category of justification ensues below.

Justification as verification.

A GE summary was categorized as justification as verification when the justification verified or validated the conjectured rule. There were a total of nine instances where verification occurred. These instances were subdivided into two subcategories. The table below presents the distribution of these verification categories along with their associated description and frequencies.

Table 44

Frequencies and Descriptions for Justification as Verification

Justification as Verification/Description	Frequency
Determining that a conjectured rule is true or false by numerically substituting given cases in that result in a true or false statement (Verification as a Numerical Check)	8
Determining that a conjectured rule is true or false by applying the recursive, figural rule to a given case to see if it produces the subsequent case (Verification as a Figural Check)	1

Table 44 presents the two types of verification. The most common type was verification as a numerical check, which consisted of substituting in numerical values from particular cases and using the truth of this statement to determine the validity of the conjectured rule. Six of the ten participants utilized this type of verification with two of the six participants using it in multiple GEs. This type of verification was used exclusively to validate conjectured rules, or a part of a conjectured rule. The following excerpt from Claire illustrates how this justification appeared in her solving of this task.

"So for week 1 you have 1 times, 2 minus 1, so you get 1, and that works (writes $1(1) =$). [Week] 2 you have 2 times, 4 minus 1, 4 minus 1 is 3, you get 6, so that works (writes $2(3)$). And then for week 3 you have 3 times, 2 times 3, which is 6, minus 1, which is 5, and then that's 15 and that works (writes $3(5)$). Okay, so this is my rule"

In this excerpt, Claire had conjectured the rule $w(2w-1)$ just prior to these statements. She then substituted in the week number and checked to see if this substitution would result in the associated number of stars in the array. She proceeded to check this for week one, two, and three, concluding after each week that the rule worked. After checking all three weeks she stated that she was satisfied with $w(2w-1)$ being her rule for the task.

Justification as explanation.

A GE summary was categorized as justification as explanation when the justification explained or provided additional details or insights for a conjectured rule. There were a total of eight instances of explanation; these instances were subdivided into two subcategories. The table below presents the distribution of the different explanations along with their associated description and frequencies.

Table 45

Frequencies and Descriptions for Justification as Explanation

Justification as Explanation/Description	Frequency
Providing insight into why a statement cannot be made or is false	6
Providing insight into why a statement can be made or is true	2

Table 45 presents frequency counts and descriptions for the two types of explanations. The most common of these was explaining why a conjectured rule was false or could not be made. Five of the ten participants drew upon this reasoning with one of these five participants using it in multiple GEs. The following excerpt from Jack illustrates how this type of explanation appeared in his solving of this task.

> "No, that's not right because here you're only adding 2 (points to 2 rows of stars in rightmost column of week 2 figure). Here I'm adding 3 plus 3 (points to 3 rows of stars in 2 rightmost columns of week 3 figure)…with week 1 we start with 1 [star sticker]. In week 2, this is 2 squared, plus 2 (points at 2x2 array and rightmost column with 2 stars in it). This is 3 squared, plus 3 plus 3 (points at 3x3 array and 2 rightmost columns with 3 stars in each). That's 15 [star stickers]…It [rule] works for these 2 (points at week's 1 and 2), but it doesn't work for [week 3]."

In this excerpt, Jack had just conjectured the rule $x^2 + 2(x - 1)$ and was trying to determine if it was appropriate. After consulting the given figures in the task, he stated that the rule was not right. He then explained that in week two, there were two unaccounted stars in the right-most column—they were not counted by the two-by-two square formed by the two left-hand columns. Using similar reasoning for the week three figure, Jack commented that there are two columns each with three stars that are unaccounted for by the three-by-three square formed by the three left-hand columns in the rectangular array of star stickers. Jack then returned to his rule and noted that it accounted for the square array of stars and the remaining columns of stars for week's one and two, but that it failed to do so for week three. Synthesizing, Jack explained that the rule was incorrect because it failed to match and count the stars in the array for all three weeks, concluding it was not an appropriate rule for this task.

A subset of these explanations of why not provided insight into why a symbolic rule did not fit or agree with a pattern observed in the task's figures. Three of the five participants provided explanations of this nature. The previous excerpt from Jack was an example of this.

In contrast, the other type of explanation provided by participants explained why a statement was true. Only one of the ten participants used this reasoning, providing it in multiple GEs. The following excerpt from Gina illustrates how this type of explanation appeared in her solving of this task.

> "1 plus 5 (writes 1+5 = 6), then 6 plus 9 is 15 (writes 6+9 = 15), 15 plus 13 (writes 15+13 = 28). So the next one would be 28 + 17, because that [rate of change] is 4 more than 13. So that's 45, I think (writes 28+17 = 45)."

In this excerpt, Gina had identified the rate of change in the number of stickers between weeks one and two, and weeks two and three as +5 and +9. She had extended this pattern to +13 to determine that there would be 28 stars in the array for week four. Gina then extended this pattern again as +17 to determine that there would be 45 stars in the array associated with week five. She explained that this was an appropriate extension because 17 was four more than 13 (i.e., the second differences were a constant four).

Observations from Task Three: The Star Sticker Task

After identifying the different types of justification, I explored the data for additional information. The results are presented below.

Separating the types of justifications by participant.

After categorizing and developing descriptions for the different types of justifications for the Star Sticker task, I separated them by participant, which is presented in the table below.

Table 46

Separation of the Types of Justifications on the Star Sticker Task by Participant

	Justification Type					
Participant	Verification as a Numerical Check	Verification as an Figural Check	Explanation For or Why	Explanation Against or Why Not	No Justification Given	**Totals**
Ashley	-------------	-------------	-------------	1	------------	**1**
Brooke	2	-------------	-------------	1	------------	**3**
Claire	1	-------------	-------------	2	1	**4**
Dane	-------------	1	-------------	-------------	1	**2**
Eli	1	-------------	-------------	-------------	2	**3**
Frank	1	-------------	-------------	1	2	**4**
Gina	-------------	-------------	2	-------------	------------	**2**
Hailey	1	-------------	-------------	-------------	2	**3**
Ian	2	-------------	-------------	-------------	2	**4**
Jack	-------------	-------------	-------------	1	------------	**1**
Totals	**8**	**1**	**2**	**6**	**10**	**27**

Looking at Table 46, one can see that every participant provided at least one justification, but no more than three. Seven of the ten participants used verification in at least one case, making it the most common type of justification provided by the participants (nine cases), with two of these six participants utilizing verification in two cases. However, explanation was nearly as common (8 cases), with six of the ten participants utilizing it. Overall, although verification was the more common occurrence, it was only slightly more common than explanation (9:8).

In comparing verification to explanation, five of the ten participants only, or more frequently, provided verifications, and four of the ten participants only, or more frequently, provided explanations. Only one participant (Frank) had an even split (1:1) between verification and explanation in the justifications he provided. Overall, participants slightly trended towards using verification compared to explanation (5:4).

Six of the ten participants had at least one case where no justification was provided, with four of these six participants having two cases where no justification was provided. Although verification was the most common type of justification provided by participants (6/10), there were just as many participants who had at least one case where no justification was provided (6/10). As a whole on this task, participant’s use of verification, explanation, and no justification provided did not dominantly trend towards any one type.

Comparing and Synthesizing across the Types of Justifications on the Quadratic Tasks

After analyzing the three quadratic tasks individually, I constructed tables to look for trends across the tasks (e.g., what was the most common type of justification given across all quadratic tasks). The account below begins with frequency tables and then progresses to an observation from comparing and synthesizing across the descriptions for the justification categories.

Comparing and synthesizing justifications by task.

A table considering verification, explanation, or no justification for the tasks was analyzed for relationships and is presented below.

Table 47

Comparison of the Types of Justification across Tasks

	Task 1	Task 2	Task 3	Totals
Verification	17	8	9	**34**
Explanation	13	3	8	**24**
No Justification	10	8	10	**28**
Totals	**40**	**19**	**27**	**86**

Table 47 indicated that providing a justification accounted for 67% (58/86) of the cases, and not providing a justification accounted for 33% (28/86) of the cases. However, it must be noted that some of the generalization episodes were coded as both verification and explanation. Of the 58 justifications provided, 59% (34/58) were verification and 41% (24/58) were explanation. Verification had a just over 40% more codes than explanation (34:24).

Looking across the three quadratic tasks, verification and explanation were used almost equally in task three, with verification occurring once more (9:8). In contrast, the skew was towards verification in tasks one (17:13) and two (8:3). Overall across all three tasks, justification as verification was more common than justification as explanation (34:24).

Looking across all three of the quadratic tasks, there was little variation in the number of justification codes when no justification was provided (10:8:10). However, there was more variation in the number of justification codes for verification (17:8:9) and as explanation (13:3:8). Verification was used nearly an equal amount of times on tasks two and three (8:9); in contrast to task one where as it was used about twice as frequently (17:8 or 17:9). The use of explanation was also varied, differing by five cases between tasks two and three (3:8), and by five cases again between tasks three and one (13:3). Stated differently, the group of participants was inconsistent in utilizing verification and explanation, but was consistent in not providing a justification. Consider Table 48 below, which further decomposed verification and explanation.

Table 48

A Finer-Grained Comparison of the Types of Justifications across Tasks

	Task 1	**Task 2**	**Task 3**	**Totals**
Numerical Check (Verification)	15	8	8	**31**
Algebraic Check (Verification)	2	--------------	-------------	**2**
Figural Check (Verification)	--------------	--------------	1	**1**
For or Why (Explanation)	5	3	2	**10**
Against or Why Not (Explanation)	8	---------------	6	**14**
Totals	**30**	**11**	**17**	**58**

Looking across all three tasks, there were a total of 34 cases of verification, but 91% (31/34) were verification as a numerical check. The remaining three were divided between verification as an algebraic check (6%, 2/34) and verification as a figural check (3%, 1/34). Considering these three different types of verification, verification as a numerical check occurred on all three tasks, ranging in frequency from eight to 15, constituting between 47% (8/17) and 73% (8/11) of the total justifications for any one task. In contrast, verification as an algebraic check occurred only twice and only on the first task, and verification as a figural check occurred only once and only on the third task, composing no more than 7% (2/30) of the total verifications on the tasks. The participants' use of verification as a numerical check varied by task. Both tasks two and three each had eight cases identified, whereas there were 15 cases noted in task one, though this is in part due to the larger number justifications provided in task one. Overall, verification as a numerical check appeared to be the preferred verification, occurring in all three tasks, though its use did vary.

Looking across all three quadratic tasks, there were a total of 24 cases of explanation, with explanation of why constituting 42% of the cases (10/24) and the remaining 58% (14/24) being explanation of why not. Explanation of why occurred on all three tasks, whereas explanation of why not occurred only on task one and three. Additionally, explanation of why varied by no more than three cases in its use across tasks (5:3:2).

Comparing and synthesizing justifications by participant.

GE summaries were reorganized based upon the participant and the major types of justification. This information is presented in Table 49 below.

Table 49

Types of Justification Separated by Participant

	Verification	**Explanation**	**No Justification**	**Totals (Excluding No Justification)**
Ashley	2	1	2	**5 (3)**
Brooke	5	3	1	**9 (8)**
Claire	5	4	1	**10 (9)**
Dane	3	1	3	**7 (4)**
Eli	6	---------------	5	**11 (6)**
Frank	3	4	5	**12 (7)**
Gina	1	5	1	**7 (6)**
Hailey	3	1	5	**9 (4)**
Ian	3	1	5	**9 (4)**
Jack	3	4	---------------	**7 (7)**
Totals	**34**	**24**	**28**	**86 (58)**

Across all of the participants, verification was the most common type of justification identified (34) and explanation the least common (24). There were 28 cases identified as not providing a justification. Participants had at least five, but no more than 12, justification codes identified from their GEs. Of those codes, at least three, but no more than nine, were codes for either verification or explanation.

All participants utilized both verification and explanation, except for one (Eli) who utilized only verification. Participant's use of verification varied, with frequencies ranging from one to six, with an average of just under three and a half verifications per participant. The participant's use of explanation also varied, with frequencies ranging from zero to five, with an average just under two and half explanations per participant. Additionally, all participants had at least one case of not providing a justification, with the exception of one participant (Jack).

Although verification was the most common justification provided, not all participants trended towards utilizing it most frequently. Although seven of the ten participants (Ashley, Brooke, Claire, Dane, Eli, Hailey, and Ian) utilized verification more often than explanation (2:1, 5:3, 5:4, 3:1, 6:0, 3:1, and 3:1), the other three participants (Frank, Gina, and Jack) utilized explanation more frequently than verification (3:4, 1:5, and 3:4). Overall, participants appeared to favor the use of verification compared to explanation, though not unanimously.

When considering the participant's verifications or explanations given on a per task basis (see Tables 38, 42, and 46), participants often gave the same sorts of justifications, regardless of task. Seven of the ten participants (Ashley, Brooke, Claire, Dane, Eli, Hailey, and Ian) trended towards or exclusively gave verifications for each task. Of these seven participants, one (Eli) only gave verifications for each task, and six (Ashley, Brooke, Claire, Dane, Hailey and Ian) trended towards giving verifications for two of the three tasks. Two of the ten participants (Gina and Jack) trended towards giving explanations for two of the three tasks. One of the ten participants (Frank) was split between verification and explanation. He trended towards giving explanations for task one (1:3), verifications for task two (1:0), and was split in giving verifications and explanations in task three (1:1). As a whole, participants tended to give the same sort of justification independent of task, with the majority being verifications.

Comparing and synthesizing the category descriptions.

The descriptions for the different justifications (e.g., verification as a numerical check, explanation for why not) were the same, or nearly the same, when they were identified in the transcripts. Thus, comparing the different justifications within a particular class (i.e., justification as verification, justification as explanation) did not yield any additional information. However, this comparison across tasks did reveal an observation worth noting. Attention to the figures appeared as a subcategory of verification (task one), but also as a subcategory of explanation for why (task two) and explanation for why not (task three). That is, considering the figure in the justification occurred in both verification and explanation. However, the frequencies associated with these categories were low, ranging between one and three.

A Synthesized Framework for the Justifications on Quadratic Tasks

Considering the justifications identified on the three quadratic tasks presented above, a framework for the different justifications on quadratic geometric-numerical patterning tasks is now presented and described. This framework contains three major categories—justification as verification, justification as explanation, and not providing a justification. Within each category, a justification may be argued through one of three different lenses—numerical, algebraic (i.e., symbolic), and figural. The two sections below unpack and describe each major category as well as the lenses that may be contained within it.

Three categories of justification.

The first component of the justification framework identifies the three major categories of justification. One category is *justification as verification*. That is, determining whether a statement is true or false. The focus of this justification category is on a statement's validity. Another category is *justification as explanation*. That is, providing insight into why a statement is true or false. Justification as explanation is composed of two subcategories—explanation for or why, and explanation against or why not. An *explanation (for or why)* provides insight into why a statement can be made, is appropriate, or true. In contrast, an *explanation (against or why not)* provides insight into why a statement cannot be made, is not appropriate, or is not true. The focus of justification as explanation category is that of explaining or unpacking a statement so it is comprehendible. The third category is *not providing a justification*. That is, not making a statement (or statements) indicating whether a previously made statement (e.g., rule) is reasonable or not. It is important to note that these three categories do not have to be used discretely. Rather, they may be drawn upon in different steps in working towards developing a rule on quadratic patterning tasks.

Three potential lenses for viewing a justification through.

The second component of the justification framework identifies three different lenses that may be drawn upon in justifying a statement. One lens that may be drawn upon to argue a justification is a *numerical lens*. That is, arguing a justification through numerical values or quantities. Another lens that may be utilized to argue a justification is an *algebraic lens*. That is, arguing a justification through the symbolism employed or present. The third lens that may be drawn upon to argue a justification through is a *figural lens*. That is, arguing a justification through figures, pictures, or other visual images. It is important to note that these lenses may be used to justify, but are not essential requirements to justify. Additionally, similar to the three categories of justification above, these different lenses do not have to be used discretely. Rather, some justifications may draw upon multiple lenses in crafting them.

Research Question Two Summary

A taxonomy and descriptions for the different types of justifications given on the three quadratic geometric-numerical patterning tasks were presented, described, and illustrated with excerpts from transcripts. Trends within and between the justifications were searched for within each task individually. Following this, a comparison and synthesis across the tasks was presented and described, noting any trends observed within and between the justifications across all three tasks. Lastly, a framework for the justifications on quadratic geometric-numerical patterning tasks was given and described.

Research Question Three

Research question three investigated the types of rules in conjunction with the types of justifications for the quadratic geometric-numerical patterning tasks. Specifically, what patterns or relationships exist between the types of rules and justifications given? The GE summaries were sorted based upon the three dimensions of the generalization framework (i.e., the rule's construction category, the kind of rule, the characteristics appealed to) and the "justification categories" identified in the justifications framework (i.e., justification as verification, justification as explanation, no justification). The comparison between the constructs of generalization (i.e., rules) and justification begins with the "construction category" dimension of the rules framework compared to the justification categories from the justifications framework.

Comparing the "Construction Category" with the "Justification Category"

The rules and justification frameworks were applied back onto the data to identify the construction category (i.e., developed, attempted, not attempted) and the justification category (i.e., verification, explanation, no justification). The codes associated with each GE were then paired and placed into corresponding cells of a table. If two justification categories (i.e., verification, explanation) were associated with a GE, then each pair of codes (i.e., the construction category with a justification category) was placed into a corresponding cell within the table. Thus, a single GE was sometimes associated with two separate pairs of codes. To note, a single GE associated with both verification and explanation does not imply that there was an equal distribution of both in the GE, but just that both were present. The table below consolidated the pairs of codes from across the three quadratic tasks.

Table 50

Comparison of the Construction Categories to the Justification Categories across Tasks

	Developed	**Attempted**	**Not Attempted**	**Totals**
Verification	26	8	---------------------	**34**
Explanation	13	9	2	**24**
No Justification	11	8	9	**28**
Totals	**50**	**25**	**11**	**86**

Recall that developed rules were more common than attempted overall (50:25). Considering verification, there were roughly triple the number of developed versus attempted rules (26:8). However, this ratio varied by task. Task two had just under twice as many developed versus attempted rules associated with verification (5:3), whereas task one had nearly five times as many (15:3) (see Appendix H). In comparison, there were roughly 50% more developed versus attempted rules associated with explanation. Again, this varied by task. Task one had nearly twice as many developed versus attempted rules associated with explanation (7:4), yet tasks two and three were much more balanced (2:1 and 4:4) (see Appendix H). Regardless of the justification, it was more common for participants to develop rules versus attempt them.

Recall that verification was more common than explanation overall (34:24). For developed rules, verification was twice as common as explanation (26:13). This was the only trend that was observed within Table 50 that could also have been observed in each task's table individually (see Appendix H). For attempted rules, there was roughly an even split between verification and explanation (8:9). However, this balanced varied by task. Task three had half as much verification as explanation (2:4), but task two had three times as many verifications versus explanations (3:1) (see Appendix H). Regardless of the rule's construction category, verification was more common than explanation.

There was little variation between the number of rules developed, attempted, and not attempted associated with no justification (11:8:9). However, this balance varied by task. Task one had almost no variation (4:3:3), but task three had a wider range of variation (6:1:3) (see Appendix H). Overall, there did not appear to be a trend towards a particular construction category when no justification was provided.

Comparing the "Kind of Rule" with the "Justification Category"

The rules and justification frameworks were applied back onto the data to identify the kind of rule (i.e., explicit, recursive, hybrid) and the justification category (i.e., verification, explanation, no justification). The codes associated with each GE were paired and placed into corresponding cells of a table. In the proposed rules framework, if a rule was not attempted it was not paired with a justification category—the kind of rule only exists when a rule was attempted or developed (see proposed rules framework from Research Question 1). Additionally, if two justification categories (i.e., verification, explanation) were associated with a GE, then each pair of codes (i.e., the kind of rule with a justification category) was placed into an associated cell within the table. Thus, a single GE was sometimes associated with two separate pairs of codes. To note, a single GE associated with both verification and explanation does not imply that there was an equal distribution of both in the GE, but just that both were present. The table given below consolidated the pairs of codes from across the three quadratic tasks for the kind of rule and the justification categories.

Table 51

Comparison of the Kind of Rule to the Justification Categories across Tasks

	Explicit	Recursive	Hybrid	Totals
Verification	22	12	---------------------	**34**
Explanation	13	9	---------------------	**22**
No Justification	10	8	1	**19**
Totals	**45**	**29**	**1**	**75**

Explicit rules were about 50% more common than recursive overall (45:29). This trend was also observed in considering each task individually (see Appendix I). For verification, there were about twice as many explicit versus recursive rules (22:12). However, this varied by task. For task one, this ratio was about the same (11:6), but in task two there were triple the number of explicit versus recursive rules (6:2) and task three there was nearly an even balance (5:4) (see Appendix I). For explanation, there were about 50% more explicit than recursive rules (13:9). Again, this varied by task. Tasks one and two had nearly the same amount of explicit and recursive rules (6:5 and 2:1), but task three had approximately twice as many explicit versus recursive rules (5:3) (see Appendix I). Regardless of the justification, explicit rules were more common than recursive.

Although explicit rules dominated recursive for verification and explanation, they were fairly balanced when no justification was provided (10:8). This trend was also observed when considering each task individually (see Appendix I). Restated, explicit rules were most common only when a justification was provided.

Verification was about 50% more common than explanation overall (34:22). However, this varied by task. Although this ratio was roughly the same in task one

(17:11), verification was more than two and a half times as common as explanation in task two (8:3), and they were nearly the same in task three (9:8). For explicit rules, verification was almost twice as common as explanation (22:13). However, this varied by task. Although this ratio was nearly the same in task one (11:6), verification was three times as common as explanation in task two (6:2), and task three had an even balance between then (5:5) (see Appendix I). For recursive rules, verification was a third more common than explanation (12:9). However, considering each task individually appeared to indicate a fairly even balance between verification and explanation (6:5, 2:1, and 4:3) (see Appendix I), but aggregating these counts resulted in verification being a third more common than explanation. Regardless of the kind of rule, verification was more common than explanation.

Verification and explanation for recursive rules were fairly balanced for each task (6:5, 2:1, 4:3) (see Appendix I). This balance contrasted the skew towards verification over explanation for explicit rules in each task (11:6, 6:2, 5:5) (see Appendix I). Restated, the use of verification and explanation appeared to be fairly balanced for recursive but not explicit rules.

Separating this comparison by rules that were developed versus attempted.

The comparison between the "kind of rule" and the "justification category" from the rule and justification frameworks were then further separated based upon whether a rule was developed or attempted. The table below presents the pairs of codes from across the three quadratic tasks considering the kind of rule and the justification categories for rules that were developed, as well as for rules that were attempted.

Table 52

Comparison of the Kind of Rule to the Justification Categories across Tasks for Developed and Attempted Rules

	Explicit (Developed/ Attempted)		**Recursive (Developed/ Attempted)**		**Hybrid (Developed/ Attempted)**		**Totals (Developed/ Attempted)**	
Verification	18	4	8	4	---------	---------	**26**	**8**
Explanation	9	4	4	5	---------	---------	**13**	**9**
No Justification	6	4	4	4	1	---------	**11**	**8**
Totals	**33**	**12**	**16**	**13**	**1**	---------	**50**	**25**

Comparisons for developed rules.

There were about twice as many developed explicit rules versus recursive (33:16). For verification, developed explicit rules were about twice as common compared to recursive (18:8). For explanation, developed explicit rules were also about twice as common as recursive (9:4). Regardless of justification, developed explicit rules were more common than recursive.

For developed rules, verification was twice as common as explanation (26:13). Verification was also twice as common as explanation when considering developed explicit (18:9) and recursive (8:4) rules individually. Overall, verification was more common than explanation for developed rules.

For both trends observed, developed explicit rules associated with verification were the most common. Stated differently, participants appeared to encounter the most success with completing the tasks with developed explicit rules associated with verification. However, these observations did not appear when considering attempts to develop a rule.

Comparisons for attempted rules.

Attempted explicit rules were roughly as common as recursive (12:13). For verification, attempted explicit rules were as common as recursive (4:4). For explanation, attempted explicit rules were also nearly as common as recursive (4:5). Regardless of justification, attempted explicit rules were basically as common as recursive.

For attempted rules, verification was about as common as explanation (8:9). Verification was about as common as explanation when considering attempted explicit (4:4) and recursive (4:5) rules individually. Overall, verification was about as common as explanation for attempted rules.

Overall, attempted explicit and recursive rules associated with verification and explanation occurred with nearly the same frequency. Stated differently, participants did not tend to become stuck with any particular kind or rule and justification more often than another. However, comparing the distributions for the developed and attempted rules did yield a difference worth noting.

Comparing developed and attempted rules.

One difference observed was that the developed and attempted rules distributions were different for explicit and recursive rules associated with verification or explanation. Developed explicit rules were more common than attempted, regardless of justification category (18:4 for developed versus attempted associated with verification, 9:4 for developed versus attempted associated with explanation). Developed recursive rules were also more common than attempted associated with verification (8:4). Synthesizing, participants' rule development was more successful for explicit rules associated with verification or explanation, or recursive rules associated with verification. Participants appeared to have the greatest success with explicit rules associated with verification. Comparisons between the other developed and attempted categories did not seem to indicate any differences between the distributions.

Comparing the "Characteristics the Rule Appealed to" with the "Justification Category"

The rules and justification frameworks were applied back onto the data to identify the characteristics the rule appealed to (i.e., figural, numerical, symbolic) and the justification category (i.e., verification, explanation, no justification). The codes associated with each GE were then paired and placed into corresponding cells of a table. In the proposed rules framework, if a rule was not attempted it was not paired with a justification category—the characteristics the rule appealed to only existed when a rule was attempted or developed (see proposed rules framework from Research Question One). Additionally, if two justification categories (i.e., verification, explanation) were identified as associated with a GE, then each pair of codes (i.e., the kind of rule with a justification category) was placed into an associated cell within the table. Thus, a single GE was sometimes associated with two separate pairs of codes. To note, a single GE associated with both verification and explanation does not imply that there was an equal distribution of both in the GE, but just that both were present. The table given below consolidated the pairs of codes from across the three quadratic tasks considering the kind of rule and the justification categories.

Table 53

Comparison of the Characteristics the Rule Appealed to the Justification Categories across Tasks

	Figural	**Numerical**	**Symbolic**	**Totals**
Verification	20	13	1	**34**
Explanation	14	7	1	**22**
No Justification	12	6	1	**19**
Totals	**46**	**26**	**3**	**75**

One theme observed was the dominance of appealing to figural versus numerical characteristics—figural characteristics were over 75% more common than numerical (46:26). For verification, figural characteristics occurred 50% more often than numerical (20:13). For explanation and no justification, figural characteristics were twice as common as numerical (14:7 and 12:6). Regardless of the justification, appealing to figural characteristics was more common than numerical.

However, the observations noted above varied depending on the task, with the dominance of figural characteristics ranging from as high as 23 times more common than numerical on task three (23:1) to less than a quarter on task two (3:13) (see Appendix J). For verification, although figural characteristics were more common than numerical overall (20:13), the ratio between figural and numerical characteristics varied widely depending upon the task, ranging from 0:8 on task two to 9:0 on task three (see Appendix J). Recall that for explanation, figural characteristics were more common than numerical overall (14:7). However, in tasks one and two this ratio was roughly the same (6:4 and

1:2), but in task three there were seven times as many figural versus numerical characteristics appealed to (7:1) (see Appendix J). Recall that for no justification, figural characteristics were more common than numerical overall (12:6). However, in tasks one and two this ratio was roughly the same (3:3 and 2:3), but in task three there were many more figural versus numerical characteristics (7:0) (see Appendix J). Essentially, whether or not figural characteristics were the most common varied by task.

Another theme observed was the dominance of verification over explanation. Across all three tasks, verification was over 50% more common than explanation (34:22). For figural characteristics, verification was about 50% more common than explanation (20:14). For numerical characteristics, verification was about twice as common compared to explanation (13:7). Regardless of the characteristic a rule appealed to, verification was more common than explanation.

Although verification was more common than explanation overall, its commonality varied by task. For example, verification was more than two and a half times as common as explanation on task two (8:3), but nearly balanced on task three (9:8) (see Appendix J). For figural characteristics, verification was more common than explanation overall (20:14). However, the ratio between verification and explanation varied from almost double on task one (11:6) to nearly an even split on tasks two and three (0:1 and 9:7) (see Appendix J). For numerical characteristics, verification was more common than explanation (13:7). However, the ratio between verification and explanation varied from four times as great on task two (8:2) to nearly an even balance on tasks one and three (5:4 and 0:1) (see Appendix J). The dominance of verification over explanation varied by task.

Separating this comparison by rules that were developed versus attempted.

The comparison between the characteristics a rule appealed to (i.e., figural, numerical, symbolic) and the justification category (i.e., verification, explanation, no justification) were then separated further based upon whether a rule was developed or attempted. The table below presents the pairs of codes from across the three quadratic tasks considering the characteristics the rule appealed to and the justification categories for rules that were developed.

Table 54

Comparison of the Characteristics the Rule Appealed to the Justification Categories across Tasks for Developed and Attempted Rules

	Figural (Developed/ Attempted)		**Numerical (Developed/ Attempted)**		**Symbolic (Developed/ Attempted)**		**Totals (Developed/ Attempted)**	
Verification	18	2	7	6	1	--------	**26**	**8**
Explanation	10	4	3	4	--------	1	**13**	**9**
No Justification	9	3	2	4	--------	1	**11**	**8**
Totals	**37**	**9**	**12**	**14**	**1**	**2**	**50**	**25**

Comparisons for developed rules.

One trend observed for developed rules was that there were just over three times as many figural characteristics appealed to versus numerical overall (37:12). For verification, figural characteristics were appealed to roughly two and a half times more frequently than numerical characteristics (18:7). For explanation, figural characteristics were more than three times as common as numerical characteristics (10:3). Regardless of the justification for developed rules overall, appealing to figural characteristics was more common than numerical.

Another trend observed for developed rules was that there were twice as many verifications versus explanations given overall (26:13). For figural characteristics, there were just under twice as many associated with verification versus explanation (18:10). For numerical characteristics, verification was more than twice as common as explanation (7:3). Regardless of the characteristic appealed to for developed rules overall, verification was more common than explanation.

The most common pairing of rule characteristic and justification was figural and verification. Stated differently, participants appeared to encounter the most success with developed rules when appealing to figural characteristics and justifying through verification. However, the trends for developed rules did not appear when considering attempted rules.

Comparisons for attempted rules.

Recall that for developed rules there were just over three times as many figural characteristics attended to versus numerical overall (37:12). However, it was less common for participants to attend to figural characteristics in favor of numerical for attempted rules overall (9:14). For verification, only a third of the attempted rules appealed to figural characteristics in contrast to numerical (2:6). For explanation, there was an even split between the attempted rules that attended to figural or numerical characteristics (4:4). Although there were as many figural and numerical characteristics appealed to when explaining attempted rules, appealing to figural characteristics declined in favor of numerical when verifying attempted rules.

Recall that for developed rules where there were twice as many verifications versus explanations overall (26:13). However, there were roughly as many verifications and explanation for attempted rules overall (8:9). For figural characteristics, only half were associated with verification versus explanation (2:4). For numerical characteristics, verification was 50% more common than explanation (6:4). Essentially, figural characteristics as associated with verification declined in favor of explanation, but the reverse relationship occurred for numerical characteristics in favor of verification over explanation.

For both trends observed, figural characteristics associated with verification were the least common pairing of characteristics. Stated differently, participants had the fewest cases of attempting to develop a rule by considering figural characteristics and verification. However, comparing the distributions for the developed and attempted rules did yield a difference worth noting.

Comparing developed and attempted rules.

One difference observed was that the developed and attempted rules distributions were different. Developed rules that appealed to figural characteristics were more common than attempted, regardless of justification (18:2 and 10:4). Even when no justification was given, developed rules that appealed to figural characteristics were three times as common compared to attempted (9:3). Synthesizing, participants appeared to be more successful at developing rules when the rule appealed to figural characteristics, regardless of justification. Participants appeared to have the greatest success with figural characteristics and verification. Comparisons between the other developed and attempted categories did not seem to indicate any differences between the distributions.

Further decomposing this comparison of rules developed versus attempted by task.

The counts for the developed and attempted rules between the characteristics the rule appealed to and the justification category were further separated by task. The table below presents the comparison between the figural and numerical counts associated with the justification categories of verification, explanation, or no justification across the three quadratic tasks for developed rules.

Table 55

Comparison between the Figural and Numerical Characteristics the Rule Appealed to, Broken Down by Task and Justification Categories for Developed Rules

	Verification (Figural / Numerical)		**Explanation (Figural / Numerical)**		**No Justification (Figural / Numerical)**	
Task 1	11	2	6	1	3	1
Task 2	0	5	0	2	0	1
Task 3	7	0	4	0	6	0

Some tasks trended towards figural characteristics and others towards numerical. In task one, developed rules appealed more often to figural versus numerical characteristics, regardless of the justification category (11:2, 6:1, and 3:1). This trend was also apparent in task three, regardless of justification category (7:0, 4:0, 6:0 for each justification category respectively). However, figural characteristics were lesson common than numerical in task two, regardless of the justification category (0:5, 0:2, 0:1). Restated, there was a trend towards figural characteristics for tasks one and three and a trend towards numerical characteristics for task two.

The differences in the distributions of figural to numerical characteristics associated with verification, explanation, or no justification prompted a comparison of tasks one and three to task two (see Figure 13 below).

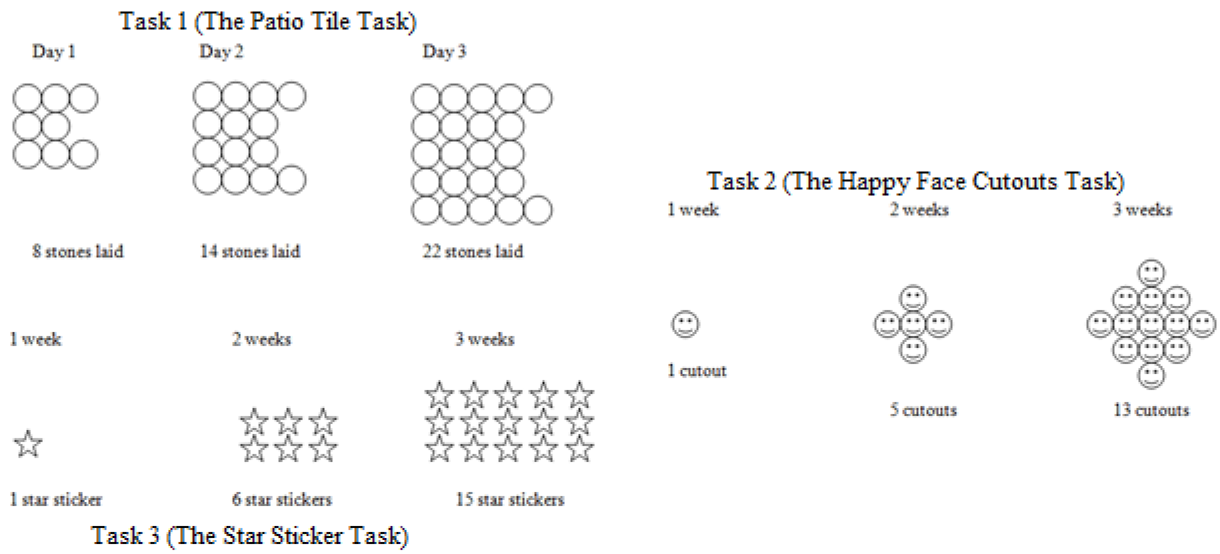

Figure 13. Comparing the Arrangements of Objects in Tasks One and Three.

One difference noticed was in the arrangement of the objects given in each task. The arrangements of objects on the left are from tasks one and three, which are in rectangular or rectangular-like arrays. That is, there are the same number of objects per row for some number of rows (with an additional tile added at the end of the top and bottom rows in task one). In contrast, the arrangement on the right is from task two and does not contain the same number of objects per row. Instead, the number of objects per row is constantly increasing or decreasing by two. Thus, one difference between tasks one and three and task two is in the arrangement of the objects. Although tasks one and three appeared to be different from task two, this difference did not appear when considering attempts to develop a rule, illustrated in the table below.

Table 56

Comparison between the Figural and Numerical Characteristics the Rule Appealed to, Broken Down by Task and Justification Categories for Attempted Rules

	Verification (Figural / Numerical)		**Explanation (Figural / Numerical)**		**No Justification (Figural / Numerical)**	
Task 1	0	3	0	3	0	2
Task 2	0	3	1	0	2	2
Task 3	2	0	3	1	1	0

When considering attempted rules, the sharp contrast between appealing to figural and numerical characteristics was not present. Regardless of justification, the difference between the number of figural and numerical characteristics was at most three. Essentially, although there were differences between the number of figural and numerical characteristics for attempted rules, they were not as sharp as the contrasts for developed rules, regardless of task.

One trend observed in comparing these two tables (Table 55 and 56) was that the distributions appeared to be different. For developed rules, tasks one and three often attended to figural characteristics, regardless of the justification. Additionally, task two more frequently attended to numerical characteristics, regardless of the justification. That is, tasks one and three were skewed towards figural characteristics and task two was skewed towards numerical. However, the distribution for attempted rules was semi-uniform across the justification categories, tasks, and characteristics the rule appealed to.

In short, when rules were developed the characteristics the rule attended to appeared to depend upon the task.

Research Question Three Summary

A comparison between the three dimensions of the rules framework and the "justification categories" from the justifications framework was presented and described for the quadratic geometric-numerical patterning tasks. The "kind of rule" and the "characteristics the rule appealed to" were decomposed and compared to the justification categories for developed and attempted rules. Additionally, a description was provided for the comparison between the justification categories associated with the characteristics the rule appealed to for developed and attempted rules on a per-task basis.

There were five major findings from this analysis. One, developed explicit rules were more common than recursive, regardless of the associated justification. However, attempted explicit and recursive rules were more balanced, regardless of the associated justification. Additionally, recursive rules were associated with a more balanced use of verification and explanation. Two, the use of verification dominated over explanation, though this did vary based upon task. Three, rules appealed to figural characteristics more often than numerical overall, though this did vary by task. Figural appeal was also more common than numerical when verifying, but more balanced when explaining. Four, participants appeared to encounter the most success (i.e., they developed rules) when appealing to figural characteristics and verifying. Five, tasks one and three appeared to be of a different variety than task two. A conclusion and discussion of this study's results now follows.

CHAPTER V

SUMMARY, CONCLUSIONS, AND DISCUSSION

Based upon the results presented in the previous chapter, conclusions were drawn to each research question. Additionally, findings were identified that could have implications for future research or practice. The following sections are organized by research questions, followed with limitations of the study. The chapter then concludes with directions for future research.

Findings, Conclusions, and Discussion of Research Question One

Research question one investigated the construct of generalization for quadratic geometric-numerical patterning tasks. Specifically, what types of rules were developed for quadratic patterning tasks presented in a geometric-numerical format? What patterns or relationships existed within or between the types of rules developed? A brief summary of the major findings is given, followed by conclusions and comments to the research question. A discussion with implications closes this section.

A Brief Summary of Major Findings

Developed rules were the most common construction category overall. The majority of participants developed or trended towards developing rules. However, the number of developed rules varied more between tasks than rules that were attempted or not attempted.

Explicit rules were the most common kind of rule overall. Nearly every participant developed or attempted at least two explicit rules across the three tasks. However, the number of explicit rules used varied more than recursive rules. Developed explicit rules were nearly always symbolized. As participants worked the tasks, 50% had cases where they identified a recursive pattern and then attempted to write this pattern as an explicit rule.

Developed explicit rules on task one and three primarily appealed to figural characteristics, and developed explicit rules on task two primarily appealed to numerical characteristics. The results for the rules on the three tasks culminated in a framework for generalization, discussed further below.

Conclusions and Comments

The types of rules identified on quadratic geometric-numerical patterning tasks were based upon the three dimensions of the generalization framework. The first dimension was the rule's construction category. The three categories of this dimension were developed rules, attempted rules, and not attempted rules. The second dimension was the kind of rule (to be) developed. The three categories of this dimension were explicit rules, recursive rules, and hybrid rules (i.e., a combination of explicit and recursive rules). The third dimension was the characteristics the rule appealed to. The three categories of this dimension were appealing to figural characteristics, numerical characteristics, or symbolic characteristics. If a rule's construction category was not attempted, then the other two dimensions did not apply.

The framework's three dimensions were developed by categorizing generalization episodes from ten participant's work on three different quadratic tasks. A framework designed from more than one quadratic task provides affordances and drawbacks. One drawback is that the framework's details are not as focused on the specifics from one task, but on the commonalities in thinking that occurred across tasks (e.g., some developed rules will be explicit, others will be recursive or hybrid). That is, the level of fine-grained detail is greater when developing a framework to a single task (e.g., there were developed explicit rules of variety A, B, and C) versus a framework that encompasses multiple tasks (e.g., there were different varieties of developed explicit rules). Thus, this framework operates as a regional framework (i.e., a framework developed from more than one task), in contrast to a local framework (i.e., a framework developed on a single task). The regional framework described may extend to other quadratic geometric-numerical patterning tasks similar to those utilized (i.e., objects are arranged into rectangular or rectangular-like arrays). This extension is an affordance of the developed framework—its use is not restricted to the task it was developed from. To note, the three local frameworks for each task can be observed in the research question one section of chapter four.

An additional caveat to this framework was that it was built from sorting 77 generalization episodes. Although this was a moderate number of GEs sorted, it is possible that additional categories may exist within a dimension (or possibly even another dimension) but were not identified due to the moderate number of GEs sorted. Thus, the proposed framework may not be in a finalized form. The analysis of student thinking on other quadratic geometric-numerical patterning tasks, and from additional pre-service teacher populations, is needed to help further establish the strength of the proposed framework.

The Framework's Relation to Existing Literature

The framework presented above built upon some ways of thinking present in the existing literature by coordinating them into different dimensions. Two of the categories from the kind of rule developed dimension were explicit and recursive rules. These two categories appeared similar to two ways of reasoning in the literature—explicit reasoning (i.e., directly relating co-varying quantities) (Lannin, 2003; Mason, 1996) and recursive reasoning (i.e., building upon a particular case in a sequence to determine subsequent cases in the sequence) (Lannin, 2003; Mason, 1996). Additionally, two of the categories from the characteristics a rule appealed to dimension were figural and numerical characteristics. These two categories appeared similar to two ways of reasoning in the literature—figural reasoning (i.e., using figures, diagrams, and other visuals to identify variant and invariant characteristic, properties, or structures in a set of objects) (Becker & Rivera, 2005; Chua & Hoyles, 2010; Mason 1996) and numerical reasoning (i.e., the use of numbers, quantities, or numerical cues to establish a rule) (Becker & Rivera, 2005, 2006; Chua & Hoyles, 2010; Healy & Hoyles, 1999). Based upon the description of the framework from the preceding section, a rule could utilize both explicit and figural reasoning in this framework, described as an explicit rule that appealed to figural characteristics. This example helps to illustrate the building upon and coordination of the ways of thinking present in the current literature for the generalization framework that emerged in this study.

Discussion and Implications

All of the developed explicit rules on task three and nearly all on task one appealed to figural characteristics. Moreover, these developed explicit rules seemed to be the same as Chua and Hoyles' (2010) additive (i.e., separating a figure into non-overlapping pieces that can be counted and used to develop a rule) and non-additive constructive generalizations (i.e., viewing a given figure as part of a larger figure and determining a rule based upon the larger figure with the sub-components of this figure removed that are not within the given figure). However, all of the explicit rules developed on task two appealed to numerical characteristics. Chua and Hoyles (2010) noted that the preservice secondary teachers in their study utilized numerical reasoning the least, which was in contrast to the preservice secondary teacher's developed explicit rules on task two in this study. Due to this contrast in appealing to figural and numerical characteristics, it appears that the task being utilized may influence the rule. Additionally, more research is needed to identify the characteristics of tasks that bring out numerical reasoning.

Upon comparing tasks one and three to task two further, I noticed that tasks one and three utilized rectangular or rectangular-like arrays of objects. That is, the number of objects in each row or column was always or nearly always the same. In contrast, task two presented an arrangement of objects that were not organized in this manner (see Figure 10). Rather, the number of objects in each row or column was increasing or decreasing by two (compared to the rows or columns next to it) for any given week. Although both types of tasks can be modeled with quadratic relationships, tasks one and three seem to more naturally appeal to figural aspects of the task due to the rectangular or rectangular-like arrays of objects, an arrangement which was absent in task two. Perhaps this absence was why the rules developed for task two appealed to numerical characteristics.

A mathematical difference also existed between tasks one and three and task two. Tasks one and three were based upon determining the product of two changing quantities (and adding a constant for task one). These tasks can be modeled as finite, *constant*, linear series (i.e., $\sum_{i=1}^{n} 2n - 1$ for task three). However, task two can be modeled as the sum of two finite, *non-constant*, linear series, one with length n (i.e., $\sum_{i=1}^{n} 2i - 1$) and the other $n - 1$ (i.e., $\sum_{i=1}^{n-1} 2i - 1$). Based upon this mathematical distinction, I seems that secondary preservice teachers may have a stronger association between quadratic relationships and multiplication than quadratic relationships and finite, non-constant linear series. Unpacked further, I think that secondary preservice teachers more frequently recognize the product of two non-constant, linearly changing factors as able to be modeled with a quadratic relationship versus a finite, non-constant linear series as able to be modeled with a quadratic relationship. This seemed reasonable not only based upon participant’s responses to these tasks, but also given that students encounter multiplication at a much earlier age than finite, non-constant linear series.

A concerning observation was that 50% of the participants (5/10) may have possessed the misconception of confounding a recursive rule with an explicit pattern.

Although two of the participants were able to work through this error to develop a rule, three did not. This implies that 30% of the participants had not reconciled this potential misconception after completing the quadratic tasks. This is a concerning number of preservice secondary teachers who may have possessed this misconception, especially given that the participants in this study were nearing the end of their student teaching and preparing to enter classrooms as full-time practicing teachers. It is important for a teacher to be able to distinguish between recursive patterns and explicit rules, as well as identify relationships between them. Additionally, it is unknown if the participants who worked through this confounding would be able to avoid this error in the future. Further research is needed to identify preservice secondary teacher's understanding of recursive patterns, explicit rules, and relationships between the two.

When comparing the descriptions between developed and attempted rules, there appeared to be a difference in focus. For developed rules, the focus appeared to be on figural or numerical characteristics in the tasks. In contrast, for attempted explicit rules, the focus appeared to be on determining if the relationship was linear, quadratic, exponential, or something else. It seems that having students shift their focus towards the figural or numerical aspects of the task may help them develop a rule to the task. Other researchers (e.g., Becker & Rivera, 2006; Lannin, 2005) have noted the importance of attending to figural aspects of patterning tasks as well. Perhaps the shift in focus from the type of relationship to how that relationship is manifest in the figures or numerical quantities is the obstacle a learner must overcome to develop a rule.

Findings, Conclusions, and Discussion of Research Question Two

Research question two investigated the construct of justification for quadratic geometric-numerical patterning tasks. Specifically, what types of justifications were given when solving quadratic patterning tasks presented in a geometric-numerical format? What patterns or relationships existed within or between the types of justifications given? A brief summary of the major findings is given, followed by conclusions and comments to the research question. A discussion with implications then closes this section.

A Brief Summary of Major Findings

Verification was the most common type of justification overall, utilized by each participant at least once. Verification as a numerical check constituted nearly all of the cases of verification. Although uncommon, verification as an algebraic check was a category of verification that stood out to me because I have not encountered it in any reviewed literature.

Explanation was less common than verification. Of the explanations provided, a subset did address figural characteristics of the tasks. Although explanation was less common than verification overall, the number of verifications and explanations given on the three tasks varied. Each participant also had at least one case where a justification was not provided. The justification results on the three quadratic tasks culminated in a framework for justification, discussed further below.

Conclusions and Comments

The types of justifications identified on quadratic geometric-numerical patterning tasks were based upon the three justification categories and lenses. The three justification categories were justification as verification, justification as explanation, and not providing a justification. Verification and explanation are also projected through one of three lenses. These three lenses are numerical, figural, or algebraic. If a justification was not provided then it could not be projected through a lens.

Unlike the independence between the different dimensions of the generalization framework, the justification categories and lenses are not independent of one another. Rather, one cannot talk about a justification without addressing the content of that justification (i.e., the lens the justification is projected through). Similarly, one cannot talk about the content of a justification without addressing the manner in which that justification operates (i.e., justification category). The justification categories and lenses are inextricably linked.

The Framework's Relation to Existing Literature

The framework presented above contained two justification categories that appeared to build upon some ways of thinking present in the existing literature. The first was verification, which appeared similar to using examples to justify (i.e., providing examples of specific cases as support for a statement, such as a rule) (Balacheff, 1988; Harel & Sowder, 2007; Lannin, 2005). In this study, participants often verified their rules by utilizing specific numerical values from particular cases in the sequence presented in the tasks. This was identified as verification as a numerical check in my analysis of the data, and captured in the framework as verification projected through a numerical lens. The second justification was that of explanation. Explanation appeared similar to a combination of using generic examples to justify (i.e., an example that describes the situation for a general, non-specified, case) (Lannin, 2005) and using contextualized information to justify (i.e., utilizing information from a given problem or task, such as figures and their characteristics, as reasoning behind why a statement is true) (Becker & Rivera, 2003, 2007; Lannin, 2003). In this study, participants sometimes explained how their conjectured rule fit each of the cases provided in the task, sometimes regarding the figures in the cases and other times the quantities associated with the cases. This way of reasoning was identified as explanation in this study. The explanation's basis upon the figures present in the cases or their associated quantities determined if it was an explanation projected through a figural or numerical lens. These two examples help illustrate building upon the ways of thinking present in the current literature for the justification framework that emerged in this study.

Discussion and Implications

Justification as verification was the most dominant category overall, as well as on each task individually. It was utilized by each participant at least once and nearly always came in the form of verification as a numerical check (i.e., determining the truth of a conjectured rule by numerically substituting given cases into the rule). I noticed that verification as a numerical check almost always occurred following a conjectured rule, often as one of the last statements made before a participant concluded that a rule had been developed to the task, or that he/she was not able to develop a rule to the task. In contrast, explanations sometimes followed a conjectured rule, but were also distributed throughout generalization episodes. The location of verifications and explanations in the generalization episodes seemed different.

Based upon this observation, I wondered if different justifications occur at different stages during the generalization process. Stated differently, where do different justifications most commonly appear? Are they distributed throughout the generalization process, or do certain justifications happen most frequently during different stages (i.e., justification as verification occurring most frequently at the end of the generalization process)? Future research is needed to investigate the distribution of the different types of justification identified in this study.

One uncommon category of verification was verification as an algebraic check (i.e., symbolically manipulating a conjectured rule such that it matches a previous rule assumed to be true). This verification stood out to me because I could not find any other studies that identified this reasoning. In comparing it to the proof literature, I noticed that it sounded similar to the transformational proof sub-scheme in the class of deductive proof schemes (Harel & Sowder, 2007). The transformational proof sub-scheme has three characteristics—generality (i.e., the argument must apply for every case, not just a subset of cases), logical inference (i.e., mathematical justifications must be based upon deduction), and operational thought (i.e., identifying goals and anticipating them when justifying). Although verification as an algebraic check satisfied the first characteristic (i.e., generality), evidence that the participant engaged in logical inference was questionable, and there was no evidence that the participant had engaged in operational thought. Thus, I claim that this study identified a previously undocumented type of justification. Verification as an algebraic check extended beyond the class of empirical proof schemes towards deductive proof schemes (Harel & Sowder, 2007), but did not possess all necessary characteristics of any one deductive proof scheme.

Some explanations from tasks two and three explained why a conjectured rule did or did not align with a pattern observable in the task's figures. That is, some explanations related a conjectured rule to the figures associated with that task. There were a total of five explanations from tasks two and three relating the rule to the figures. In contrast, there was only one verification associated with figures, which occurred in task three. Researchers (e.g., Becker & Rivera, 2006; Richardson, Berenson, & Staley, 2009) have commented on the importance of having students relate their symbolic rules to the figures being modeled. It seems that explanation may have provided a natural mechanism for the participants to relate their rules to the figures. Thus, having students provide explanations for their conjectured rules (in contrast to verifications) may help students develop relationships between the rules they conjecture and the figures from which their rules originate from.

Findings, Conclusions, and Discussion of Research Question Three

Research question three investigated the interaction between the constructs of generalization and justification for quadratic geometric-numerical patterning tasks. Specifically, what patterns or relationships exist between the types of rules developed and justifications given? A brief summary of the major findings is given, followed by conclusions and comments to the research question. A discussion with implications then closes this section.

A Brief Summary of Major Findings

Developed rules were most frequently associated with verification. For developed rules, justifications were most frequently associated with figural characteristics. In contrast, attempted rules were most frequently associated with appealing to numerical characteristics, with the use of verification and explanation evenly split in these attempts. Participants appeared to be most successful at developing a rule when the rule was justified through verification and appealed to figural characteristics.

Conclusions and Comments

Justification as verification was twice as common compared to justification as explanation for developed rules overall. However, the trend towards verification did not exist when considering attempted or not attempted rules. Instead, verification and explanation were evenly split for attempted and not attempted rules. Additionally, explicit rules were more frequently associated with verification versus recursive rules. This association was especially prevalent for developed explicit rules and verification. These findings seem to agree with other researchers (e.g., Kirwan, 2013; Rivera & Becker, 2003) who observed students often using examples to justify their rules.

Rules appealing to figural characteristics and justified by verification were 50% more common than those justified by explanation overall. Considering only developed rules, figural characteristics and verification were associated more than twice as often as figural characteristics and explanation. This strong association between figural characteristics and verification was surprising given that researchers (e.g., Rivera & Becker, 2003; Stacey, 1989) have noted that verifying rules is often based upon numerical arguments (e.g., proof-by-example). However, given that verification was most frequently associated with developed rules and the tasks contained more developed than attempted rules, perhaps this is why there was the more frequent association between figural characteristics and justification as verification.

Discussion and Implications

Although this study identified relationships between the constructs of generalization and justification, what is still unclear is *how* or *why* these constructs were related. For the relationships given in the conclusions and comments section, why did verification have strong ties to developed rules? How and why was verification related to rules that appealed to figural characteristics? Future research of the relationships identified is needed to unveil the interaction between the constructs of generalization and justification.

Another relationship between the constructs of generalization and justification was that the characteristics a rule appealed to (i.e., figural, numerical, symbolic) also occurred in the lenses a justification could be projected through (i.e., figural, numerical, algebraic). This seemed curious to me, and I wondered why this same theme occurred in both the generalization and justification frameworks. Is this commonality a medium that can be utilized to transition between generalization and justification? What kinds of interactions occurred between the rule and its justification regarding this common occurrence? Future research is needed to unpack this potential relationship.

Distributions for the justification categories and associated characteristics a developed rule appealed to were similar on tasks one and three, but appeared different for task two (see Table 55). The ratios of figural to numerical characteristics appealed to that were associated with verification (11:2 and 7:0), explanation (6:1 and 4:0), or no justification (3:1 and 6:0) were similar on tasks one and three, but these ratios seemed to be reversed when considering task two (0:5, 0:2, and 0:1 for verification, explanation, and no justification). Based upon these differences in the distributions, I wondered if tasks one and three may be of a different variety than task two. A similar question regarding the influence of the task was raised when discussing research question one above.

Uncommon Ways of Reasoning about Generalization and Justification

One of the major goals of this study was the frameworks that emerged from categorizing and synthesizing across the categorizations regarding the common rules and justifications given. A consequence of this analysis is that it did not adequately capture ways of reasoning about generalization and justification that were uncommon. There were two uncommon ways of reasoning that are worthwhile discussing because they did not exist in any of the literature reviewed. The first uncommon way of reasoning discussed is the hybrid rule developed by Ian on task three, followed by verification as an algebraic check by Eli on task one.

Uncommon Reasoning One: Hybrid Rule

Recall that there was one hybrid rule given by one participant (Ian) on task three (The Star Sticker Task). This rule counted the number of rows in each array by directly coordinating the rows with the week number, but counted the number of columns by referencing the number of columns from the previous week and increased that value by two. It was the product of these two factors (i.e., number of rows and columns) that constituted Ian's rule to the task. This rule was unique because a) it was provided by only one participant on a single task, and b) it combined explicit and recursive components to form a single rule that utilized both ways of reasoning. This rule is worth noting because it provides evidence that not all students may view explicit and recursive reasoning as mutually exclusive. Additionally, it contributes to the literature because rules that utilized both explicit and recursive reasoning concurrently when generalizing rules from patterns was not identified in the literature reviewed.

Uncommon Reasoning Two: Verification as an Algebraic Check

Recall that there were two GEs counted where verification as an algebraic check was given. Both GEs were associated with task one and the same participant, Eli. Verification as an algebraic check was described as symbolically manipulating a conjectured rule such that it matches a previous rule assumed to be true. This justification was unique because a) it occurred only twice, both times by the same participant on the same task, and b) it was the only justification given in this study that was projected through a symbolic lens. This justification is worth noting because it provides evidence that some students' justifications may be based upon the semiotic nature of the argument. That is, the justification is grounded in the symbols, rather than the referents of what those symbols are associated with. Additionally, this justification contributes to the literature because verification as an algebraic check was not observed in the literature reviewed.

Significance of this Study

This study was significant for two primary reasons. I illustrated that a variety of ways of reasoning about generalization exist through my review of the literature (e.g., explicit reasoning, figural reasoning). However, few studies have coordinated these different ways of thinking. The generalization framework that emerged from the data in this study coordinated different ways of thinking about generalization. This framework contributes to the literature by organizing different ways of thinking previously identified into a more cohesive framework. For example, explicit and recursive reasoning, as well as figural and numerical reasoning, have been identified as ways of reasoning about generalization in the literature, but were not previously related. This framework coordinated these was of reasoning as different dimensions. Thus, a participant might use both explicit and figural reasoning when generalizing, which would be identified as an explicit rule that appealed to figural characteristics in the framework from this study. Moreover, the framework presented was specific to that of quadratic geometric-numerical patterning tasks (see Chapter 1 for details and clarification of quadratic patterning tasks), which are limited in the literature (Kieran, 2007; Vaiyavutjamai & Clements, 2006).

In my rationale I argued that generalization and justification were related (Ellis, 2007a; Lannin, 2005; Radford, 1996), yet few studies have invested their interaction (e.g., Richardson, Berenson, & Staley, 2009). This study was significant because it coordinated the consolidated ways of thinking from the literature in the generalization framework with associated ways of thinking that emerged in the justification framework. That is, this study identified potential relationships between ways of thinking about generalization and justification. For example, participants most often developed rules that appealed to figural characteristics and justified them by verification, with many verifications projected through a numerical lens. Thus, the use of figural characteristics to develop rules and numerical characteristics to verify the rule may be related. Again, these potential relationships between generalization and justification were specific to that of quadratic geometric-numerical patterning tasks (see Chapter 1 for details and clarification of quadratic patterning tasks), which are limited in the literature (Kieran, 2007; Vaiyavutjamai & Clements, 2006).

Recommendations

Based upon the analysis and findings from this study, I would provide the following recommendations. Preservice secondary teachers should have more exposure to and working with geometric-numerical patterning tasks. Specifically, more work with non-linear and non-exponential patterning tasks, such as quadratic patterning tasks. Geometric-numerical patterning tasks may be a type of task that can help students transition from arithmetic to algebra (Lee, 1996; Blanton & Kaput, 2011), so it is important for preservice teachers to have opportunities to become familiar with these types of tasks and develop a deep understanding of the content relative to them. Work with quadratic patterning tasks helps improves preservice teachers content knowledge, as well as raising preservice teachers awareness of these types of tasks should they choose to draw upon them pedagogically.

When preservice secondary teachers are working with quadratic patterning tasks during preparation coursework, they should not develop just one rule to the task, but as many rules as they can. Having preservice secondary teachers exhaust all of the ways they may think about quadratic patterning tasks informs both the teacher and student about the flexibility in thinking about this type of task and content contained within the task. For example, if a student quickly develops an initial rule to a task, but struggles to develop a second rule, then this may inform the student that perhaps he/she knows an algorithm to determine. Similarly, this information may inform a teacher that selecting a patterning task that draws upon non-algorithmic thinking may be useful.

More exposure to and work with justification regarding its role in geometric-numerical patterning tasks may also be useful. Researchers have argued that students possess a limited understanding of justification (e.g., Harel & Sowder, 2007), utilizing insufficient methods for establishing a claim, such as the use of specific examples to justify (Balacheff, 1988; Becker & Rivera, 2007; Lannin, 2003). This study was no exception to that trend, with verification (i.e., determining the truth or validity of a statement) being the most common justification provided. Having students relate their rules to the figures or context of the problem may help establish that a justification needs to more than verify the truth of a claim, it must also explain and unpack why that claim is true. Such a recommendation has also been made by other researchers (e.g., Becker & Rivera, 2006; Richardson, Berenson, & Staley, 2009).

Limitations

This study was conducted from the interpretivist paradigm with the purpose of developing a deeper understanding of the phenomenon (i.e., generalization and justification). One trademark of operating under this paradigm was that the focus was not on extending the developed understanding to other samples or populations. Rather, the focus was on developing that understanding of the phenomenon within the parameters of the study. However, although the desire to generalize findings is not relevant under the intrepretivist paradigm, stating that generalization of a study's findings is not relevant does not make this desire to generalize go away. Thus, the limitations that follow are given as cautions in generalizing this study's findings.

Researchers have commented that the degree to which the findings from qualitative research studies can be extended (i.e., generalized) is commonly a concern (Glesne, 2011; Miles, Huberman, & Saldana, 2014). This study is no exception to that statement. The participants in this study were non-randomly chosen and came from a single teacher preparation program during the 2013-2014 academic year. Additionally, this study sought to *describe* participant's understanding of generalization, justification, and relationships between the two. Although the findings characterize the sample, to what degree these findings may be extended to other samples or populations is a debatable question.

Another limitation of this study is that the frameworks produced are developed from using only geometric-numerical (Radford, 1996) patterning tasks. Thus, the frameworks developed for generalization and justification strategies only apply to these types of tasks. Although desirable to create frameworks to encompass other types of tasks (e.g., writing a quadratic that passes through three given points), this is beyond the scope of this study.

Another limitation of this study was that it was conceptualized with generalization as a product/object, rather than a process/action. As a consequence of this conceptualization, only the rules (i.e., generalizations) were focused upon in the development of the generalization framework. Thus, the generalization framework proposed extends only to the view of generalization as a product/object. However, the process of how those rules were developed is an important component of generalization as well. Again, this is beyond the scope of this study. Future research is needed to understand generalization as an activity or process on quadratic geometric-numerical patterning tasks.

Directions for Future Research

Many studies have been conducted regarding generalization of linear relationships. However, it is unclear whether the relationship being generalized (e.g., linear, exponential, quadratic) influences the generalization. That is, do different types of generalizations occur on linear and quadratic tasks? Ellis (2004) noted that generalization with linear relationships may not extend to other contexts. One direction for future research is to compare the generalizations made on quadratic geometric-numerical patterning tasks to those on linear tasks.

To begin addressing this, data was collected and analyzed on a linear task utilizing the same methodology described in chapter three. The preliminary results of this analysis seemed to indicate that there were some similarities and differences between the linear and quadratic tasks. One difference was that 90% (27/30) of the GEs contained a developed rule, with 81% (22/27) being developed explicit rules and the other 19% (5/27) being developed recursive rules. That is, rules were nearly always developed to linear tasks, with the majority being explicit rules. This was in contrast to the results in this study which had larger numbers of attempt and not attempted rules. However, the developed rules on the linear task appeared to consider figural characteristics more frequently than numerical, a similarity to the findings in this study. Based upon these observations, there may be differences and similarities between the generalizations on linear and quadratic tasks. Further research is needed to determine if linear and quadratic tasks elicit different types of generalizations, if only a subset of some linear tasks do this, or if this initial observed difference between the linear and quadratic tasks was due to random chance.

Another direction for future research would be to consider each individual participant's progression as he/she worked through the three tasks in this study. The purpose would be to look for patterns and trends regarding the rules and justifications given. Stated differently, treat the individual participant as the unit of analysis, and considering the string of generalization episodes coded by the frameworks that emerged in this study regarding generalization and justification. This could allow for investigating questions about the reasoning individual participants utilized. For example, do participants utilize different types of rules on the same task? On different tasks? Do the associated justifications vary? How do the generalizations and justifications vary per participant? Considering the string of generalization episodes for each participant allows for the exploration of each participant's experience with these tasks.

Another direction for future research would be the "correctness" that is associated with given generalizations and justifications. This study helped to describe aspects of generalization and justification. However, depending upon if the rules or justifications are correct or incorrect, what a teacher may want to promote in their classroom could be influenced. Thus, the correctness of a generalization or justification needs to be addressed. Moreover, the issue of correctness is noticeably absent in the literature regarding generalization. Correctness does appear to be at least semi-present in the literature on justification, but seems to be convoluted with different tiers of sophistication with regards to justification. Research regarding how correctness is related to generalization and justification is needed.

Despite this seemingly simple question, determining correctness is challenging because of the need to be explicit about what is meant by correct. For example, does correctness mean a solution identified as correct or incorrect by the teacher? That is, does an authority determine what solutions are correct and incorrect? Does correct mean that there is a consistent and logical flow to the solution given, even if it did not result in the desired outcome as identified by the teacher, such as in the case of a computational error? These two examples highlight different facets that make correctness a complex construct. Research regarding the multifaceted construct of correctness with regards to generalization and justification is needed to inform teachers of which ways of thinking they may want to promote, depending upon what facets of correctness are identified as desirable.

Concluding Remarks

This study identified that secondary preservice teachers were able to generalize and justify on geometric-numerical patterning tasks in a variety of ways. These generalizations and justifications were captured in frameworks that emerged from the data. The data also indicated that secondary PSTs encountered the most success generalizing when appealing to figural characteristics and verifying the generalization made. Future research is needed to explore the role of the relationship in generalizing and justifying, as well as the "correctness" of the generalization made.

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APPENDIX A

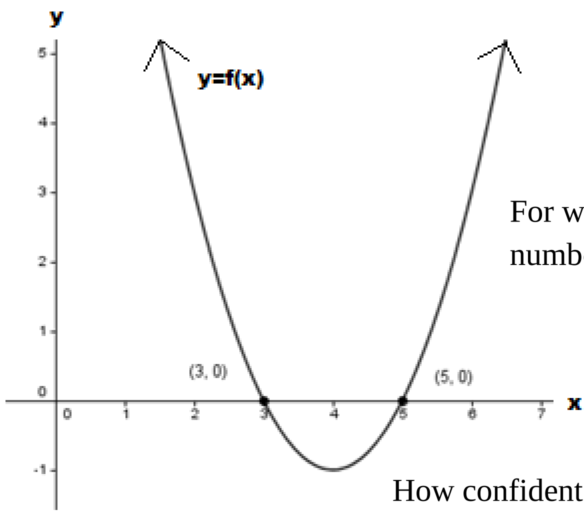
PARTICIPANT SELECTION SURVEY

<u>Directions:</u> For each of the questions you should work out your answer in the space beneath the question. Then write your answer in the column marked "Answer." For every question, you must mark an X in one column to show how you feel about your answer.

Directions: For each of the following equations or inequalities, state all real-number value(s) of *x* which make the statement true.	Answer	I'm certain I'm right	I think I'm right	I've got a 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
Example: $x - 1 = 2$	3	X				
1. $x^2 = 9$						
2. $\frac{1}{x^2} = 16$						

3. $x = \frac{4}{x}$						
4. $x^2 + 6 = 0$						
5. $x > \frac{16}{x}$						
6. $x^2 + 2 > 0$						

7.

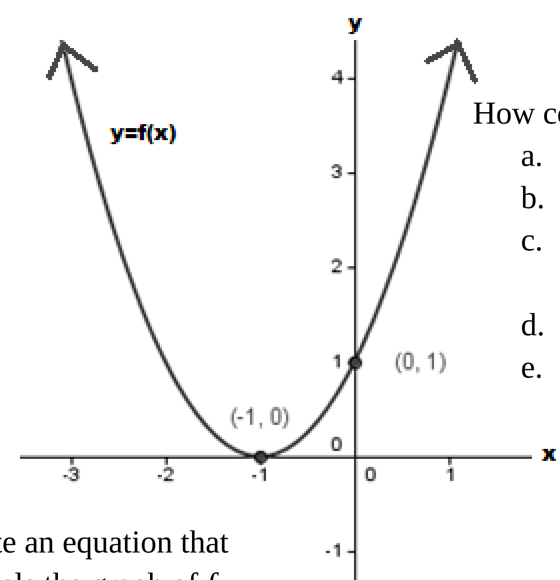


For what real-valued number(s) of x is $f(x) > 0$?

How confident are you in your solution?

a. I'm certain I'm right
b. I think I'm right
c. I've got a 50-50 chance of being right
d. I think I'm wrong
e. I'm certain I'm wrong

8.



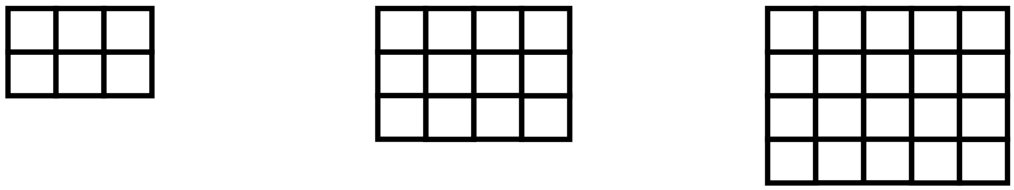
How confident are you in your solution?

a. I'm certain I'm right
b. I think I'm right
c. I've got a 50-50 chance of being right
d. I think I'm wrong
e. I'm certain I'm wrong

Write an equation that models the graph of *f*.

9. The first three steps of a pattern built from shower tiles are shown below. Determine and write a rule that relates the **step number** to the **total number of shower tiles** required to construct the pattern. Please explain why your rule is correct.

Step 1 Step 2 Step 3



6 shower tiles 12 shower tiles 20 shower tiles

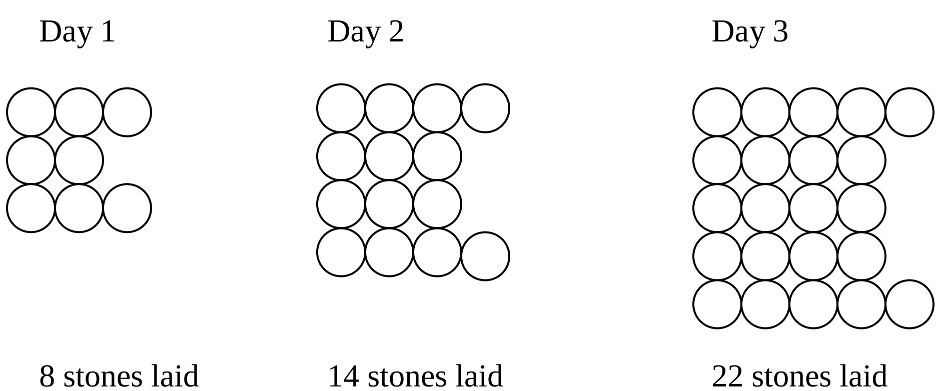
10. After you write and explain your rule in question 9, complete the task a second time to write and explain a DIFFERENT rule that relates the **step number** to the **total number of shower tiles** required to construct the pattern.

APPENDIX B

INTERVIEW TASKS

The Patio Tile Task

You are laying circular stones for a patio you are building. You take a picture of the patio each day to capture your progress, which is shown below. Determine and write a rule that relates the **day number** to the **total number of stones laid** in your patio for that day. Please explain why your rule is correct. Solve this task in as many ways as you can.



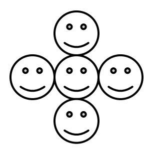
The Happy-Face Cutouts Task

To help positively reinforce good homework completion habits, a teacher puts happy-face cutouts on her classroom's bulletin board in the arrangement shown below. The teacher adds more cutouts to the board for each week all of the students complete all homework assignments. Determine and write a rule that relates the **number of weeks** the students have completed all homework assignments to the **total number of cutouts** required to construct that pattern. Please explain why your rule is correct. Solve this task in as many ways as you can.

1 week

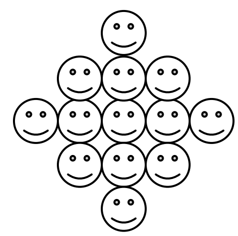
1 cutout

2 weeks



5 cutouts

3 weeks



13 cutouts

The Star Sticker Task

Sophia has been collecting star-shaped stickers. At the end of each week, she arranges all of the stickers she has collected, forming the pattern shown below. Determine and write a rule that relates the **number of weeks** Sophia has been collecting stickers to the **total number of star stickers** she has collected for that week. Please explain why your rule is correct. Solve this task in as many ways as you can.

1 week | 2 weeks | 3 weeks

1 star sticker | 6 star stickers | 15 star stickers

APPENDIX C

THINK-OUT-LOUD TASK PROTCOL

1. Thank the participant for being willing to participate in this study. Inform the participant that they will be working patterning tasks where they will *create a rule for the pattern and explain why their rule is correct*. Inform the participant that they will repeat this process until they cannot create any new rules. Inform the participant that I am not looking for a specific answer, but am interested in how they create their rules and how they explain why the rules fit the pattern. **Inform the participant to be sure to verbalizing their thinking as they work the task.** Inform the participant that as they are working the tasks in this interview, there will be times when I interpret their work to make sure I understand their thinking. Inform the participant that these interpretations do not support nor dismiss the correctness of their work. Emphasize that it is equally as important for the participant to disconfirm my interpretations as it is to confirm them.
2. Have the participant read the task's directions out loud. Ask the participant to explain what the task is asking them to do in their own words. If he/she understands the problem's directions, proceed with solving the task. If the participant misinterprets the task's directions, provide additional clarification of the task's directions.
3. As the participant works on the task they should be verbalizing their thinking. If more than 30 seconds passes without any verbalization, ask the participant what they are thinking.
4. After the participant give a rule for the pattern and they have explained how they know that their rule is correct, proceed with the post-task interview protocol. Once this protocol has been completed, ask the participant if they can create a different rule for the pattern. If they say they cannot, proceed with the next task.
5. If the participant is unable to develop a rule for the task, proceed with the next task in the sequence. If the participant cannot create a rule for the three tasks or they appear frustrated, provide them with an easier task (i.e., The Box Problem). Once they have developed a rule and explained how they know their rule is correct, proceed with the post-task interview protocol and then end the interview.

APPENDIX D

POST-TASK INTERVIEW PROTOCOL

1. Please describe what your rule means.
2. What influenced you in developing your rule for this task? Explain.
    a. If participant indicates "picture", probe…what about the picture?
        i. *Potential related strategies*: visually (de)composing figures
    b. If participant mentions "numerical values", probe…what about the numerical values?
        i. *Potential related strategies*: recursive reasoning, explicit reasoning, proportional reasoning (with adjustments)
3. On a scale of 1 to 5, how confident are you that your rule is correct? Why?
4. Please explain why your rule is correct.
5. What influenced you to give that explanation for why your rule is correct?
    a. *Anticipated responses*: it works for certain cases (i.e., proof by example), problem context, was able to modeling the problem with a generic case, inductive argument
6. On a scale of 1 to 5, how confident are you in *why* your rule is correct? Explain.
7. (**At the completion of the interview**) Do you have any questions for me?

APPENDIX E

GENERALIZATION EPISODE SUMMARY SHEET

GE:		
Categorization	**Notes/Brief Description**	**FOCUS**
RULE:		
Evidence:		
JUSTIFICATION 1:		
Evidence:		
JUSTIFICATION 2:		
Evidence:		

APPENDIX F

DESCRIPTIONS FOR TYPES OF RULES ACROSS TASKS

Explicit Rules Developed

Task 1	**Task 2**	**Task 3**
Count stones in the top and bottom row, then add stones counted in the middle rectangular array. Symbolized as $2(d + 2+d(d+1)$	Count the number of cutouts in the array by squaring the week number and adding the remaining portion, realizing that the remaining portion is a perfect square of one less than the week number. Always symbolized as $w^2 + (w-1)^2$	Determine the number of stars in the array by multiply the dimensions of the array. The number of rows and columns is directly related to the week number. Always symbolized as $w \times (2w - 1)$ or $w \times (w + w - 1)$
Count stones assuming there is a full square array, then remove stones not present in right column. Symbolized as $(d+2)^2 - d$	Count the number of cutouts in the array by adding one less than the week number of multiples of four to the initial one cutout from week one to determine the number of cutouts in the array for that week. Symbolized as $1 + 4(w-1)$	Count the number of stars in the left columns that for a *n* x *n* square, and then count the number of stars in the rectangular array formed by the remaining columns. Always symbolized as $w^2 + w(w-1)$
Count stones in rectangular array formed by full left columns, then add remaining two stones in right column. Symbolized as $(d+1)(d+2)+2$		Count the number of stars as if the rectangular array was a full square where the height is a great as the width, and then subtract the overcounted (i.e., missing) rows at the top of the array. Always symbolized as $(2w - 12-w-1(2w-1)$

Count stones in the middle rows, then add in stones in the top and bottom rows, assuming the 6 stones from day 1 are always there. Symbolized as $d^2 + d + 6 + 2(d - 1)$		
Count stones in square array in upper-left corner of figure, then count *n* stones below square array, then count the remaining three stones.		
Simplify $(d + 2)^2 - d$ to standard form.		
Model part of the stones laid with the exponential 2^{d+1}, and then correct for the number of stones not counted with a quadratic that passes through the points (1,4), (2,6), and (3,6). Partially symbolized as 2^{d+1}		

Recursive Rules Developed

Task 1	**Task 2**	**Task 3**
Add a stone to the end of the top row and each middle row. Then add a new bottom row that has as many stones as the new top row. Most often symbolized as (e.g., $d_1 = 8; d_n = d_{n-1} + 2n + 2$)	Given the number of cutouts in an array for a particular week, add on four times the week number associated with that array to determine the number of cutouts for the subsequent week. Symbolized in one of the two cases as $c_1 = 1, c_n = c_{n-1} + 4(n-1)$	Add two columns of stars to the right of the array where each column height is as great as any other column, and then add a row of stars across the top of all columns. This rule was never symbolized.
Identify the current number of stones laid on day *n*. Determine how many stones were added to day *n-1* to give the number of stones laid on day *n*, and increase that number of stones added by 2, then add that quantity to the number of stones laid on day *n*. Most often symbolized as (e.g., $d_n = d_{n-1} + 6 + 2^{n-2}$)	Assuming the pattern is a square array with an extra cutout above/below the middle column and to the left/right of the middle row, to determine the number of cutouts for a week (i.e., week *n*), one takes the number of cutouts on the previous week (i.e., week $n-1$) and increases this amount by 2^n. Symbolized as $a_{n-1} + 2^n$	Add a row of stars across the top of all columns, and then add two columns of stars to the right of the array whose height is a great as any other column. This rule was symbolized once as $a_n = a_{n-1} + 3(n-1) + n$
Add a stone to the end of the top and bottom rows, as well as each middle row. Then add a new middle row with as many stones as each new middle row has. Not symbolized.		

Hybrid Explicit/Recursive Rules Developed

Task 1	Task 2	Task 3
None	None	Determine the number of stars in the array by multiplying the dimensions of the array. The number of rows is directly related to the week number, whereas the number of columns is based upon how many columns there were the preceding week. Always symbolized as $w \times (c_{w-1} + 2)$

Explicit Rules Attempted

Task 1	Task 2	Task 3
Identified that the relationship is quadratic. The participant then tried to use the Vertex Form of a quadratic to symbolize this relationship, but was unable to adequately develop the appropriate symbolism to describe this relationship.	Conjecture an explicit rule, and then adjust this rule based upon whether it accurately relates the week number with its associated number of cutouts.	Rearrange the rectangular arrays of stars into triangular arras of stars (i.e., triangular numbers—a sequence of columns where each column has one more star than the preceding column), but is unable to coordinate the week number and the number of stars when trying to count them.
Identified that the relationship was linear, and that the goal was to directly relate the total number of stones laid each day with the day number. However, the participant was not able to determine a linear relationship that satisfied all three given cases.	Attempts to count the number of cutouts in the middle column and row (the "cross"), and then adds this to the number of cutouts not on this middle column and row (not on the "cross"). Able to count the cutouts on the cross but becomes stuck when trying to count cutouts not on the cross.	Conjectures a quadratic, then linear rule, unsatisfied with each conjecture after it is made.
Identified that the goal was to develop a direct (i.e., explicit) exponential rule that over-counts the number of stones laid on a given day, and then remove the over-counted amount. The participant was able to develop the exponential portion of the rule, but became stuck in trying to correct the over-counting.	Assumes the pattern is a square array with an extra cutouts above/below the middle column and to the left/right of the middle row, and attempts to count the number of cutouts in the square array and then add on the four cutouts on the top/bottom and left/right of the middle column/row. Able to count all portions except the number of cutouts in the square array.	
Identified that the relationship should be an explicit rule. The		

participant then conjectured linear, quadratic, and exponential rules but was never satisfied with them.		

Recursive Rules Attempted

Task 1	**Task 2**	**Task 3**
Identifies the recursive pattern +6,+8,… and attempts to develop a recursive rule based upon this pattern, but the participant was not able to develop the appropriate symbolism to capture this pattern as a recursive rule.	Recognizes the recursive pattern of +4, +8, … in the change in the number of cutouts between successive weeks, but confounds this recursive pattern, which builds off of the number of cutouts from the previous week, with trying to write an explicit rule and cannot develop the symbolism to capture this pattern.	Attempts to develop a recursive rule based upon the figures by adding a border of stars along the left, top, and right sides of the array, but cannot develop the appropriate symbolism to capture counting the given cases.
Identifies the recursive pattern +6,+8,… but confounds this recursive pattern, which builds off of the number of stones laid the previous day, with the symbolism associated with explicit rules and is unable to capture this recursive pattern as a recursive rule symbolically (i.e., no use of subscripts to denote terms in a sequence).	Counts the number of cutouts in the array and then adds a "border" of cutouts around the outside of the array, counting the number of cutouts in this border, with the sum of these two quantities being the number of cutouts in the subsequent array. Does not conclude that this rule is correct or developed though.	Attempts to develop a recursive rule based upon the figures by adding two columns to the right of the array and a row across the top of all of the columns, but cannot develop the appropriate symbolism to capture this pattern for the given cases. Symbolized in one case as $5w - 4$

No Attempt Made

Task 1	Task 2	Task 3
Searching for useful information or problem characteristics to develop a direct rule by investigating the rates of change	Searches for useful information by identifying the change of +4, +8, … between successive cases in a single variable	Searching for useful information by identifying the changes of +5, +9, … between successive cases for the number of star stickers in each array.
Searching for useful information or characteristics of patterns when assuming the relationship is linear or exponential	Searches for useful information by rewriting the number of cutouts as a sum of the square of a week number and what amount remains	Searching for useful information using the slope formula, concluding the relationship is not linear.
Searching for useful information to develop a direct rule by rearranging the stones in the figures into columns based upon values in the Fibbonacci sequence		Searching for useful information by determining the ratio between the number of star stickers and the week number.

APPENDIX G

CONSTRUCTION CATEGORY AND JUSTIFICATION CATEGORY COMPARISON

FOR EACH TASK INDIVIDUALLY

Table 57

Comparison of the Construction Categories to the Justification Categories for Tasks 1, 2, and 3

	Developed (Task 1/2/3)			**Attempted (Task 1/2/3)**			**Not Attempted (Task 1/2/3)**			**Totals (Task 1/2/3)**		
Verification	14	5	7	3	3	2	-----	-----	-----	**17**	**8**	**9**
Explanation	7	2	4	4	1	4	2	-----	-----	**13**	**3**	**8**
None	4	1	6	3	4	1	3	3	3	**10**	**8**	**10**
Totals	**25**	**8**	**17**	**10**	**8**	**17**	**5**	**3**	**3**	**40**	**19**	**27**

APPENDIX H

KIND OF RULE AND JUSTIFICATION CATEGORY COMPARISON

FOR EACH TASK INDIVIDUALLY

Table 58

Comparison of the Kind of Rule to the Justification Categories for Tasks 1, 2, and 3

	Explicit (Task 1/2/3)			**Recursive (Task 1/2/3)**			**Hybrid (Task 1/2/3)**			**Totals (Task 1/2/3)**		
Verification	11	6	5	6	2	4	**-----**	**-----**	**-----**	**17**	**8**	**9**
Explanation	6	2	5	5	1	3	**-----**	**-----**	**-----**	**11**	**3**	**8**
None	4	2	4	3	3	2	**-----**	**-----**	1	**7**	**5**	**7**
Totals	**21**	**10**	**14**	**14**	**6**	**9**	**-----**	**-----**	**1**	**35**	**16**	**24**

APPENDIX I

CHARACTERISTICS OF A RULE APPEALED TO AND JUSTIFICAITON

CATEGORY COMPARISON FOR EACH TASK INDIVIDUALLY

Table 59

Comparison of the Characteristics the Rule Appealed to Versus the Justification Categories for Tasks 1, 2, and 3

	Figural (Task 1/2/3)			**Numerical (Task 1/2/3)**			**Symbolic (Task 1/2/3)**			**Totals (Task 1/2/3)**		
Verification	11	-----	9	5	8	-----	1	-----	-----	**17**	**8**	**9**
Explanation	6	1	7	4	2	1	1	-----	-----	**11**	**3**	**8**
None	3	2	7	3	3	-----	1	-----	-----	**7**	**5**	**7**
Totals	**20**	**3**	**23**	**12**	**13**	**1**	**3**	-----	-----	**35**	**16**	**24**