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Investigating Pre-service Secondary Mathematics Teachers' Reasoning When Learning Trigonometry Using the Line Segment Approach

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INVESTIGATING PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' REASONING WHEN LEARNING TRIGONOMETRY USING THE LINE SEGMENT APPROACH

LAWRENCE SSEBAGGALA

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Review of the history of trigonometry content and pedagogy indicates the necessity and importance of trigonometry in the school curriculum (e.g., van Brummelen, 2009; van Sickel, 2011). For example, understanding trigonometric functions is a requirement for understanding some other areas of science, such as Newtonian physics, architecture, surveying, and several branches of engineering. Research indicates that teachers have a narrow and inadequate understanding of trigonometry, researchers (e.g., Hertel, 2013; van Sickel, 2011; Weber, 2005) posited that learning trigonometry in a way that fosters and advances quantitative reasoning can help alleviate difficulties faced in the teaching of trigonometry. Examining participants' reasoning is one way of determining if an instruction sequence promotes the desired understanding (Hiebert, 2003; Tall, 1996). Missing from the literature is investigative work that uncovers what kind of reasoning teachers use when engaged in a trigonometry instruction sequence that promotes quantitative reasoning.

In this dissertation, I will examine pre-service secondary mathematics teachers' reasoning about trigonometric functions when an instructional sequence (Hertel & Cullen, 2011) of trigonometric activities was used. Ultimately, this research is intended to shine more light on how a particular approach (line-segment) can influence prospective secondary mathematics teachers' knowledge of trigonometric functions and will seek to benefit the pre-service teachers by developing a strong comprehension of trigonometric functions.

KEYWORDS: Quantitative Reasoning; Creative Reasoning; Imitative Reasoning; Trigonometry; Line-segment Approach; Teaching and learning

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CHAPTER I: INTRODUCTION

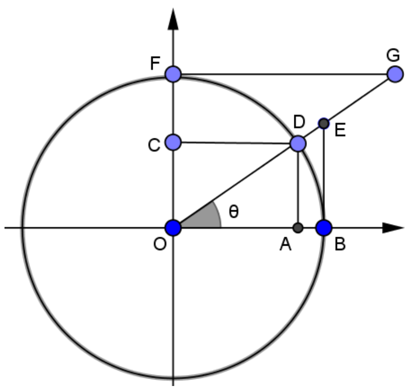
Background

Initially arising from the need to study and understand astronomy, trigonometry like other strands of mathematics (e.g., geometry, algebra) has a rich history from earlier centuries to the present day. The Greeks are considered the pioneers in the study of trigonometry and trigonometric functions (Kennedy, 1969; van Sickel, 2011). Particularly, Hipparchus of Nicaea, a Greek astronomer (190–120 B.C.) is regarded as the originator of modern trigonometry (van Sickel, 2011). Later, Ptolemy of Alexandria (85–165 A.D) made significant findings, like the sum and difference formulas, and the gnomon—used in learning the tangent function—that are still in use in the modern era (Maor, 1998; van Brummelen, 2009).

Around 500 CE, with the Greek civilization declining, the ascending civilizations of the time—Indians and Arabs—continued to study trigonometry whilst focusing on the study of the heavens and modelling their natural world, and this contributed to the advancement of trigonometry as a subject (Hertel, 2013). With the successful invasion of the Arab world by Christians from Western Europe in the fifteenth century, European mathematicians (e.g., Johannes Muller von Konigsberg, Leonhard Euler) once again took the leadership mantle in developing trigonometry as a distinct branch of mathematics and analysis (van Sickel, 2011).

Pedagogical Changes

As a subject, trigonometry was taught in North American colleges, such as Harvard and Yale, as early as the eighteenth century (Gordy, 1933; Hertel, 2013), although there is evidence that some high schools may have offered it earlier to some students (Hertel, 2013). Primarily, from its discovery up to the late sixteenth century, trigonometry problems were solved using a geometric *line-segment* concept. That is, solutions were obtained without complicated calculation but through geometric constructions. This approach defines the trigonometric functions as relationships between line segments in relation to a circle (see Figure 1) as opposed to being thought of as functions of angles.



$\overline{AD} =$ sine

$\overline{BE} =$ tangent

$\overline{OE} =$ secant

$\overline{CD} =$ cosine

$\overline{FG} =$ cotangent

$\overline{OG} =$ cosecant

Figure 1. The line-segment definitions of trigonometric functions

In the later part of the sixteenth century, due to developments in symbolic algebra and analytic trigonometry, trigonometry started to shift from a geometric form to an analytic form or ratio approach (Maor, 1998, van Sickle 2011). As far as pedagogy is concerned, this change was ultimately realized in secondary school mathematics curriculum by the late nineteenth century (Allen, 1977; van Sickel, 2011). Even though this shift from line-segment trigonometry to the *ratio* —in which “trigonometric functions for angles between 0 and 90 degrees are defined as ratios of sides of a triangle” (van Sickel, 2011, p. 11)—was not without resistance from teachers (Cajori, 1890; van Sickel, 2011; Hertel, 2013), the ratio approach was the primary form of instruction by the twentieth century.

Mathematics education researchers’ (e.g., Hertel, 2013; van Sickel, 2011; Weber, 2005; Hoachlander, 1997) reviews of the history of trigonometry content and pedagogy indicates the necessity and importance of the subject in the school curriculum. For example, understanding trigonometric functions is a requirement for understanding some other areas of science such as

Newtonian physics, architecture, surveying, and several branches of engineering (Hoachlander, 1997). Trigonometry also serves as a link between algebraic, geometric, and graphical reasoning, as well as acting as a significant precursor to understanding pre-calculus and calculus. Within other fields like surveying, the students' deficiency in the knowledge of trigonometry has been cited to hinder instruction in these fields (Elgin, 2007).

The modern approach to teaching this crucial course in the middle and secondary school curriculum mainly relies on the algebraic ratio approach, which unfortunately fosters memorization and very little in the way of reasoning (Blackett & Tall, 1991). The line-segment approach, which circumvents several issues that are related with the ratio approachapproach, is not often taught beyond the sine and cosine functions (Hertel & Cullen, 2011).

Statement of the Problem

Unfortunately, numerous researchers have described teachers' understandings of trigonometry as narrow and inadequate (Akkoc, 2008; Fi, 2003, 2006; Thompson, Carlson, & Silverman, 2007; Topçu, Kertil, Akkoç, Kamil, & Osman, 2006). Researchers have identified issues with some of the foundational ideas needed to completely understand trigonometry. For example, teachers' understanding of the radian as a unit of angle measure was not adequate, and consequently, they were much more inclined to use degree angle measures. Even in cases where they had some grasp of radians, secondary mathematics teachers were reported to be unable to describe a meaning of radian measure beyond certain conversion procedures—for converting between radian and degree angle measures—that they employed, even though the procedures were not meaningful to start with. (Fi 2003, 2006).

Additionally, a study of pre-service teachers (Akkoc, 2008) found that teachers with less developed understandings of the radius as a unit of measurement in the teaching and learning of trigonometric functions relied on only using a right triangle while explaining the concepts of trigonometry. Akkoc (2008) concludes by imploring mathematics educators to tailor their teaching of trigonometric concepts in ways that encourage concepts that promote understanding the radian as a unit of measurement.

Teachers' deep and strong attachment to their current high school curriculum and the meanings they have ascribed to that curriculum over time has also hindered successful teaching and learning of trigonometry concepts. For example, Thompson, Carlson, and Silverman (2007) investigated teachers working on several teacher-related tasks in trigonometry and found teachers to always commence with right triangles, as opposed to angle measure and the unit circle. Other teachers contended that trigonometry is primarily concerned with solving for measurements of a triangle. Such perceptions, notwithstanding their lack of coherence, dictated what teachers envisioned themselves teaching in a real class setting.

A recurring theme in the studies discussed above is that of secondary teachers being heavily attached to meanings that did not echo reasoned understandings of trigonometry. Most of the teachers' underlying but necessary understanding of trigonometry basics was insufficient, probably adding to the incoherence of their meanings and significances.

Rationale

The literature accessible has described teachers as holding very limited understandings of trigonometric functions. It is not farfetched to conclude that if teachers' understandings are limited, their students' understandings of trigonometry functions will most likely be limited as well. Although there are cited difficulties with learning trigonometric functions, the mathematics education community is grappling with sparse literature and educational research in this area (Moore, 2013).

Documented studies on this topic typically concern themselves with difficulties students find with trigonometry and suggestions to overcome them (e.g., Blackett & Tall, 1991). Other related literature on trigonometry education mainly presents studies comprised of teaching techniques for enhancing or replacing typical instructional techniques (Weber, 2005). However, Barnes (1999) notes that, while these instructive recommendations are fascinating and possibly valuable, they are principally not grounded in research or theory or their effectiveness is seldom evaluated. This study endeavored to steer away from that route and instead investigate the reasoning learners (in this case pre-service secondary teachers) of trigonometry engage in during instruction.

There is a wide-ranging agreement in the mathematics education fraternity that mathematics needs to be moved away from teaching techniques that promote passing paper-and-pencil assessments to instead focus on teaching that fosters understanding (National Council of Teachers of Mathematics [NCTM], 2000) including instruction in trigonometry (Weber, 2005). This informal agreement is based on numerous researchers' findings that existing teaching practices of trigonometry courses do not appear to promote students' understanding of these trigonometric and related functions (DeJarnette, 2014; Kendal & Stacey, 1997).

With calls such as one from Hirsch, Weinhold, and Nichols (1991), that mathematics educators must desist from promoting "memorization of isolated facts and procedures and proficiency with paper-and-pencil tests [and move towards] programs that emphasize conceptual understanding, multiple representations and connections, mathematical modelling, and problem-solving" (p. 98), some mathematics education researchers have recently begun to investigate this need in the teaching and learning of trigonometry. In these studies, (e.g., Moore, 2014; van Sickel, 2011; Weber, 2005), researchers have posited that one way to overcome this and other difficulties faced in the teaching of trigonometry is to promote teaching that fosters and advances quantitative reasoning (see definition below).

Moreover, as noted earlier, literature has emerged suggesting that using the line-segment definition of the trigonometric functions can be instrumental in promoting a relational understanding of trigonometry (Bressoud, 2010; Hertel, 2013; Hertel & Cullen, 2011; Maor, 1998; Moore, 2010, van Sickle, 2011). In particular, Bressoud (2010) opined that:

In the mid-nineteenth century, when those studying trigonometry were most likely to use it in navigation and surveying, defining these functions as ratios made sense. There is convincing evidence that this approach does help students working on this type of problem (Kendal and Stacey 1998). But today students are more likely to encounter the sine and cosine as periodic functions rather than as navigational aids. Biological, physical, and social scientists use them more often to model periodic phenomena than to find the unknown side of a right triangle. If we want our students to understand trigonometric functions as functions, then the historical definitions that describe them as relating two lengths—arcs and line segments—are more transparent. (Bressoud, 2010, p. 112)

Furthermore, stemming from Skemp's (1976) distinction between instrumental and relational understanding, researchers (e.g., Balacheff, 1988; Lithner, 2000; Sfard, 1991) have posited that one way to investigate if the desired understanding has been achieved or not is to analyze the students' reasoning when justifying their solutions to the tasks given. Missing from the literature is investigative work that uncovers what kind of reasoning teachers use when engaged in a trigonometry instruction sequence that is based on the line segment approach. This study sought to contribute to the limited body of research literature on teachers' understanding of trigonometry by broadly answering this question while working with pre-service secondary mathematics teachers.

Purpose Statement

The main purpose of my research is to investigate the types of reasonings that pre-service secondary mathematics teachers exhibit and use in the process of learning trigonometry with instructional emphasis on quantitative reasoning through a line-segment definition of trigonometry.

Research Question

The research question that guided this study is:

- What types of reasoning do pre-service secondary mathematics teachers engage in while participating in an instruction sequence on trigonometry that focuses on a quantitative reasoning approach.

Definition of Terms

Mathematical Reasoning

Among mathematics educators, there is no clear definition for the term reasoning (Ball & Bass, 2003; Lithner, 2008; Martin et al., 2009). For example, Ball and Bass (2003) state that "mathematical reasoning is no less than a basic skill" (Ball & Bass, 2003, p. 28), while others (e.g., Duval, 2002; Harel, 2006) consider only strict proof as mathematical reasoning. On one hand, it is assumed that there is a universally agreed upon implicit meaning for the term mathematical reasoning (or simply reasoning) (Yackel & Hanna, 2003), and on the other hand, the term is used in conjunction with and often interpreted as 'proof' (e.g., NCTM, 1989, 2000). And as such, it is crucial that we declare a particular meaning for reasoning in this study.

It should be noted that when we express reasoning as being based on mathematical logic in the form of mathematical deduction and/or creating or analyzing proof, it may encourage students to regard mathematics as a mere set of procedures and drills, yet the goal of mathematical reasoning is to foster understanding (Martin et al., 2009; NCTM, 1989, 2000). Articulating reasoning as an act of creating and revising conjectures (Ball & Bass, 2003; Lithner, 2008), or the process and product of student thinking (e.g., Martin et al., 2009) partly guided our adoption of the definition and later the theoretical framework from Lithner (2008).

In this study, *reasoning* "is the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it" (Lithner, 2008, p. 257). The attempt by students to make use of their current knowledge and understanding to justify how they solve a task at hand is what is central to the whole idea of reasoning. Put differently, reasoning is regarded as strategies and guiding principles students employ in order to make sense of a task. (Lither, 2008).

Quantitative Reasoning

Quantitative reasoning involves not only the quantity (which is the object of the reasoning) but also the actions and procedures (Piaget, 1970) that one performs during the thought process (Thompson, 1994) of solving a task. However, as is the case with other concepts in mathematics educations, there is no clear agreement on what is considered to be quantitative reasoning. Mayes, Peterson, and Bonilla (2013) defined quantitative reasoning as an application of one's knowledge. This pertains to how one applies their knowledge to "mathematics and statistics applied in real-life, authentic situations that impact an individual's life as a constructive, concerned, and reflective citizen" (Mayes, Peterson, & Bonilla., 2013, p. 6).

A clear distinction between the two definitions stems from what is regarded as key to quantitative reasoning. Thompson's definition is concerned with quantification, whereas Mayes et al.'s definition emphasizes application. Inasmuch as application of knowledge is important, when referring to quantitative reasoning in this study, we adopted the definition from Thompson (1990), as quantification needed to be central to our approach. Therefore, *quantitative reasoning* (Thompson, 1990) refers to a learner mentally visualizing a situation, conceptualizing measurable attributes (called quantities) within this imagined state, and constructing relationships between these quantities. Central to this approach is the view that the analysis of the mental actions involved in conceiving a situation is primarily void of numbers and numeric relationships. Rather, the relationships among quantities is emphasized.

Covariational Reasoning

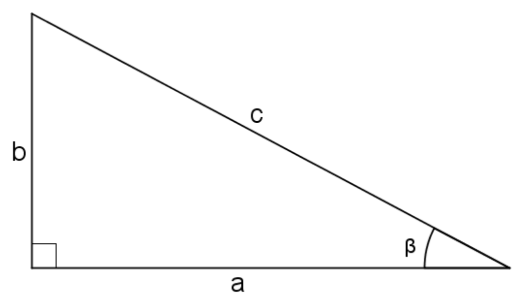
Thompson (2011) refers to covariational reasoning as an essential component of quantitative reasoning., It is characterized with relating two quantities whilst focusing on how a change in one affects the change in another (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008; Saldanha & Thompson, 1998). He further states: "The importance of covariational reasoning for modeling is that the operations that compose covariational reasoning are the very operations that enable one to see invariant relationships among quantities in dynamic situations" (Thompson, 2011, p. 46). In this study, some of the instructional tasks required students to engage in varying of quantities and reasoning about how the quantities covaried.

Line Segment Trigonometry

Line segment definition of trigonometry is categorized as a form of quantitative reasoning as it brings the relationship between two quantities, in this case line segments and arcs, to the focus of instruction. A detailed definition for this context is provided in Figure 1 above

Ratio Trigonometry

Adopting the definition used by van Sickel (2011), ratio trigonometry also known as right triangle trigonometry is an approach in which "trigonometric functions for angles between 0 and 90 degrees are defined as ratios of sides of a triangle" (van Sickel, 2011, p. 11). The definitions of the basic functions are shown in Figure below. In the approach, the relationships between trigonometric functions are determined algebraically by some general relation of these quantities in connection with a triangle.



$\sin\beta = {}^{b}/_{c}$ $\qquad$ $\cos\beta = {}^{a}/_{c}$

$\sec\beta = {}^{c}/_{a}$ $\qquad$ $\csc\beta = {}^{c}/_{b}$

$\tan\beta = {}^{b}/_{a}$ $\qquad$ $\cot\beta = {}^{a}/_{b}$

Figure 2. The definitions of trigonometric functions in ratio trigonometry

Unit Circle Trigonometry

In this approach to trigonometry, functions derive their meaning from a circle of whose radius is one. In particular, the sine and cosine functions are respectively defined as the y and x coordinates of a point on a unit circle.

Outline of Study

In this dissertation I will examine pre-service secondary mathematics teachers' reasoning about trigonometric functions when an instructional sequence (Hertel & Cullen, 2011) of trigonometric activities was used. Ultimately, this research is intended to shine more light on how a particular approach (line-segment) can influence prospective secondary mathematics teachers' knowledge of trigonometric functions and will seek to benefit the pre-service teachers by developing a strong comprehension of trigonometric functions.

The participants in this study were twenty secondary pre-service teachers enrolled in a technology-based mathematics content course in a medium sized Midwestern University in the United States. The trigonometry portion of the course was designed in such a way as to promote quantitative reasoning, by the professor assigned to teach it, who was also the principle investigator.

All the participants volunteered to take part in the study in which I (the researcher) was a participant-observer during the six weeks of the study. The instructional sequence for the study consisted of six 2-hour and 40-minute instruction sessions open to all twenty-three students enrolled in the course (three students did not agree to participate). With the data collected, I categorized participants' responses as either imitative or creative mathematically founded reasoning using Linthner's conceptual framework for creative and imitative reasoning (Lithner, 2008). Later analyses consisted of looking for patterns in reasoning as well as relationships between reasoning and the approach to trigonometric employed (i.e., ratio, line segment, unit circle).

In Chapter 2, I describe a conceptual framework used for the study, and an overview of the research literature on trigonometry, and quantitative reasoning. In Chapter 3, I report on the methods for data collection, the rationale for choosing to study pre-service secondary mathematics teachers, and data analysis used for this study. In Chapter 4, I present the findings from each of the four tasks that were used to investigate students' reasoning, as well as the general findings from this investigative study. A summary of the general discussion about the findings is also provided. In Chapter 5, I provide an analysis of the results and conclusions drawn about patterns or themes that developed after I explored the types of reasoning that pre-service secondary mathematics teachers use in the process of learning trigonometry with instructional emphasis on quantitative reasoning through a line-segment definition of trigonometry. I then discuss the implications, limitations of the study and suggestions for the teaching and learning of trigonometry. Finally, directions for future research will be suggested.

CHAPTER II: THEORETICAL FRAMEWORK AND LITERATURE REVIEW

In this chapter, I present the theoretical framework for the study. To investigate students' knowledge of trigonometry, I examined the kind of reasoning they engaged in while acquiring knowledge of trigonometry. In this chapter, I also review the pertinent research literature on the teaching and learning of trigonometry, which to a large extent contends that learners and sometimes their teachers have inadequate understandings of this strand of mathematics.

Theoretical Framework

The main purpose of my research is to investigate the types of reasoning that pre-service secondary mathematics teachers exhibit and use in this process of learning trigonometry with instructional emphasis on quantitative reasoning. To examine pre-service teachers' reasoning about trigonometry, I need to be able to detect and describe different types of reasoning in the realm of trigonometric functions.

Mathematical Reasoning

There are numerous theoretical frameworks that focus on mathematical reasoning, e.g. KOM (Niss, 2003). However, none particularly embody the detailed description of reasoning presented by Lithner's (2008) framework used to distinguish between imitative and creative reasoning.

Skemp (1976) offers a framework that aims to categorize the comprehension that facilitates students' reasoning, in terms of instrumental and relational understanding. However, Skemp's work did not present any additional specifications about reasoning. Similarly, almost two decades later, Wyndhamn and Säljö (1997) analyzed children's mathematical reasoning while focusing on the content and rules in the students' reasoning, as the students were working on tasks involving word problems. The results from the study contrast from the widely held opinion that students generally succeed in resolving standard problems but labor to solve in new challenging situations. The researchers assert that “students are able to deal with a particular kind of difficulty” (Wyndhamn & Säljö, 1997, p. 381) and in so doing they initiate discussions involving different types of reasoning. Nonetheless, in their framework the authors provide no definition of reasoning, and they neither stipulate the different categories of reasoning, nor highlight any mathematical properties used as a basis for the reasoning. Fransisco and Hähkiöniemi (2011) differentiate between various types of foundations when investigating techniques students use to reason about algebra. The essential events were identified and interpreted whilst searching for evidence of algebraic ideas and the various ways of reasoning. Similar to Wyndhamn and Säljö’s (1997) study, a definition of what constitutes reasoning was not articulated, even though the authors concluded that according to the results, students were engaging in different types of algebraic reasoning.

Other researchers (e.g., Krummheuer, 2007) have argued that exploring argumentation is one way of obtaining information about reasoning. In this study, he investigated students learning mathematics through partaking in activities that promoted collective argumentation. As much as the framework developed by Krummheuer (2007) effectively centers on data, deduction, warrant and supporting arguments, it neglects to address the different types of reasoning. Synonymously, in other studies (e.g., Voigt, 1994) in which researchers hold the view that reasoning is part of social interaction, there is no mention of the different types of reasoning. Moreover, there is also a possibility that argumentation may be overemphasized hence blurring the difference between argumentation and reasoning. To draw a distinction between argumentation and the different types of reasoning requires one to first define the terms.

Ball and Bass's (2003) definition of reasoning when referring to mathematical understanding was premised on mathematical reasoning. They stated that reasoning "comprises a set of practices and norms that are collective not merely individual or idiosyncratic, and rooted in the discipline." (Ball & Bass, 2003, p. 29). However, in their framework, they posit that reasoning is rooted in logic, thus making it more objective than subjective. This view of reasoning was not appropriate for the data in this dissertation because I needed to subjectively identify reasoning. A different framework, that deals with reasoning grounded in an understanding that is subjective, would be more appropriate for our study. Vinner (1997) does offer such a view of reasoning, by focusing on analytical and pseudo-analytical behavior. However, the concept of analytical behavior is not defined, rendering it challenging to use this framework when attempting to single out which of the students' reasoning is or is not analytical.

Sfard's (2001) focal analysis framework was used by Farmaki and Paschos (2007) in studying the interaction between intuitive and formal mathematical thinking. Focal analysis offered Farmaki and Paschos an avenue to examine students' mathematical thoughts, and cognitive operations. Results from analyzing data—from six activities—collected from one of the students who partook in the study revealed that it necessitated students to carry out multiple cognitive operations to be able to evolve from intuitive reasoning to formal argumentation. In the end, Farmaki and Paschos (2007) suggested a further investigation of this transition in reasoning if one were to make substantive conclusions concerning the mechanisms of knowledge construction. Inasmuch as Sfard's (2001) analytical tool permits for a clear distinction concerning choices and arguments in students' thinking and reasoning, it stills fall short of affording the researcher who opts to use it, an avenue to classify mathematical reasoning in different categories.

Although the different frameworks discussed above were successfully applied and used in the stated settings and studies, I required a framework in which reasoning is defined and the different categories of reasoning are clearly stated and described. Therefore, I elected to use Lithner's (2008) research framework for imitative and creative reasoning because it not only defines reasoning but it also allows the classification of participant reasoning. This framework proposes a wide conception of mathematical reasoning inspired by Pólya (1954).

In opting for Lithner's framework, there were three underlying motivations. Primarily, the framework helps to distinguish between what is creative, mathematically, well-founded reasoning and what is not. Secondly, the framework considers *reasoning* as a line of thought adopted to produce assertions and reach conclusions in task solving. It does not have to be based on formal logic, and it may even be incorrect. It features the argumentation for the choice of methods and substantiations learners make while solving mathematical tasks. The classification and description of imitative and creative reasoning in this framework facilitated a thorough examination of the geneses and significances for each reasoning statement. Finally, in comparison to other frameworks, Lithner (2008) provides well defined concepts of reasoning in a well-formulated conceptual framework (Bergqvist, 2012).

Several researchers have used Lithner's (2008) framework to analyze data from different studies. For example, Sumpter (2009) employed this framework to investigate how beliefs influence upper secondary school students' reasoning, and to find out whether reasoning and beliefs were gendered. Sumpter used Lithner's (2008) and two other frameworks to explore affect and gender as aspects of mathematical reasoning. Prior to examining how beliefs influence students' reasoning, the researcher used the framework to initially find out and characterize the “type of mathematical reasoning students perform when solving mathematical tasks” (Sumpter, 2009, pp. 3-4).

Bergqvist (2012) also set out to understand the rationale used by university calculus instructors while creating calculus examination. The researcher was particularly interested in the kind of reasoning the instructors expected the students to display in their solutions to these questions. Lithner’s (2008) framework was used to classify the views of the instructors concerning the students’ reasoning. The results revealed that at the time of writing the examination questions, imitative reasoning dominated the instructors’ expectations from the students, as questions that may require creative reasoning were deemed “too difficult and lead to too low passing rates” (Bergqvist, 2012, p. 399).

Finally, Jonssona, Norqvist, Liljekvist, and Lithner (2014) used the framework to investigate the learning of mathematics through algorithmic and creative reasoning by upper secondary school students in Sweden. The framework was used to distinguish between two teaching models, one based on students’ own creation of knowledge, denoted creative mathematically founded reasoning (CMR), and a procedure-based model of teaching, referred to as algorithmic reasoning (AR). In their study, students who went the CMR-based model were found to have performed better than those who underwent procedure-based instruction (AR).

Lithner’s Conceptual Framework for Mathematical Reasoning

With efforts by curriculum reform advocates geared towards teaching mathematics for understanding (Pirie & Kieren, 1994), there is need for the mathematics education community to know what typifies this understanding. One way to achieve this is to capture, categorize and analyze the types of mathematical reasoning that students exhibit when engaged in a particular form of instruction (Hiebert, 2003; Tall, 1996). Lithner's (2008) framework goes further to describe the reasoning that underlie the participants' conceptions.

To be able to investigate these justifications, explore and capture the different forms and key characteristics of mathematical reasoning, I draw from Lithner's (2008) conceptual framework for creative and imitative reasoning (see Table 1) to describe what kind of reasoning pre-service secondary mathematics teachers engage in while participating in an instruction sequence in trigonometry that uses a line segment definition. More precisely, Lithner's (2008) framework will be used as a classification tool for the sequences of reasoning the participants produced while solving tasks on trigonometry.

The framework affords the researcher a basis to investigate students' reasoning, primarily with regards to the difference concerning using existing (memorized or given) solution methods and creating or structuring the solution. The framework depicts the reasoning exhibited by the students as being dependent on the one's erstwhile knowledge, guidance, or examples that are accessible when resolving the task. The framework lays out a sequence for reasoning that commences with the given task and proceeds to a solution, and the resulting reasoning is dependent on the task, the student's opinions, and the social setting.

Reasoning sequences. When resolving a task, a decision where to start from is necessary and crucial. Researchers (e.g., Schoenfeld, 1985) have reported that novice problem solvers, unlike the experts seldomly invest time into preparing and selecting appropriate reasoning sequences. The novices where noticed to quickly "dive in" and repeatedly adopted problem-solving procedures that were inappropriate for the tasks. This process of solving a task, Lithner (2008) submitted that it can be viewed as a directed graph where accomplishment of a solution strategy (edges) are linked by a subtask and the reasoner's momentary state of knowledge (represented as vertices in Figure *3*). The edges of the graph contain solution procedures that the reasoner chooses from, and implements to reason about and solve a specific subtask. Therefore, a particular task can be resolved by taking different paths throughout the graph.

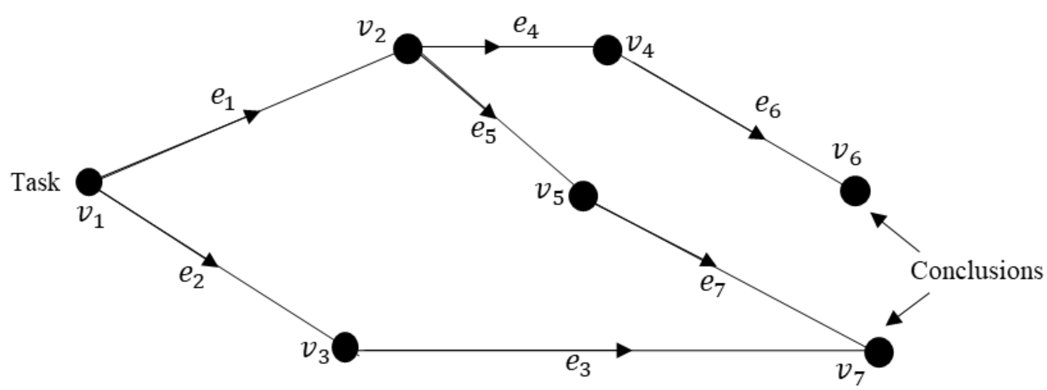


Figure 3. Reasoning sequence as depicted in Lithner (2008)

Creative mathematically founded reasoning. Lithner (2008) classified reasoning in two major categories, creative and imitative reasoning, with the later having different sub-categories. When solving mathematical tasks, students gravitate towards a familiar solution method (i.e., by an algorithm or by recalling memorized answers). In cases where they cannot use known strategies, the student may attempt to re-create a forgotten one or construct a completely new reasoning sequence and linking it to the intrinsic mathematical properties essential for the task at hand. Such reasoning that involves both novelty and mathematically founded arguments, Lithner called it CMR. Lithner (2008) concluded that reasoning can be categorized as CMR if it fulfills the following three conditions:

1. Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.
2. Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.

3. Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. (Lithner, 2008, p. 266)

Lithner stressed that the creativity referenced here can be as simple as construction of known (to others) argument but a new reasoning sequence to the student solving the task.

Imitative reasoning. Analogously, reasoning that is connected to either recalling a completed answer and simply writing it down (memorized reasoning) or performing a recalled procedure without connecting it to mathematical properties (AR) is referred to as Imitative reasoning (IR). Students that exhibit memorized reasoning (MR), recall complete answers to given tasks, coupled with just writing down the answers. On the other hand, AR was defined by Lithner (2008) as follows:

Algorithmic reasoning (AR) fulfils the following two conditions.

1. The strategy choice is to recall a solution algorithm. The predictive argumentation may be of different kinds, but there is no need to create a new solution.
2. The remaining reasoning parts of the strategy implementation are trivial for the reasoner, only a careless mistake can prevent an answer from being reached. (Lithner, 2008, p. 259)

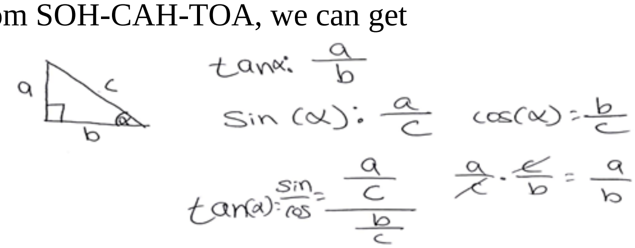
A reasoning is considered to be AR if the student retrieves the algorithm that leads to the solution either from memory, or it is availed in the instructions or retrieved from a worked example (Lithner, 2008). It is on this basis that Lithner further classifies AR into different sub-categories. In familiar algorithmic reasoning (AR-F), the student chooses to use a particular algorithm because the task at hand is "of a familiar type" (Lithner, 2008, p. 262) and a known algorithm can be employed, whereas engaging in delimiting algorithmic reasoning (AR-D) entails the development of a set of algorithms depending on the surface knowledge the student has about the task, and tries to use one algorithm at a time until one from the delimited set checks out. The final strand of AR is called guided AR. External help is key in this category. The help may be directly from another person, (e.g., instructor) in which case it is called person-guided (AR-PG) or from a textbook in different forms (e.g., worked example, theorem, etc.). This is referred to as text-guided (AR-TG). Noteworthy, in all cases of AR, the algorithm is not verified by the student before using it.

To demonstrate the distinction between the two main different types of reasoning, IR and CMR, and to situate the reader as to what these reasoning types may look like in the context of this study, I present below the different cases in which different responses to the same prompts from the students in different tasks were coded differently.

Question: Explain how you know that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Response 1: I will go like for the unit circle stuff, sin(θ)=y/r, cos(θ) = x/r and we know tan(θ) = y/x. If a student came to me and asked why cos(θ) = adj/hyp, I would say, SOH-CAH-TOA.

Response 2: From SOH-CAH-TOA, we can get



Response 1 was classified as IR. The student was only able to reproduce what was learnt from a prior trigonometry course. However, there is an implicit declaration that they simply memorized SOH-CAH-TOA and thus could not reason beyond the mnemonic. Response 2 was classified as CMR. The student was able to re-create a reasoning sequence that went beyond the mnemonic and used it to define the different ratios while comparing the different outcomes from the quotients.

Table 1

Framework for Reasoning

Type of Reasoning	Variants	Conditions
Imitative	1. Memorized (MR)	1. The strategy choice is founded on recalling a complete answer. 2. The strategy implementation consists only of writing it down.
	2. Algorithmic (AR)	1. The strategy choice is to recall a solution algorithm. 2. The remaining reasoning parts of the strategy implementation are trivial for the reasoner, only a careless mistake can prevent an answer from being reached.
	• Familiar AR	1. The reason for the strategy choice is that the task is seen as being of a familiar type that can be solved by a corresponding known algorithm. 2. The algorithm is implemented.
	• Delimiting AR	1. An algorithm is chosen from a set that is delimited by the reasoner through the algorithms' surface relations to the task. The outcome is not predicted. 2. The verificative argumentation is based on surface considerations that are related only to the reasoner's expectations of the requested answer or solution. If the implementation does not lead to a (to the reasoner) reasonable conclusion it is simply terminated without evaluation and another algorithm may be chosen from the delimited set.

(Table continues)

Type of Reasoning	Variants	Conditions
	• Guided AR	
	○ Text-guided AR	1. The strategy choice concerns identifying surface similarities between the task and an example, definition, theorem, rule, or some other situation in a text source. 2. The algorithm is implemented without verificative argumentation.
	○ Person-guided AR	1. All strategy choices that are problematic for the reasoner are made by a guide, who provides no predictive argumentation. 2. The strategy implementation follows the guidance and executes the remaining routine transformations without verificative argumentation.
Creative	Creative mathematically founded reasoning (CMR)	1. Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created. 2. Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible. 3. Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

Note. Adapted from "A research framework for creative and imitative reasoning," by J. Lithner, 2008, *Educational Studies in Mathematics, 67*(3), pp. 255–276.

The table below summarizes the different types of reasoning, the affiliated codes that were used in categorizing the data, with descriptive examples of each code from the data. From the data reported, the only coded reasoning was that directly related to the identified task. For CMR I will identify each of the cases of CMR by chronologically labelling them (e.g., CMR1, CMR2).

Table 2

Codes for Reasoning Types and Descriptive Examples from the Data

Description	Example from the data
A. Imitative Reasoning	N/A
(a) Memorized Reasoning (MR)	What is the secant? Ans : $1 + tan^2\theta = sec^2\theta$
(b) Algorithmic Reasoning	N/A
1) Familiar AR (AR-F)	Where is the secant function? (see)
2) Delimiting AR (AR-D)	Does anybody know what secant means? It is this line that cuts through? Or may be something to do with cosine? I give up.
3) Guided AR (AR-G)	N/A
(i) Text-guided AR (AR-TG)	Wait, let me check (*opens textbook*). Secant line must cross the circle twice.
(ii) Person-guided AR (AR-PG)	Eron: I think this green segment is the secant, but I am not sure why.
	Dr. Kay: What kind of triangle is created here with the radius, the tangent and the green segment? Does anyone remember the Pythagorean identities?
	Eddie: Ah, I see now! It is the secant because this is a right triangle with the radius (equal to 1), the tangent, and the green segment. That means we could use the Pythagorean Theorem which would tell us 1^2+tan^2 = (green segment)^2 and I remember that $1 + tan^2\theta = sec^2\theta$ so the green segment must be equal to the secant

(Table continues)

Description	Example from the data
B. Creative Reasoning	
(a) Creative mathematically founded reasoning (CMR)	Henry: You could go with the same thing we did for tangent. We know that this is secant must lie on this secant line, and if we translate this line to the graph and covary it with the arc, the resulting graph looks like this, (*shows the graph of secant*). Since the graph behaves like what we normally know as the graph of the secant, this line segment must be the secant.

Literature Review

The available research on students' and teachers' conceptions of trigonometry although limited set the foundation and rationale for how I designed this study. A popular theme was that both students and teachers' understanding of trigonometry was not only limited but also fragmented.

Students' Understandings of Trigonometry

There is scanty research literature on students' understandings of trigonometry. Nonetheless, there has been significant research done in specific areas like trigonometric functions (Brown, 2005, 2006; Hardison, 2017; Weber, 2005), angle measurement (Hardison, 2017; Moore, 2013), or the sine function (Demir & Heck, 2013; Wood, 2011). In these studies, researchers have mainly documented students as possessing limited and constricted understanding of trigonometry and a weak comprehension of angle measure (Brown, 2005). Many of the studies paint a grim picture of the status of trigonometry, however, recent studies show an emerging change in the trend. For example, in his work with a ninth-grade student, (Hardison, 2017) found that the student was able to develop an extensive quantification of the angle measure.

Weber’s (2005) comparative study on students’ understanding of the trigonometric functions revealed that students who were taught using Gary and Tall’s (1994) concept of procept in an experimental instruction developed a deep understanding of trigonometric functions in comparison to those who experienced a traditional definition-theorem-proof lecture method, whose understanding was reported as being limited after the semester long course. The experimental instruction required participants to reason about the values of trigonometric functions, by reasoning about properties of the anticipated output irrespective of the step-by-step algorithm involved. This required the students to make constant connections to the definition of the trigonometric functions in the tasks. Weber contends that it is this connection that helped students view tasks as entire objects to which they developed a reasoning processes that let to them having a better understanding of trigonometry when compared to those in the lecture-based group.

Furthermore, Weber (2005) also reported that in the lecture-based group all students were unable to make progress on a question that required them to explain why “sin(x) is a function”. Moreover, after declaring their lack of knowledge of what a function was, the interviewer restated the question as “how do you know that sin (x) can only have one value for a given x-value?” Other researchers (e.g., Carlson, 1998; Carlson & Oehrtman, 2004; Harel & Dubinsky, 1992; Oehrtman, et al., 2008) have reported similar findings of students struggling to comprehend functions as a process, thus impeding their reasoning about the same. In contrast, their counterparts who underwent the experimental instruction managed to answer the same question by making a connection between the input and output of the sine function. Inasmuch as those in the experimental instruction group were taught using the unit circle approach, Weber cautions that emphasis should be on having students understand the unit circle procedures as *processes.*

Much of the research on the teaching and learning of trigonometry is rooted in examining particular trigonometry stances (e.g., right triangle, unit circle) or comparisons between two stances. For example, Weber's (2005) caution had earlier been articulated by Kendal and Stacey (1997) who found that in teaching students using either the right triangle approach or the unit circle approach, those students using the latter model did not learn as much as those using the right triangle approach. Brown (2005) documents students' difficulties interpreting within the unit circle stance, pointing out that they were unable to make a connection between a point on the unit circle to the graph of a corresponding function (e.g., the graph a of sine function). This disconnect between two representations implies a lack of conceptual understanding of trigonometry which researchers suggest benefits in using multiple representations. Schnotz and Bannert (2003), for example, found that if trigonometry instruction allowed for use of multiple representations, students were more likely to successfully solve the tasks presented to them. In fact, Kendal and Stacey (1997) found that even if students were taught using an algorithm, they would ably learn the concept as long as the algorithm was accompanied by a representation.

Teachers' Understandings of Trigonometry

The struggle with trigonometry is not limited to secondary school students. Even pre-service teachers with a record of having successfully completed a trigonometry course prior to the study, were found by Fi (2003) to be struggling with concepts of trigonometry. Specifically, Fi's (2003) examination of these pre-service teachers pertained to their pedagogical and trigonometry content knowledge. Through using concept maps, interviews, card sorting, and a set of trigonometric problems, Fi encountered teachers having difficulties working with several trigonometric functions. Fi’s interviews led him to attribute the teachers’ struggles to the instrumental approach they took while initially learning trigonometry. Moreover, on specific sections (e.g., trigonometric identities) Fi reported that the pre-service teachers have misconceptions and with the exception of fundamental Pythagorean identity, they labored to derive other identities, leave alone applying them. The pre-service teachers were also unable to reason about how the different attributes of trigonometric function (e.g., phase shift, vertical translations) can affect the nature of the corresponding graph.

Other researchers (Akkoc, 2008; Thompson, et al., 2007; Topçu, Kertil, Akkoç, Kamil, & Osman, 2006) have not only reported teachers’ knowledge of trigonometry as being narrow, and limited, but also cemented. The researchers reported teachers as being deficient in dealing with the radian as a unit of angle measure. Rather, most teachers often times opted to using only degrees while measuring angles. For instance, Topçu, Kertil, Akkoç, Kamil, and Osman (2006), and Akkoc (2008) described teachers in their studies as viewing π radians as being the number 180 not the angle 180 degrees.

The teachers’ difficulty with trigonometry are also documented by Moore, LaForest and Kim (2012). Before carrying out the intervention, the researchers found the two pre-service secondary mathematics teachers unable to describe the radius of a unit circle as being “one radius length”, even though they acknowledged that a unit circle has a radius of one. For example, one of the pre-service teachers specifically quantified the radius of a unit circle as being 1 inch. The teachers struggled to apply the given central angle and radius to determine the arc length.

Mathematical Reasoning

A dictionary (Webster) definition of reason is “the ability to think coherently and logically and draw inferences or conclusions from facts known or assumed” (Guralnik, 1982, p. 1183). As stated earlier, in particular mathematical reasoning in this study “is the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it” (Lithner, 2008, p. 257).

Sfard (1991) argues that students need to acquire both the operational and structural knowledge of mathematics in order to succeed at the subject, as one type of knowledge complements the other. In other words, although rote learning (operational knowledge or IR) is known to contribute to student difficulties, it is necessary in some cases.

The influence of mathematical reasoning on mathematical learning has been examined and investigated from several viewpoints and at different levels of education. In Rowan, Chiang, and Miller’s (1997) study, the researchers established a connection between tracking how a student forms their knowledge and their performance in mathematics. In fact, Ball (2003) found it crucial for teachers to have knowledge of their students’ mathematical reasoning. Other researchers (Maher & Martino, 1996; 1998; Maher & Davis, 1995) have discussed how even at a young age, children are able to exhibit evidence of reasoning capability. This reasoning and justification are propelled by the children’s desire to make sense of a given problem, and to discern and develop patterns (Maher & Martino, 2000). It is this desire to provide convincing mathematical justifications by refining their solutions through discussion and negotiation with classmates that structures their reasoning (Weber, Maher, Powell & Lee, 2008; Maher, 2005). At a higher level, the NCTM (2000) clearly pronounces its stance on the significance of mathematical reasoning. The standards stress that learning to reason is vital to understanding mathematics. By the end of high school, students need to be capable of employing mathematical reasoning to "produce mathematical proofs, and should appreciate the value of such arguments" (NCTM, 2000, p. 56).

Mansi (2003) also argues that students should be helped to develop their mathematical reasoning well before they are tasked to engage in other forms of mathematics like writing proofs. Perhaps, Mansi (2003) proficiently captured the general rationale for why mathematical reasoning is important. While investigating the complexities concerned with learning to write proofs Mansi stated:

> Students' overall ability to reason about mathematical ideas and make justifications for why a mathematical concept makes sense or why a procedure should be used is a powerful and necessary part of learning mathematics. Students who are not forming these reasoning and justification abilities throughout their learning of mathematics will most likely struggle with the notion of proof. (Mansi, 2003, p. 9)

Although all the research presented here emphasizes the need for fostering mathematical reasoning in students, the types of reasoning to be stressed were hardly mentioned. Hiebert (2003) points to this as one of the main problems in mathematics education. There is tremendous effort invested in developing students as problem solvers, but with all the research and reforms put in place, the majority of students still engage in rote thinking (Hiebert, 2003). Furthermore, Lithner (2008) also argues that it is the rote thinking—later characterized as MR—that is at the center of the "learning and achievement difficulties [in mathematics]" (Lithner, 2008, p. 255). Basing on a number of empirical studies (e.g., Lithner 2003; 2004) the researcher was able to address the issue of rote learning by identifying the central attributes of IR and CMR.

Mathematical reasoning being dynamic and unique and individual-based, we adopted Lithner's (2008) characterization of the different types of reasoning to investigate the types of reasoning pre-service teachers engage in while learning trigonometry. Several researchers have contended that mathematical reasoning is best evaluated by how a learner's performs on geometric tasks (Mason & Moore, 1997; Wu, 1996). In other words, geometric reasoning follows from successfully establishing mathematical reasoning (Clements &Battista, 1992). In this study, we therefore engaged students in tasks that were based on line-segment trigonometry, which is geometry based and promotes quantitative and covariational reasoning. This set-up of the study we anticipated would be encourage the participating students to not merely complete given tasks, but also engage in some form of reasoning at the same time.

Instructional Strategies

Numerous studies that have demonstrated techniques for teaching specific trigonometry topics (e.g.., Borba & Confrey, 1996, Clements & Battista, 1989, 1990; Keiser, 2000, 2004), while others have investigated instructional strategies for improving the teaching and learning of trigonometry as a whole (e.g., Hertel & Cullen, 2011, Moore, 2013; Weber, 2005). For example, while teaching function transformation, Borba and Confrey (1996) used the *rubber sheet* method to help students visualize graphical representations as being made of two transparent, ductile rubber sheets. Transformations were either defined as movements on the sheet of axes (i.e., horizontal transformation) or as movements on the sheet containing the curve of function inputs (i.e., vertical transformations). However, this approach had some limitations like difficulty in representing the shrinking and stretching effects on a function.

Kendal and Stacey (1998) measured progress from pre- to post-test of tenth graders who were instructed using either the unit circle approach or the ratio approach. From their findings students who used a ratio approach had a statistically better performance than their peers who were taught using the unit circle approach. Moreover, they contend that the former group is well positioned to correctly identify trigonometric expressions required in solving given tasks. Furthermore, beyond not providing low achieving students an outlet to overcome their struggles in basic trigonometry, Kendal and Stacey found the unit circle method to actually make it harder for students to specifically learn trigonometric ratios.

On the flip side, Weber (2005) reported that college students' understanding of trigonometric functions is hinged on introducing them to the unit circle first before the ratio method is used. He described the registered improvement in the experimental instruction group as being a direct result of their use of the unit circle model. Weber (2005) resolved that, "for the students who received the experimental instruction, … understanding the *process* used to create unit circle representations of trigonometric expressions appeared to be an integral part of their understanding of these function" (p. 107). Weber further claims that initially using the ratio method is of limited value for the learning trigonometric functions.

Other researchers (Hertel & Cullen, 2011; Moore, 2013, 2014) have proposed that using quantitative and covariational reasoning significantly fosters learning of trigonometric concepts like angle measurement, and trigonometric functions. Hertel and Cullen's (2011) directed length approach while investigating twenty-three pre-service teachers working on trigonometric functions in a dynamic geometry environment (DGE) registered statistically significant growth in the students' performance from the pre-test to post-test. This progress, the researchers contend was due to use of quantitative reasoning (i.e., directed length interpretation) and the DGE. Similarly, Moore (2013, 2014) suggested that an arc approach to angle measure can advance coherent experiences for students, and improve their thinking, which facilitates conceptualization of trigonometric functions and trigonometry in general.

Although the literature review has revealed several instructional strategies and their registered successes in aiding students' learning of trigonometry, the synthesized studies seem to fall short of conveying the importance of sense making. For example, Kendal and Stacey (1998) support the use of the ratio method while other researchers suggest the unit circle (e.g., Weber, 2005) or the directed length approach (e.g., Hertel & Cullen, 2011), but in their study Kendal and Stacey (1998) appear to only be testing the students on procedural tasks. It is on the basis of the results from these tasks that the researchers categorize students as "successful" or not. I argue that investigating students attempting such tasks does not tell the whole story of why the students are "successful" at learning trigonometry. As noted by Doyle (1983), the nature of tasks is important in students' work with mathematics that they "influence students by directing their attention to particular aspects of content and specifying ways of processing information" (Doyle, 1983, p. 161). Therefore, further insights about how the participants reason would be helpful in drawing more inferences about the different types of approaches to learning trigonometry.

The same could be said about Hertel and Cullen's (2011) directed length approach or Moore's (2013) arc approach to angle measure. Inasmuch as in both studies the researchers employed instructional sequences that promoted quantitative and/or covariational reasoning, the kinds of reasoning students exhibit while doing the tasks seem not to have been a central focus of their investigations. The type of reasoning students engage in is more important than their success on particular tasks. Moreover, the students could as well be engaging in rote learning or superficial reasoning (Hiebert, 2003), or as this study may later reveal, memorized reasoning for the most part.

In attending to a gap in the current research and owing to the wide-ranging use of trigonometry, I therefore find it valuable to examine students' reasoning as they work with trigonometric functions, and in particular what type of reasoning is exhibited when students elect to use a given approach (i.e., unit circle, ratio, or line-segment).

Summary of Chapter

This chapter provided an overview of the conceptual framework (Lithner, 2008) that I use a basis to categorize students' reasoning. A central component of this framework is that reasoning is not constituted of correct justifications utterances.

The research available has documented the struggles both teachers and students experience in their quest to teach and learn trigonometry. Additionally, students and teachers were reported as lacking introductory reasoning abilities that are prerequisites to understanding trigonometric functions, although few research studies have examined the role of these foundational understandings. This chapter concluded by highlighting some suggested instructional strategies that can be used to improve the teaching and learning of technology. In the next chapter, I provide the methods for this study, which proposed ways explore the reasoning abilities of the students as they investigate and hypothesize about different trigonometric functions and relationships.

CHAPTER III: METHODS

The principal focus of this investigative study is to examine the types of reasoning that pre-service secondary mathematics teachers exhibit and use in the process of reasoning about trigonometry with instructional emphasis on quantitative reasoning through a line-segment definition of trigonometry. With this goal in mind, it is imperative to consider the conceptions developed by pre-service teachers and their reasoning while working with the activities that are typically used for instruction in a secondary school trigonometry course. These were leveraged to help us understand the prospective teachers' struggles, and hence be in a position to suggest ways to not only improve trigonometric instruction but to also add to the growing body of literature on the learning and teaching of trigonometry.

Ultimately, this dissertation is intended to shed more light on how a particular approach (line-segment) influences prospective secondary mathematics teachers' reasoning about trigonometric functions. This may benefit the pre-service teachers by developing a strong comprehension of trigonometric functions while exploring their reasoning when attempting trigonometry-based tasks

Participants and Setting

The targeted population of this study is pre-service secondary mathematics teachers who were in advanced stages of their teacher training program. This population was yet to engage in student teaching, but gained some field experiences, working with cooperating teachers in area high schools. All the potential participants had completed almost all their mathematics content, mathematics methods, and other required courses. The only courses they had pending were the ones they were attending at the time of this study, including the course in which data for this study was collected. Additionally, all but one participant—who was a junior and was not selected to be part of the focus groups—were seniors, teaching some mathematics lessons to high school students in real classroom settings during their field experiences in the same semester. This targeted population had therefore acquired the familiarities of both an undergraduate mathematics major and a pre-service secondary school mathematics teacher at the end of her/his teacher training.

Recruitment

This study took place in a semester long technology-based mathematics content course for pre-service secondary mathematics teachers at a medium-sized Midwestern University in the United States of America. Participation was solicited from all 23 students enrolled in the course. All 23 students attended class for all 15 weeks of the semester, including the seven weeks in which data was collected. However, only 20 officially consented to having their data collected and released in publishable form. There were 11 male and 9 female students who participated in the study.

The participants (henceforth referenced as students) were chosen on a volunteer basis and were not be compensated for their time. In order to acquaint myself with the students, and the classroom environment in the aforementioned course, I attended all the class sessions—as a non-participating observer—prior to the seven weeks in which I collected the data for this study. This interaction prior to data collection was helpful in informing the selection of one of the focus groups. I selected these four members to be in the first group (henceforth referenced as Group A) because I had observed them thinking out loud and being willing to participate when working in a small group setting. The second group (henceforth referenced as Group B) was selected out of convenience as it just happened to be put together based on their willingness to participate in a different study that was running at the same time as my study.

Participants. From the 20 students who agreed to participate in the study, eight students were selected to be part of the focus Groups, A and B. Because the members of Group B were selected as a set group from another study, I did not sit with these students during class and thus did not get to query their thinking while they were working. With this in mind, I will only highlight those in Group A based on my interaction with them over the six weeks of the study. Unlike Group B, which consisted of one male and three females, namely: Ben, Monica, Annaliese, and Andrea, Group A consisted of two males and two females namely: Suzie, Eron, Mark, and Eddie. All these students were in the first semester of their senior year and I provide more background about each below.

Suzie was outgoing and always willing to speak her mind. However, she sometimes seemed unsure of her solutions and reasoning. Suzie routinely questioned her responses. Additionally, she intimated that prior to her college education she had not attended a trigonometry course. Mark was a very confident student who enjoyed working alone. Even in tasks that required students to work in pairs, he rarely engaged with Suzie his partner. Mark was always a step ahead of the rest group in terms of accomplishing the tasks. He was constantly exploring other concepts and going beyond what the instructor assigned. Eron and Eddie thrived in bouncing ideas off each other. Eddie's mastery of the trigonometric ratios and identities was also notable.

Course Overview

The course was a semester long technology-based mathematics content course for pre-service secondary mathematics teachers. The course was designed to focus on four roles of technology in the teaching and learning of mathematics: "(a) promoting cycles of proof (i.e., explore←→conjecture←→test/revise←→prove); (b) presenting and connecting multiple representations; (c) supporting case-based reasoning; and (d) serving as a tutee" (Cullen, Hertel, & Nickels, p. 5, in Press). Prior to the weeks which were the focus of this study, the students had completed three main activities; Exploring Quadratics, Series Problem Solving Task, and Excel can Solve Quadratics (Cullen, Hertel, & Nickels, in Press). These tasks were designed to engage students in promoting cycles of proof (Exploring Quadratics), supporting case-based reasoning (Series Problem Solving Task), and using technology as a tutee (Excel Can Solve Quadratics). The technology used during these activities was mainly GeoGebra, spreadsheets, and graphing calculators while the content covered was quadratics and sequences series.

During class meetings students were working in cooperative four-person teams assigned by the instructor for some classroom tasks, discussion, and projects in which they were expected to be active participants. Students were seated at round tables with a laptop provided for each student. The instructor gave students times to engage in small group discussions, and later share their ideas on the board. I attended all the class sessions from which I used the first five weeks to acquaint myself with students, in order to get an idea of who among the potential participants exhibited positive dispositions towards working in a group and so that they would be familiar with me. In the subsequent weeks in which I collected data; I assumed a participant-observer role. My involvement in the focus group (I was always stationed at Group A's table) was primarily to guide any discussion to a direction that was purposeful to this study while being careful not to overshadow the participants.

The instructor, who had taught high school for six years, was in his ninth year of teaching at a university, working with pre-service secondary mathematics teachers. The goals for the course, as stated in the course syllabus were to (a) extend and enrich understanding of selected mathematics topics, (b) enhance knowledge of and fluency with mathematics technologies,

(c) address the roles of technology in the teaching and learning of mathematics, and (d) gain fluency in investigating mathematics topics beyond what is obvious. This was the ninth consecutive fall teaching this course, and he had conducted a prior study using the line-segment definition of trigonometry in a prior teaching of this course. The instructor's focus when designing the lessons for trigonometry portion of this course was on quantitative/covariational reasoning. Employing lecture and direct instruction minimally, he aimed to give the students in the course an opportunity to deepen their conceptual understanding of trigonometry by engaging them in learning about the line-segment approach and making connections to geometry (i.e., chord, secant, tangent, complimentary) as well as connection among the different approaches (i.e., right triangle, unit circle, ratio, line-segment).

The focus of the instruction was on quantitative reasoning (Thompson, 1990). According to the instructor this focus was highlighted in a few ways. At the beginning of the instructional sequence, when the students were working to determine if the given dynamic display of the chord and arc was describable as a functional relationship, the instructor insisted that the students did not use any numbers in their explanations. The purpose of this was to deter them from reasoning indirectly about the quantities by referencing the measures of the quantities. By doing this, the instructor hoped to push the students to reference the actual arcs and segments (i.e., quantities) when they described or explained their reasoning rather than reducing the quantities down to an abstracted numerical representation (e.g., it seems this is function is the sine function because when the angle is $\frac{\pi}{2}$ the segment is 1). Additionally, when the students began to use numbers the instructor worked to direct their attention to the units being used. In particular, he had students build their own grid, in Geogebra, using the radius of a circle on both axes, as well as a coordinated display of the radius being dynamically marked off around the circle. Finally, the instructor's choice of using the arc angle as a quantity was purposeful in his mind. He was trying to avoid the use of angle as a quantity because it is more difficult to identify the object (i.e., quantity) than when using the arc as an object (i.e., quantity).

Why Study Pre-service Secondary Mathematics Teachers?

The following reasons underlie the decision to choose pre-service secondary mathematics teachers as the focus of this study.

First, results of this study may have implications for teacher education programs concerning how future secondary mathematics teachers are trained as far as teaching and learning trigonometry is concerned. The mathematics education world agrees that what teachers know influences what they ultimately do in their classroom (Wilson, 1992), and therefore having a grasp of pre-service mathematics teachers' reasoning may provide support and convalesce to the education of future teachers of mathematics. As Wilson (1992) stated, "analyzing the images of mathematics and mathematics teaching held by pre-service teachers is important because these teachers will significantly impact upon the nature of mathematics that will transpire in the future classrooms" (p. 1).

Secondly, researchers who have conducted synonymous studies on students' or teachers' reasoning with trigonometric functions (e.g., Moore, 2010, 2014), have recommended that similar studies should be carried out with different populations from the ones they studied. In particular, Moore (2014) recommended that, "the research base on trigonometry would benefit from studies that extend the current work to other populations including pre-service teachers" (Moore, 2014, p. 51).

Lastly, I have chosen to work with secondary mathematics pre-service teachers rather than pre-service teachers of lower grade levels because by the common core standards (CCSSI, 2010), trigonometric functions are part of high school standards (e.g., CCSS.MATH.CONTENT.HSF.TF.A.X).

Study Design

The study was conducted in a course that met for two hours and forty minutes, once a week for 15 weeks. The content of interest for this study was covered over a 7-week period, with the professor assigned to the course working as the instructor for all the sessions. The classroom instruction consisted of lecturing, individual and group work, and whole class discussion. The classroom was set up with a teacher chart that included a document camera, an interactive white board and a computer. Several topics including trigonometry were covered whilst highlighting the role of technology in achieving a deeper understanding of the topics. This sequence of instructional activities was delivered using a dynamic geometry environment (DGE) with the purpose of promoting quantitative reasoning through the line-segment definition of trigonometry. A synopsis for the 6-week instruction period is presented in Table 3, giving what happened in each week. The main topics covered in this study are included in Appendix B.

Table 3

Weekly Instruction Outline

Week	Instruction Goals
Week 1	• Create geometric quantities in a DGE and examine the relationships between pairs of covarying attributes. • Interpret the meaning of an ordered pair in a given context. • Explain the benefits of using the radius of a circle as a unit in a trigonometric setting.

(Table Continues)

Week	Instruction Goals
Week 2	• Extend student understanding of the chord function to the tangent and secant function. • Explain two methods for measuring an angle, one for radians, one for degrees and explain the units in each method.
Week 3	• Extend student geometric understanding of the chord and tangent to the secant, and to the three co-functions. • Explain the benefits of using the radius of a circle as a unit in a trigonometric setting.
Week 4	• Reason about the six trigonometric functions when the radius is not the unit. • Compare and contrast the use of radians and degrees in trigonometry. • Use a geometric interpretation of trigonometry to derive basic trigonometric identities.
Week 5	• Compare and contrast the use of radians and degrees in trigonometry. • Use a geometric interpretation of trigonometry to derive other trigonometric identities.
Week 6	• Use a geometric interpretation of trigonometry to derive the sum and difference formulas for sine, cosine, and tangent trigonometric identities.

Written Work

In order to classify students' solution schemes, their relevant written classwork and weekly reflections were collected and analyzed. This included classroom notes, weekly reflections, assignments and group projects.

Data Collection and Analysis Overview

The data collected for the study included:

- Videotaped teaching sessions (six 2-hour and 40 minute-sessions)
- Student written work from the teaching sessions.

Data Analysis Procedure

I first organized the collected data in the order it was gathered. My preliminary analysis consisted of viewing all recorded videos in the order they were captured. To gain an overview of each participant's behaviors, understanding and reasoning pertaining to trigonometric function, I took notes while viewing the videos. I did a loose transcription of all the data in order to identify segments of the video irrelevant to the study at hand (e.g., segments in which the students are off task, students are engaged in learning to use the technology).

After my initial analysis, the relevant portions of the classroom were fully transcribed and reanalyzed to include the participants' and researcher's statements, and actions. This was followed by creating possible explanations and descriptions for the reasoning and understandings exhibited by participants through their actions and utterances.

Using Lithner's (2008) framework, I examined classroom videos to classify reasoning exhibited by students, and then associations were sought between participant's actions, by comparing, and contrasting the group's actions over the progress of the six weeks for homogenies cognition, and abstraction. This analysis, was to reveal the "critical reasoning abilities needed for constructing connected and coherent understandings of trigonometric functions" (Moore, 2010, p. 52).

Overview of Coding

Unit of analysis. While classifying the different reasoning types based on Lithner's (2008) framework to identify the types of students' reasoning and strategies exhibited, I categorized the unit of analysis as a single sentence (utterance) by any participant, with or without justifying their statement. In the event that a block of statements was stated about the same concept, the several mathematical statements contained therein were coded independent of each other.

I also found any patterns and themes that stemmed from associating the different types of reasoning revealed in either having come from line-segment trigonometry, unit circle trigonometry or right triangle (ratio approach trigonometry). Eventually, a count of each type of reasoning, and teaching strategies about trigonometry was made, the emerging themes analyzed, and conclusions drawn.

Summary of Chapter

In this chapter, I described the research methods of this investigation into pre-service secondary mathematics teachers' (referred to as students) reasoning about trigonometric functions. An instruction sequence that was based on the line-segment approach was implemented in order to promote quantitative and covariational reasoning.

In the next chapter, I present the findings from selected tasks from the instruction sessions. In each task, I highlight the different types of reasoning exhibited by the students in their strategy choices, as well the trigonometry approach (ratio, unit circle, or line-segment approach) they may adopt.

CHAPTER IV: RESULTS AND DISCUSSION

In this section, I present the findings from the instruction sessions. A typical session was a hybrid of instructor-led demonstration, group discussions, and working in pairs on a computer to construct a given task using a dynamic geometry software.

Details about the general and group discussions, and the resulting forms of reasoning exhibited from two of the six groups will be presented. Data reported here were taken from five episodes identified by the researcher to have been focused on mathematical content rather than learning to use the technology itself. These episodes are focused around key tasks, which occurred during the study.

Task 1: Identifying the Sine Function

The goal of this task was to have students build geometric objects and then identify different attributes on/about the objects and observe the relationships between them as well as changes in the attributes. In this session, instruction focused on identifying functions in general, with particular emphasis on exploring the "chord function" using the connection between the arc length and the corresponding directed length as shown in Figure 4.

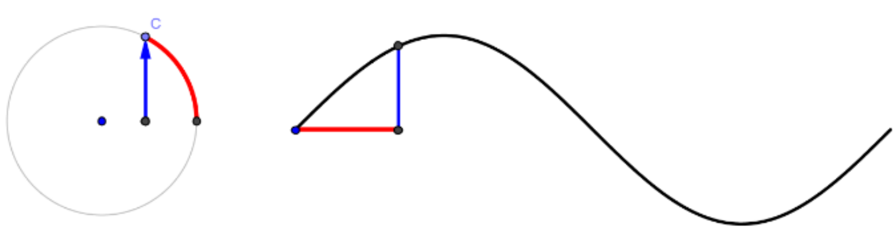


Figure 4. Varying the arc length and the directed length, and the resulting graph

An overview of the mathematical content registered from the students as they discussed the four parts of this task about functions is summarized in Table 4 below. Percentages of each reasoning type used by the students are included, as well as a general view of the process that was taken to resolve this task at hand.

Table 4

Resolving Task 1: Is this a Sine Function?

Percentages of Reasoning Strategies		Task Part	Math Content from students
CMR	Imitative		
25%	75%	What is a function?	• f(x), g(x) • Table of values • Vertical line test • Dependent/Independent variables
20%	80%	Chord: Is it a function? (see Figure 5 [a])	• x-coordinate and y-coordinate • Number of inputs • Parabola • Dependent/Independent variables • Sine function • Relation
29%	61%	Directed chord: Is it a function? (see Figure 5 [b])	• Sine function • Arc length and chord length • Dependent/Independent variables • Stretched sine function • Amplitude • Negative cosine
		Directed half-chord: Is it a function? (see Figure 5 [c])	• Sine function • Arc length and chord length

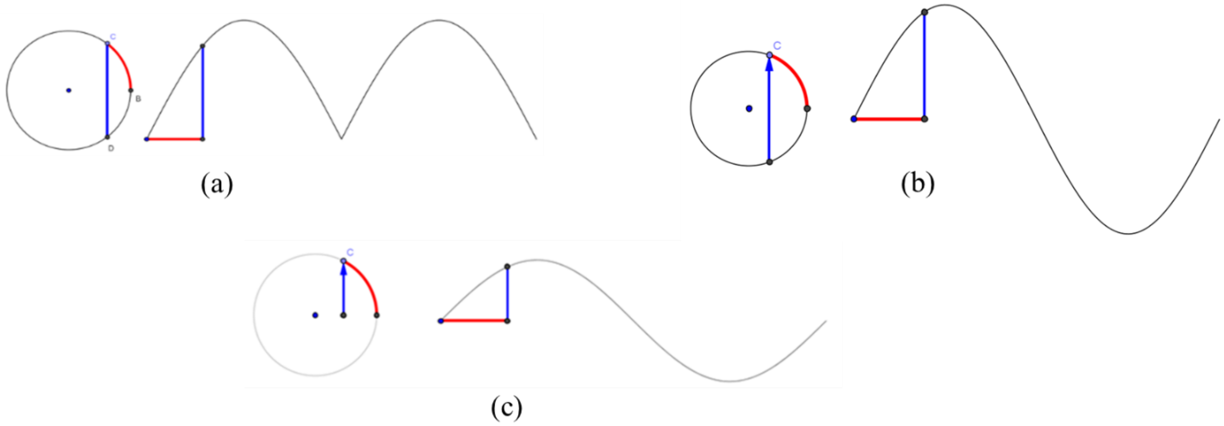


Figure 5. Investigating the chord function

Beyond highlighting the history behind the word sine from the mistranslation of the word "chord" (Bressoud, 2010), this exploration engaged the students in reasoning about what constitutes the independent variable—the angle size represented by the red arc—and the dependent variable—the blue directed half-chord—in relation to a sine function.

This description first provides Group A's – consisting of Suzie, Eron, Mark, and Eddie – reasoning and justification and then the general classroom's reasoning and justification about the broad concept of functions. This is followed by a discussion of the dialogue about identifying and characterizing the sine function, and the various types of reasoning employed in this learning activity.

To start the task, the instructor declared that he would be moving slowly in this session, with a focus on multiple representations as a role of technology. He also situated the topic of functions as one of the strands in the Common Core State Standards (CCSSI, 2010) for High School mathematics, namely: Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics & Probability. The instructor then elicited ideas from the students. The last column in each excerpt shows the codes to represent the types of reasoning (see Table 2). The statements are coded as either Imitative (I) or Creative reasoning (C). Those coded as Imitative, are further categorized as either Memorized (MR) or Algorithmic (AR). AR has variants that include Familiar (AR-F), Delimiting (AR-D), and Guided (AR-G). The last hierarchy consists of subcategories of Guided Reasoning, which are AR-TG, and AR-PG.

The instructor commenced the session by declaring his interest in focusing on the idea of functions, with the hope that the students would be pushed to think of them differently.

Instructor: In my mind when someone says functions, I immediately think of something like $f(x) = 3x + 2$. Today I want us to think about geometric functions and I want to avoid using any numbers as long as we can. What are functions about?

From the group discussions and later transitioning into individual responses, the students were able to recall certain information, and made reasonable claims about functions, but were unable to present lines of thought that would be justified as reasoning. The responses are given below

Group A: Like the table, with x and y

Ben: Vertical line test

Mark: Like f(x) and g(x)

Participants working in their groups described the concept of function based on what they could recall from earlier knowledge about functions. In particular, Group A members did not have a detailed discussion on the question posed by the instructor. When one of the participants (Eron) presented her view of functions as tables with x and y, the rest of the group agreed and had no further discussion.

When prompted further by the instructor—during the whole classroom discussion—all the participants' responses indicated that their perception of functions included tables, vertical line test and notations like *f(x)* and *g(x).* These were some of the notions related to functions that they presented.

After this brief introduction about functions, using GeoGebra, the instructor created a 1-dimension geometric object that served to generate examples used to examine the relationship between components on a circle. The objective was to lay the foundation for exploring the chord function, purposefully avoiding the use of numbers. A half an hour into the teaching session, the instructor having constructed the circle, an arc along this circle from a fixed point (B) to a moving point (C) on the circle, and a vertical chord through the moving point, the participants constructed a similar object in GeoGebra (Figure 6).

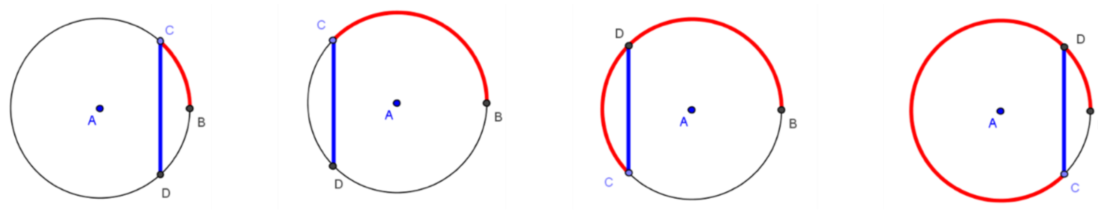


Figure 6. Covariation of arc BC and chord CD at different positions

Students then deliberated on how altering the arc length affected the chord. With varying point C now animated counterclockwise, the instructor asked if what was on display could be regarded as a function. According to the instructor, this question was intentionally vague to force students to explicitly identify what it was they were considering as the attributes of the diagram that they were relating. After two minutes of group discussions (Excerpt 1), there was a whole class discussion (Excerpt 2) to address this question. Again, adopting Lithner's (2008) definition of reasoning, from the data reported, I only coded reasoning was that directly related to the task of identifying a function, but for purposes of context, other statements are also included in the excerpts.

Excerpt 1

1	Eron:	It could be like the output. The point being…, I don't know if I can use the point again.
2	Suzie:	The line being the y-coordinate and the point being x-coordinate.
3	Mark:	Thinking about the dependent and independent, the length of the arc is the independent variable and the length of chord is the dependent variable.
4	Dr. Kay:	*(With the instructor now at their table)* Is the length of an arc a number?

5	Eddie:	I don't see it that way. There are two spots on either side of the center, which means two inputs of arc lengths having the same chord as output.	AR-F
6	Mark:	I would say that is fine. Think about the parabola. Two x-values will give one y-value and it is still passing the vertical line test. So, it is okay if you have one output for two inputs.	CMR1
7	Eddie:	Oh yeah, I see what you mean.	AR-PG
8	Mark:	But yeah, I agree with you if you are thinking about the inverse of the function. Think about the unit circle. At 0, we are zero, at 90, bang, at 1, at 180, we are at zero, and 270, bang.	AR-F

During this interaction, the students' reasoning on functions were littered with strategy choices that seemed to be familiar and memorized except for one case. In her interaction with Eron to determine if the object presented by the instructor constituted a function, Suzie explained that it was a function, with "the line being the y-coordinate and the point being x-coordinate." This response seemed to come from her prior knowledge about similar tasks but without any justification. However, when Mark's claim that "the length of the arc is the independent variable and the length of chord is the dependent variable", was contested by Eddie, Mark was able to present a plausible argument to support his assertion (Line 6, Excerpt 1).

Other students we also able to make progress on this activity. From their responses in the whole class discussion (Excerpt 2), they appeared to make a case for what would and what would not constitute a function.

Excerpt 2

1	Dr. Kay:	Let us try to chat about this as a group. It seems a bit open ended or odd to just say, is this a function? What do we think, is this a function? (puts thumb is the air for a thumbs-up and a thumbs-down)	
2	Class:	[Thumbs up in agreement]	
3	Dr. Kay:	Okay, Ben, tell me about that function (points to diagram).	
4	Ben:	I believe it is a sine function. I believe … If we're talking about, um, I'll have to get this right. Position of the dot, the intersection would be the independent variable.	AR-F
5	Monica:	He is crazy	

6.	Ben:	And that length of the chord is the dependent variable. I believe that would be the sine function. Independent is the uh it's referring to the position of the point C in radians and then the dependent is the length of that chord which would, should give you the sine function I believe.	CMR2
5	Smith:	Can we say arc length?	
6	Victor:	Independent variable: Length of arc Dependent: Length of chord	AR-F
7	Dr. Kay:	How about using numbers? [There was agreement not to use numbers]	
8	Victor:	There are four different arc lengths that give you the same chord length.	AR-PG
9	Dr. Kay:	Is this a function?	
10	Class	[Yeah.] [Yes.]	AR-F

When the instructor solicited more observations from the whole class, Ben claimed that the function in the task was a sine function. However, he reluctantly justified his argument backed up by numbers, using radians to quantify in his justification. With the students free to use numbers in their reasoning, the majority of the students were able to conclude that there was an interpretation of the relationship between the two one-dimensional objects that constituted a function.

However, the instructor purposefully insisted that the class continue to think about this activity in terms of the "red thing" and the "blue thing" (referring to the colors of the arc and the chord). Things should be viewed as the "input, output, independent, and dependent variable", he said. Furthermore, in an effort to confirm if the students were making mathematically founded arguments, there was dialogue about relations and functions. From their unison submission, it seemed that they had been reminded of what generally can be viewed as a function.

Dr. Kay.	Can we switch the dependent and independent variables and still call it a function?
Class	[No.]
Dr. Kay:	Do we still get a relation?
Class	[Yeah.] [Yes.]

In order to draw on this foundation about functions, the instructor and later the students—working in pairs on their laptops— made copies of the two quantities (arc and the chord) and mapped them on the horizontal and vertical axes respectively and created a permanent record of the associated lengths. This resulted in a graph shown in Figure 7 and a subsequent change of thought from the participants. For example, equipped with hindsight, Ben stated that, "this is not a sine function as earlier predicted (Line 6, Excerpt 2). It is the absolute value function. It (the graphical representation) helps see one point and the two associated relationships (quantities)."

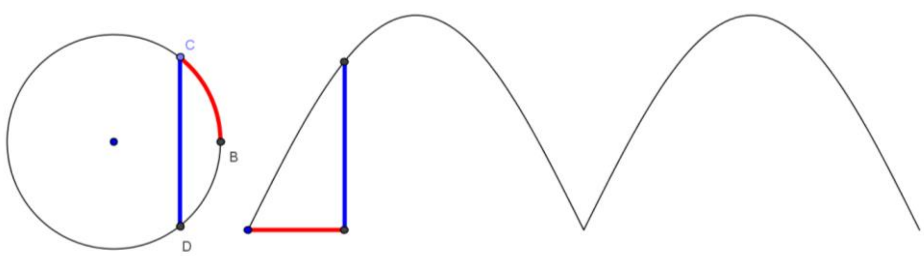


Figure 7. The graph generated by relating the arc and the chord

After this interaction, the chord was replaced by a directed chord displayed as a vector from D to C, and with the students mapping the arc on the horizontal axis and the directed chord on the vertical axis of a cartesian-coordinate grid, a locus of the copy of the different chords and positions resulted in the associated graph of the function $y = 2\sin(x)$ (assuming the radius of the circle is the unit) as shown in Figure 8. Again, students were tasked to investigate how the change in the size of included arc affected the directed chord (Excerpt 3). This then transitioned into a connection to the origin of the word "sine". Noticing that the directed chord did not yield the intended graphical representation, attention was then drawn to building a display that is consistent with the standard unit circle representation of sine where half of the directed chord was used as the value of the sine of the associated arc. In this case, the resulting geometric function (Figure 8) offered the students an avenue to adequately explain their earlier claims about the sine function. Most explanations referred to a diagram of the circle illustrating how when the arc changes as the arc moves along the circle in a counter-clockwise direction, the half-chord goes back and forth between increasing and decreasing.

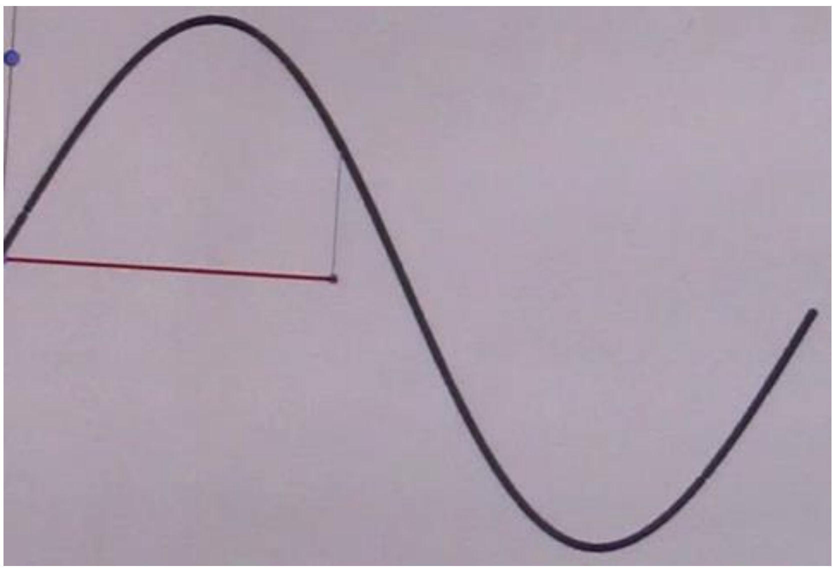


Figure 8. Amy's graph for comparing the arc length and directed chord length

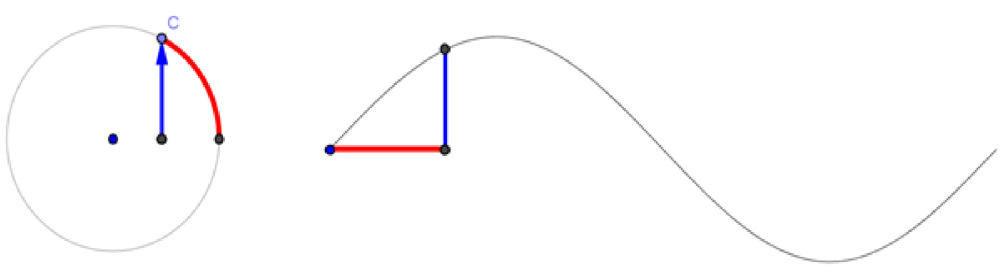


Figure 9. The sine ("directed half-chord") function

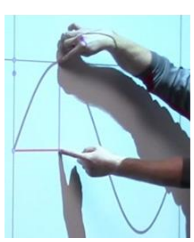
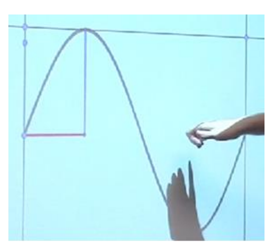
Excerpt 3

1	Dr. Kay:	What do you think? (Pointing to a transformed version of the sine function stretched vertically, as shown in Figure 8).	
2	Ben:	It looks like the sine function. A vertically stretched version of the sine function.	AR-F

3	Smith:	The amplitude is not 1	MR
4	Andrea:	I have known always the sine function to be measuring the length from the center to the circumference of the circle. Since we are dealing with a chord from one point to another on the circumference, it wouldn't be the same sine function that we think of but it would be a stretched one.	CMR3
5	Dr. Kay:	What is the problem with this picture? Without the circle, looking at only the graph, would someone be able to say if this is a sine function?	
6	Smith:	Yes. Because the sine function has a few key characteristics, one being there is a period before it starts over. It starts between the top and bottom and they are equidistant from the line, and the curve hits the horizontal line 3 times.	AR-F
7	Andrea:	I agree, but we have to restrict this to a certain domain. Because this could also be a negative cosine.	CMR4

To further the discussion, the instructor lifted the restriction on using numbers in their explanations to justify their claims of whether or not the displayed geometric construction represented a sine function. From the discussions in their different groups, the students initially had diverging views but ultimately realized that the graph represented a sine function or a function analogous to the sine function.

Excerpt 4

1	Dr. Kay	What is the length of this? (pointing to the amplitude of the graph) How many of these will fit (referring to how many lengths equivalent to the amplitude will fit in the horizontal distance equivalent to a full revolution)?  	
2	Eddie & Drew	2π	AR-PG
3	Henry	I think it is the sine function because, the values for which it is zero are shown as being at zero, and since we have no scale, we can assume that those points represent 0, π and 2π. And it goes up to 1 and down to -1.	AR-F

4	Dr. Kay:	Henry said that lets call this (maximum of the graph – from the horizontal axis to the crest of the graph) 1 and call this (the horizontal) $2\pi \approx 6$. Will 6 of those fit here?
5	Smith	3 not 6
6	Dr. Kay:	We can't fit. I feel uncomfortable to just call this 1 (points to amplitude) and call that 2π (points to the horizontal axis). You can choose one and call it what you like but you don't get to choose both. I would say that is not the sine curve. Because when you get to call that 1, you don't get to call this 6.28 and some change. Does anybody know what sine means

The participants' approaches to the task of justifying their understanding of the chord function indicated some considerable alterations in their strategy choices. There is evidence that participants had initially not regarded angle measure and by extension the reasoning about the chord function, as a procedure that engaged in using the arc length and thinking beyond the attributes of a familiar function, they had seen in prior encounters with the sine function. For example, even when the chord segment was changed to a directed one, and Andrea pointing out that the sine function measures the "length from the center to the circumference of the circle", Smith and Ben did not utilize the length, position, and orientation of the directed chord to justify their reasoning.

In summary, during this first instruction session in which students were tasked to identify what constituted a function, their reasoning strategies comprised of referencing various mathematical concepts (e.g., tables of values, f(x), and g(x)) from prior knowledge of the concept of functions, and implementing strategy choices of familiar algorithms (e.g., coordinating the value of one variable with changes in the other to create and reason about geometric objects like line segments and arcs). The task's next focus of comparing the chord length and arc length resulted in students reasoning imitatively by only focusing on the amount of change in the chord length in relation to the arc length without showing any discernment for the direction of the segments involved.

The students' engagement in the directed chord-length part of the task resulted in creating imageries of measuring the arc length in terms of the number of radians. Beyond verbalizing that inputs for several arc lengths have the same chord lengths but with different directions, the students implemented AR-F to identify key characteristics of the sine function. While working on the task of constructing a geometric representation of the covariation between the directed chord and arc length, the students transitioned from algorithmic to CMR (see Excerpt 3). For example, in the same breath, while engaging in the directed-chord task, Ben was able to transition from claiming the it was a sine function to then asserting that it was "a vertically stretched version of the sine function" (Line 2, Excerpt 3).

The directed half-chord part of the task offered a context that some students leveraged to reexamine their earlier reasoning and reconstruct forgotten reasoning strategies and in turn created a reasoning sequence that led to a plausible conclusion that the function was a "chord" function. For instance, when working with the graph of $y = 2\sin(x)$, Henry explained that it was "the sine function because, the values at which it is zero are shown as being at zero, and since we have no scale, we can assume that those points represent 0, π and 2π. And it goes up to 1 and down to -1". However, Smith reasoned that if it were the sine function ($y = \sin x$), six not three [as it were the case for $y = 2\sin(x)$] lengths equivalent to the amplitude would fit in the horizontal distance equivalent to a full revolution.

C (4)	**I (12)**				
4	MR (1)	AR (11)			
	1	AR–F	AR–D	AR–G	
		8	0	AR–TG	AR–PG
				0	3

Figure 10. Counts of students' reasoning while investigating the "chord" function

The counts in Figure 10 indicate that participants for the most part justified their reasoning strategies about the chord function using variations of IR. As this was the first task, AR-F was dominant as students heavily depended initially on guidance from the instructor, and subsequently from their own reflection on the whole process of attaching meaning to quantities. Concentrating on getting familiar with the technology, and with the varying quantities in the diagram, while at the same time trying to interpret and cope with the instructor's direction to avoid numerical values, might have encouraged students to heavily rely on their prior knowledge of functions, which led to 60% reasoning falling in the category of AR-F.

While investigating this task, in fourteen scenarios students offered mathematically valid justifications concerning the concept of a function. Of the fourteen instances, in four of them the reasoning was creative with students' utterances that were novel, supported by plausible arguments, and founded on intrinsic mathematical properties, while ten of them were based on IR. There was one case of MR, with the other nine being AR, and more specifically AR-F. This is in tandem with Lithner's (2008) assertion that "Familiar AR is common" (p. 268).

A deeper look at the four cases of CMR exhibited in this task reveals three key features that supported students in these instances. The first relevant feature of the milieu is the students' prior knowledge. Of the four instances of CMR in task one, three of them (CMR2, CMR3, CMR4) depended on the students' prior knowledge. For example, when Mark invoked the vertical line test (Line 6, Excerpt 1) to explain the concept of function to Eddie, his reasoning was considered CMR because he was able to re-create a reasoning sequence from his prior basic knowledge without necessarily just stating complete answers like in the case of MR. Also, Mark's arguments maintained a logical validity in his plausible reasoning. For example, while explaining why even with two inputs and one output a function still exists, he went beyond stating—"think about the parabola"—something he knew beforehand, and guided the discussion in a direction that re-created (to Eddie) a correct reasoning. Moreover, Mark's arguments were mathematically founded on the intrinsic mathematical properties of an example of a parabola, which he from his prior knowledge knew was a function.

In all the three identified cases (CMR1, CMR2, CMR3), the students (Mark, Ben, and Andrea respectively) wouldn't have been able to reason creatively about this task without the prior knowledge. Mark and Ben both invoked the idea independent and dependent variables, with Mark also pointing to an example of a parabola, while Andrea leveraged her knowledge of geometry of the circle and its properties to be able to make valid and plausible arguments about the concept of a function. In fact, as evidenced in Excerpt 3, it is only on the basis of her prior knowledge that Andrea was able to pull the majority of the class back from concluding that the relationship between a directed chord length and an associated arc length results in a sine function by making plausible arguments (Line 4, Excerpt 3) about the circle properties and the attributes of the sine function. Without that prior knowledge the students wouldn't have been able to reach the plausibility and mathematical foundation requirements for CMR. The students were able to draw on this knowledge to navigate the task.

The second key feature that supported students moving into CMR was particular instructional techniques. Of the four instances of CMR in task 1, the instructional techniques played a role in one instance (CMR1). In this instance (and for much of the rest of the instruction session) the instructor not only encouraged students to work in groups but also used prompts that were purposefully vague so as to force students to explicitly identify what it was they were considering as the attributes of circle that they were relating. Furthermore, another of the instructor's technique was to not permit students to use numerical values in their explanations. With the students limited from their go-to strategy it pushed them to think of the concept of functions differently, and in the due process Mark prevailed with arguments that did not depend on solution algorithms as in AR. Even when the instructor asked whether the "length of an arc is a number" (Line 4, Excerpt 1), Mark instead provided a valid argument by anchoring in the vertical line test as a property that can be used on graphs of functions. Perhaps Mark could not have achieved the level of creative reasoning he did at this stage in the task without some of the instructor moves identified here.

Notably, the four instances of CMR did not all occur at the same time, rather at different intervals during the instruction session while students were working in a group. By taking a step back from leading the discussion and letting the students work in groups, in these instances the instructor not only avoided drawing students towards person guided IR but also afforded them access to other group members which helped them to be able to bounce ideas off of one another and produced some cases of creative reasoning. For example, when Smith stated "the sine function has a few key characteristics, one being there is a period before it starts over. It starts between the top and bottom and they are equidistant from the line, and the curve hits the horizontal line 3 times" (Line 6, Excerpt 3), Andrea's response anchored in intrinsic trigonometric properties, that, "… but we have to restrict this to a certain domain. Because this could also be a negative cosine" (Line 6, Excerpt 3) was borne out of the freedom students had to work in a group and share ideas and also not having the instructor as the center of the discussion.

Task 2: Investigating the Tangent Function

After Task 1, the students were asked "which function should we investigate next?", and they all opted for the cochord (cosine) function. However, after emphasizing the necessity to explore a function not derived from other functions (i.e., cofunction), the instructor suggested investigating the tangent function next. With the understanding that geometric objects (e.g., chord and arc) can be used to develop an alternative view of trigonometric functions, for Task 2, the instructor requested the students to draw from the previous task (Task 1) to explore, and reason about the tangent function.

Mathematical content registered from the students as they discussed the three parts of the task, namely: (a) identifying the tangent function in the diagram (with limited guidance or no guidance), (b) identifying attributes of the tangent function, and (c) identifying the tangent function in the diagram (Instructor-guided), is summarized in Table 5 below.

Table 5

Resolving Task 2: Investigating the Tangent Function

Percentages		Task Part	Math Content from students
CMR	IR		
8%	92%	• Identifying the tangent function in the diagram (With limited guidance)	• SOH-CAH-TOA • Tangent to a circle • Slope

(Table Continues)

Percentages		Task Part	Math Content from students
CMR	IR		
	100%	• Identifying attributes of the tangent function	• Sine is zero • Cosine is zero • Undefined function • ASTC (All Students Take Calculus
40%	60%	• What constitutes a tangent • Identify the tangent function in the diagram (Instructor-guided)	• Opposite over Adjacent (SOH-CAH-TOA) • exsecant • Tangent to a circle • See Figure 11 below

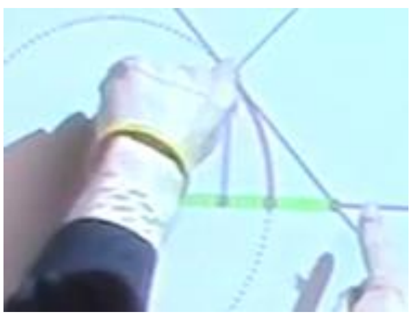


Figure 11. Example of a tangent function

In the first part of the task, the students had to identify the tangent function without any guidance from the instructor. They were tasked to answer the following question: In your group, discuss all that you know about the tangent function AND the word tangent from a geometry perspective. What do you need to add to your construction so that we can create the tangent function? The second part of the task was to use the attributes of the tangent function to help the students refine their conjectures made in the first part. The instructor asked the following questions. When is the tangent function 0, and when is it undefined? When is the tangent greater/less than 0? How do you know?

Based on their now refined conjectures, the students were requested to construct a graphical representation of the tangent function as the instructor revisited the earlier questions about the attributes of the tangent. Percentages of each reasoning type exhibited by the students are also included. In order to leverage knowledge that the students developed from the previous task about the chord function, the instructor asked them to “think about the picture we have, and what do we need to add to it to get the tangent function?” Having earlier been tasked to identify the tangent line in their groups, small group discussions ensued.

Group A members seemed to lack the ability or inclination to reason about the tangent function without the instructor’s guidance. There was no discussion on the task from the members in Group A. Two of the four group members (Eddie and Eron) were catching up with building the previous task to its conclusion, while the others (Suzie and Mark) were not proceeding with the task at hand. In order to get them working, I probed Mark about the task, and all he could say was “we are really stuck on the tangent”. I probed further by asking Mark, what he understood by the term tangent, but before he could air his views, the instructor decided to refocus the direction of the discussion on the task by posing other questions about the attributes of the tangent function to the entire class (Excerpt 7).

Conversely, while working on the same first part of the task—identifying the tangent function in the diagram—Group B members (Ben, Monica, Annaliese, and Andrea), also working in pairs (Ben with Monica, and Annaliese with Andrea), had a substantial dialogue as they attempted to describe the concept using geometry. In Excerpt 5, I present the interaction between Ben and Monica.

Excerpt 5

1	Dr. Kay:	Where is the tangent in that thing? (Pointing to the circle representation). Start figuring it out. Go!	
2	Monica:	So, if we think about tangent in geometry, isn't that like tangent to a circle? It's like where it hits the circle at one point. So, are we gonna have two…?	MR
3	Ben:	Yeah but when you talk about tangent, I, I usually think of slope.	MR
4	Monica:	Uhm …	
5	Ben:	Which, not to hold, this is the slope. (moving his mouse over the segment between the origin and the point on the ray). This (referring to the blue segment) over this (referring to the segment again between the origin and the point of the ray) is your slope.	AR-F
6	Monica	Yeah	
7	Ben:	Now how to do that here. So, one thing is for sure, we need a ray. No, we need a vector. From this (referring to origin) to this (referring to the point on the blue segment). And if I may, if I may, color this um	AR-D
8	Monica:	Couldn't you have done like this and then make it perpendicular to that. That would give you an …	
9	Ben:	That would give you the tangent line.	AR-PG
10	Monica:	The what exactly?	
11	Ben:	But how do you, we like 'cuz I assume the goal is to still do this without using numbers, right? We're supposed to be able to ignore these (referring to the left column with numbers). Right? And if you make the line perpendicular to this then I don't know how you're other than calculating the slope (*air quotes*), I don't know what's that's gonna do.	AR-D

After Monica stated that the tangent "it's like where it hits the circle at one point" Ben's contention about the tangent being related to the slope altered the direction of the discussion. The students again described the concept of the tangent function without any connection to the geometric concept of a tangent line (i.e., touching but not crossing). Although the students' line of thought represents the idea of tangent being defined as a ratio of sine to cosine, their explanation did not exhibit perceiving of the tangent in geometric terms.

In an effort to initiate a dialogue about the geometric view of tangent, the instructor moved to their table and sought clarification on the end result of their current line of though. To this end, their concept of the tangent was still anchored on the concept of slope. Ben claimed that Monica wanted to add a ray to go through the variable point on the circle and then "do the perpendicular which is the tangent line". He continued by reiterating the need to "talk about the slope of that tangent line if I'm not mistaken", much to the chagrin of Monica who responded with a mere "Ugh".

To justify his view point, Ben further described the need for considering the idea of a slope of the line segment when defining the tangent (Excerpt 6)

Excerpt 6

1	Ben:	You do because the tangent function has zero at zero. So, the slope of this (*referring to green ray*), well it's technically undefined because it's (*makes a vertical hand motion with the hand*). Uh, and then at 90, wait hold up. That's getting confusing because at 90 it's gonna be zero. So, at 90, the tangent function is like asymptotic, right? There're asymptotes here (*positions his hands in a vertical way and shifts them to represent the tangent asymptotes*), alright. And...	AR-F
2	Monica:	At π and ... no at zero and π.	AR-F
3	Ben:	At zero and π?	MR
4	Monica:	Because tangent at 90 is ... So, it's y over x. and then x is 1 at… y is zero and x is one. So that's already zero so yeah at 90 it's zero. Zero and π it's undefined. Those are its asymptotes.	CMR1
5	Ben:	Oh, I think I got confused because I thought it started at zero but this is an asymptote (*pointing to the grid*) and that's an asymptote (*pointing to the grid*) and at this peak is where it crosses the x-axis. So, it is gonna (*makes a motion to indicate a graph of the tangent function with his finger*) something like that.	AR-PG
6	Monica:	Yeah	
7	Ben:	So, then what did you want to do with the tangent line is my question.	AR-D
8	Monica:	Uh-huh, yeah. Well you just said … think back to geometry and that's what I thought of.	AR-PG
		So, I don't … but I, because I just don't know what else you would do.	AR-D

Up to this point, Monica and Ben's responses were typical of the dialogue in the other groups. The students' reasoning thus far was devoid of the role the geometric figures played in their understanding of the tangent function. At this moment, the instructor interrupted the dialogue between Ben and Monica even though Ben still wanted more time to continue with the task. The instructor promised to set the students "free again" after asking a "couple of questions" to guide the discussion on the task. There was then a small side task in which the instructor had the students (as an entire class) think about the attributes of the function and hopefully lay a solid foundation for the students to fully grasp the idea of using geometry to define and investigate the tangent function (Excerpt 7). Additionally, the instructor wanted to be able to establish a baseline for the students' reasoning related to answering these questions using their current knowledge of tangent function. This would provide an opportunity to compare this reasoning to their reasoning after being introduced to the tangent function defined as a direct line segment.

Excerpt 7

1	Dr. Kay:	When is the tangent zero, undefined, positive, negative, and how do you know?	
2	Joshua:	The tangent is zero when the sine is zero.	AR-F
3	Dr. Kay:	How do you know that?	
4	Joshua:	Because tangent is defined as sine over cosine, so when the numerator (sine) is zero, the tangent will be zero.	MR
5	Dr. Kay:	When is it undefined?	
6	Class:	[Chorus answers] When the cosine is zero	AR-F
7	Dr. Kay:	Because?	
8	Class:	[Chorus answers] When the denominator is zero the function is undefined.	MR
9	Dr. Kay:	So, you guys are just thinking of the tangent as sine over cosine. How about when is the tangent positive or negative?	
10	Lauren:	[Yells out] All students take calculus	MR
11	Dr. Kay:	Tell me more about your idea	
12	Lauren:	The first quadrant is when all the trig functions are positive, the second quadrant is when sine is positive, the third is when tangent is positive, the last is when cosine is positive.	MR
13	Suzie:	[Seemingly surprised] What is this? Why is it even a thing?	

14	Dr. Kay:	I do not know about you guys, but this does not scream conceptual understanding to me, but it is what I talked about when I taught (*high school*).

To further guide the discussion into viewing the tangent function through the geometric lens, the instructor purposefully decided to press the students on the attributes of the tangent function and the output of the tangent function at different points in a circle or for a varying angle measure. The students described the tangent being zero by looking at when the "sine is zero" and undefined by identifying when the "cosine is zero". One student further described how she used the mnemonic ASTC (All Students Take Calculus) to determine when the tangent function is positive or negative. At this point, the instructor pointed out that from using the mnemonic ASTC, one would not exhaustively mention the other functions (e.g., secant) that are positive or negative in each of the four quadrants.

With the instructor's guidance, students were able to reason about the varying positions of the tangent line, attributes of the tangent function, and how the segment representing the tangent function covaried with the arc length. Although somewhat not totally immersing themselves in the mechanical step-by-step analysis that characterized the initial part of the task, the students while working on this part of the task largely exhibited MR.

Next, when the instructor asked the class where the tangent line was, Mark described what the tangent means. Mark offered the following response:

Dr. Kay: Where is the tangent line in the circle? You have a basic start (*pointing to the circle representation*).

Mark: Is there a way that if you draw a segment or a line from the center to the point that's being manipulated, could we, I know it's still that same idea as sine over cosine, but could we look at the slope of that for the tangent?" [*Mark then goes to the front to demonstrate what he wants to be done*]. You get 0 at 2π.

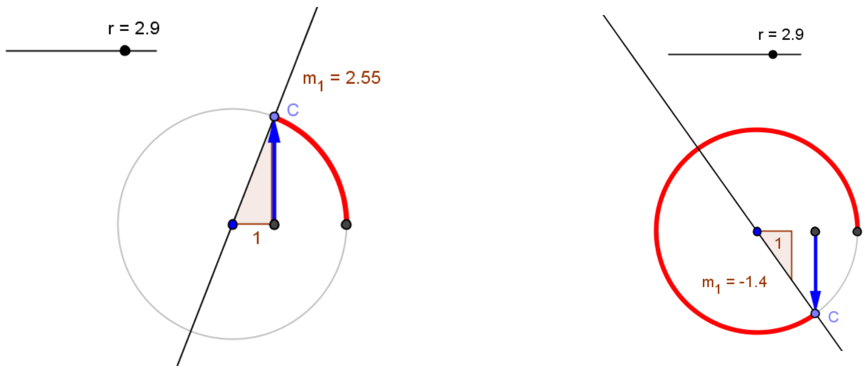


Figure 12. Mark's idea of defining the tangent using slope with point C at different positions

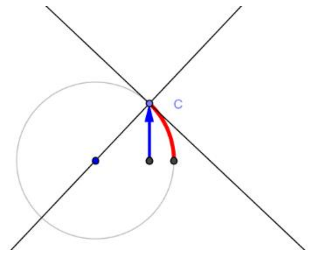
Adding to his submission, when the variable point, C, was animated, leading to different values of the slope, Mark made further descriptions of the tangent but involving the use of numbers. To this end, the instructor interjected and reminded the class about his desire to avoid numbers as earlier stated and also focus on the tangent from a geometric perspective. In response to Mark's conjecture, the instructor further stated, "this is cool, I think it might work, I have a lot of faith it may work but the question is, you used numbers and also we want to make a connection to the word tangent in geometry".

With the students input on how to add the tangent on the existing circle all debated and challenged by either their peers or the instructor, the instructor again asked the class to connect the word tangent to some part on the circle as they did for the chord while investigating the chord function in Task 1. Andrea then suggested using a perpendicular line to make a geometric tangent to the circle. After the instructor added the suggested segment to the circle, some students used this external cue to chime in while alluding to the word tangent in geometry (Excerpt 8), while some were still not escaping from using numbers in their justifications and reasoning. At this point in the lesson, only one student (Andrea) had mentioned using the tangent line to the circle as guide to locate the tangent function rather than the slope or ratio of sine to cosine.

Excerpt 8

1 Dr. Kay: Andrea what do you think?

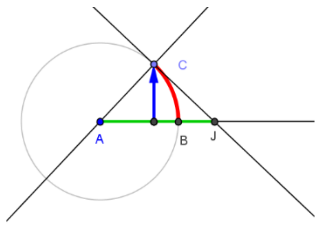
2 Andrea: To connect this to geometry, I drew a geometric tangent to meet that line at the variable point. CMR2



3 Dr. Kay: How do you connect the word tangent similar to the way we connected the word chord?

4 Smith: That is the tangent line to the circle, but it is not the tangent line itself. The tangent is the slope of that line. MR

5 Henry Let us add a horizontal line from the center to meet the tangent line. AR-F



6 Dr. Kay: Is that segment the tangent? (Pointing to the green segment, $\overline{AJ}$). Chat in your groups for 20 seconds.

Group A spent no time talking about the task at hand. Mark paired up with Henry from an adjacent group and brainstormed what they thought might be the tangent. After about 30 seconds of chatting, Henry signaled to Mark (while pointing at the board) what he seemed to now consider to be the tangent. He further moved to the board and pointed at the exsecant ($\overline{BJ}$) as the tangent (Figure 13). At this point, Henry's argument is based on his prior knowledge of ratio (right-triangle) trigonometry. The idea that segment $\overline{BJ}$ diminishes as the arc length diminishes, and non-existent at the 90-degree mark, Henry's argument can be thought of as being synonymous to the outcome of Ben and Monica's earlier dialogue (Excerpt 6). This is an example of creative mathematical reasoning (CMR) which does not necessarily yield a correct answer.

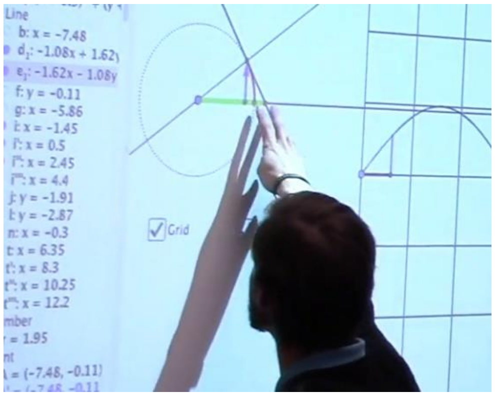


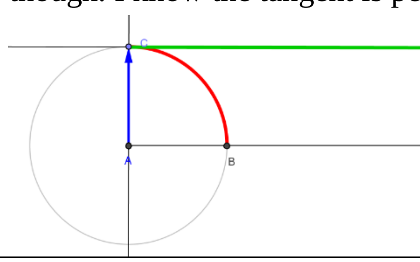
Figure 13. Henry's second idea of the tangent

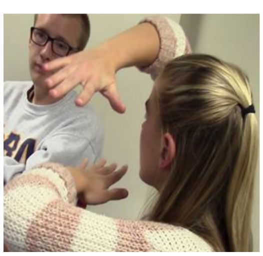
Immediately following this, the instructor halted the group discussions and solicited students' input on the task. The instructor's decision to suggest that students focus on the meaning of the word tangent as they did for the chord function was intentional. He hoped that students would identify the line segments that represents the tangent function to be part of the tangent lines to the circle, rather after this group activity, some referred to the exsecant as the tangent, while others from their attempts to recall past knowledge of right triangle trigonometry, they were still reasoning about the tangent from only the stance of ratio trigonometry. (Excerpt 9).

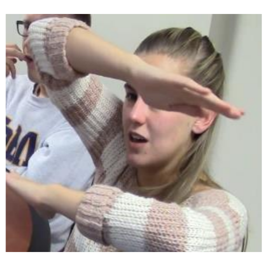
Excerpt 9

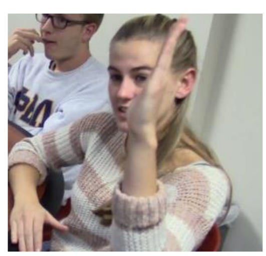
1	Dr. Kay	Okay! What do you think? Ben?	
2	Ben:	I am thrown off because when the green thing ($\overline{AJ}$) is at zero degrees, the length is one, sine is zero at zero degrees which means tangent should be zero. I don't think that is the tangent because when the variable point is at the starting point, the chord is zero but in this case the tangent is 1. So, I say the tangent is that short piece. I think the tangent is the difference between the edge of the circle and the extended part of that tangent (referring to $\overline{BJ}$ - the exsecant).	CMR3
3	Dr. Kay:	So, Ben's update to Henry's conjecture is maybe we should just be looking at this piece *(pointing at* $\overline{BJ}$).	
4	Smith:	Oh yeah! That is a good one.	AR-PG

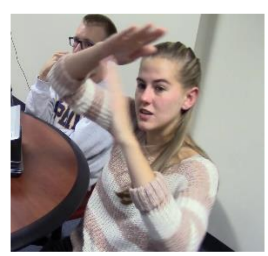
5	Henry:	I think the tangent is on the tangent line. (*Goes to the board to demonstrate*, see Figure 15)	CMR4
6	Class:	Oh! Yeah!	AR-PG
7	Suzie:	Yeah that makes sense. How did you know that?	AR-PG
	Eron	Oh! Now I am convinced. Because if there are to be defined that way, they are not anymore, and that makes sense it is undefined, but they are not meeting anymore either, I am confused. Oh, wait, you need to go the opposite way and then be parallel like this (*gesturing with her hands horizontally)* (*Figure 14*).	CMR5
8	Dr. Kay:	Is that piece the tangent? What things are you checking?	
	Class	Yes	AR-PG
9	Eron:	I think if you get to the perpendicular part, the lines are going to be parallel, so it is undefined (*gesturing both the perpendicular and parallel positions*) … Is that math, is that numbers? (Figure 14) If we think of tangent as opposite over adjacent, our angle is the central angle, which makes the opposite that green segment and the adjacent is equal to the radius which is one, so the green segment must be the tangent.	CMR6
10	Jackie:	That makes sense. If you think of the length of the green line as it gets closer to the point at the top of the circle is going to be increasing infinity, which is what a tangent line does. I am concerned with what happens when we go past that point though. I know the tangent is periodic.	CMR7











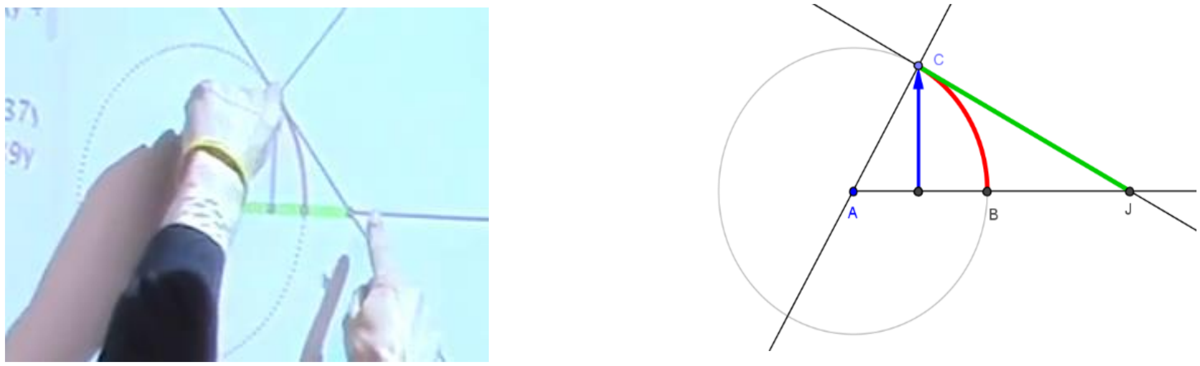


Figure 15. The tangent function as identified by Henry

Inasmuch as the majority of students seemed to agree that the “green line” ($\overline{CJ}$) was the tangent, Mark wanted to additionally venture into the idea of restricting domains for the functions, but the instructor pointed out to him that, “it is unnecessary at this stage.” Next, to move the discussion forward, the instructor suggested to construct a graphical representation of the tangent and check if the resulting graph matched what they expected (AR-F). As this was being done, Smith posited that, “Ben’s previous idea (*Line 2, Excerpt 9*) will also produce the tangent graph.” However, Mark pushed back on this assertion by stating that, “like we had for the chord function, the chord gave us a graph similar to that of the sine function but it was not actually the sine function (*it was* $y = 2sin(x)$), so even this one is close to the idea but it will not be the same.” Although Smith did not sound convinced, he settled with the idea that the tangent is “one of those two ($\overline{BJ}$ or $\overline{CJ}$) but I don’t know which one” (AR-D).

Analogously, Jackie suggested that, “let us go back to the unit circle with the concept of TOA (Tangent is Opposite over Adjacent),” and followed through by identifying ΔACJ as the one to be used to justify that $\overline{CJ}$ is the tangent.

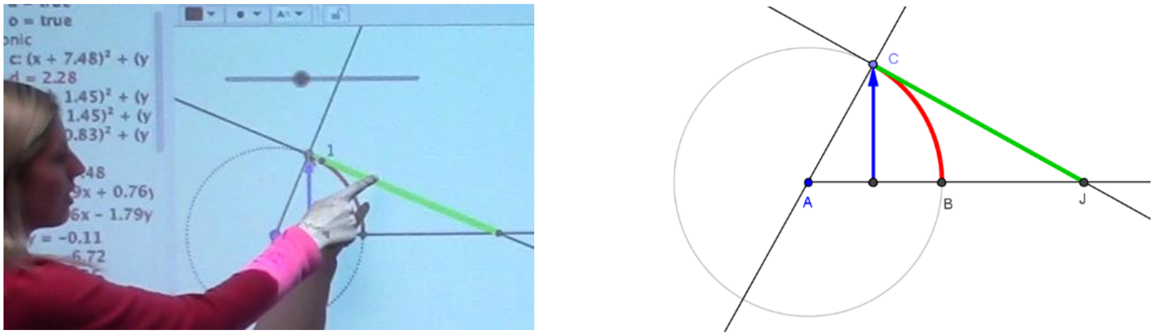


Figure 16. Jackie justifying why $\overline{CJ}$ is the tangent, and a labelled model showing the tangent

In this the eighth instance of CMR in this task, Jackie oriented herself to the situation by demonstrating on the board, tracing out what was the opposite, and adjacent sides relative to the angle at the center ($\measuredangle CAJ$), and explained that "the green line is the tangent because when referring to the angle at the origin, this is 1 (pointing to the radius) the adjacent, and this is the opposite (pointing to the "green line" $\overline{CJ}$) so the green line is the tangent since with opposite over adjacent, it is $\overline{CJ}$ over 1". Her submission was embraced by the rest of the students as she assumed her seat while other students were applauding. To move the discussion forward, the instructor responded by stressing the idea that the tangent should be part of the tangent line, and therefore Smith's earlier submission about the exsecant, $\overline{BJ}$, being the tangent was shelved. As a consequence of brief deliberation on how to account for the changing behavior (i.e., positive or negative) of the tangent function at different arc lengths, after the instructor proceeded by purposefully choosing to use a tangent line that was oriented vertically to generate a graphical representation of the line segment said represent the tangent function. The resulting graph is familiar to what the students expected (Figure 17), and the rest of the session was spent with students constructing the graphical representation of the tangent function (Figure 18).

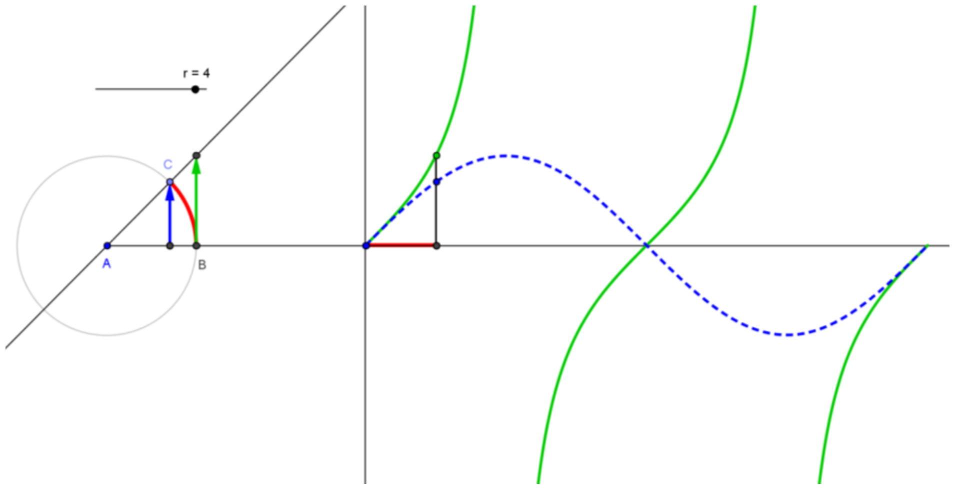


Figure 17. Instructor's geometric representation of the chord (blue) and tangent (green) functions

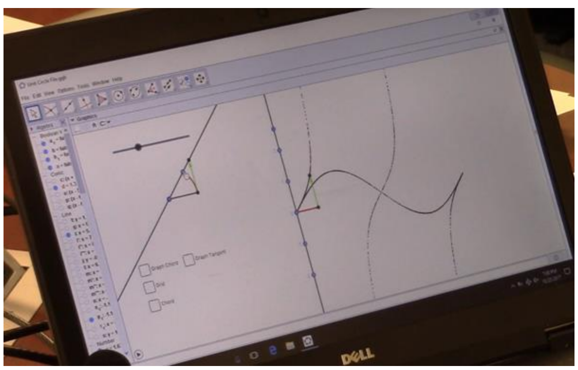


Figure 18. Ben and Monica's graphical representation of the tangent function

The counts (Figure 19) indicate that participants for the most part justified their reasoning strategies about the chord function using a variety of reasoning strategies.

C (8)	**I (26)**				
8	MR (11)	AR (18)			
	8	AR–F	AR–D	AR–G (7)	
		7	4	AR–TG	AR–PG
				0	7

Figure 19. Counts of students' reasoning while investigating the tangent function

On thirty-three occasions, students were able to articulate mathematically valid explanations concerning the concept of a tangent function. In eight of these cases, the reasoning was creative, with students either developing "new (to the students) reasoning sequences or re-creating forgotten [ones]" (Lithner, 2008, p. 266). There was also a considerable number of instances (8) where students resorted to MR as a strategy choice to support their reasoning. Taken together, the rest of the students' reasoning on this task was focused on AR in general, with all but one of the variants of AR evenly exhibited. As in the first task, getting around some aspects of the technology seemed to impede some students from engaging in any meaningful discussion. In such situations whenever the students where asked for a contribution towards the general debate, they opted for MR. The cases of guided reasoning resulted from students having either a blurred understanding or no knowledge of the concept of a tangent as a geometric entity and/or as a function.

The registered number of cases of CMR registered in this task can all be traced back to, (a) the support of having access to the technology (CMR2, CMR3, CMR4, CMR5, CMR7), (b) students' knowledge of other stances of trigonometry (CMR1, CMR6, CMR8), and (c) the support of working in a group (CMR5, CMR6).

Although in some instances getting acquainted with the technology was somewhat of a challenge, in a number of instances in this task, having access to a computer and the Dynamic Geometric Environment (DGE) to work with, provided data that supported the students in reasoning creatively. The DGE presented students with a unique way of exploring and interpreting the tangent function using a directed length, by affording them an avenue to construct figures, identify specific attributes, quantify these attributes, and analyze the relationship between directed segment and the arc length by graphing them in a coordinate plane.

This connection between the directed length and the graph of the tangent function was easily made within a DGE. Similarly, students leveraged technology to realize how to connect the idea that parallel lines in the construction correspond with asymptotes in the graphical representation of the tangent. Beyond what they could easily conceive, technology helped "students transcend the limitations of the mind in thinking" (Pea, 1987, p. 91). For example, Eron and indeed the rest of the class, was able to creatively reason about the line segment, $\overline{CJ}$ as being tangent (see Figure 14) only after Henry's demonstration using the DGE. In such a scenario, technology shifted the students' focus of mathematical reasoning (e.g., Suzie, and Eron in Excerpt 9). In fact, even Henry's salient identification of this novel (to Henry and the rest of the class) representation would not have been possible without the explicit feedback from the DGE. Because Henry finished the initial task earlier, he had time to explore while the rest of the class was completing the previous task, working on his own, he was able to generate graphical representations of different line segments until he identified one whose graph tallied with what he knew was the graph of a tangent function. In this instance, Henry leveraged the technology as a reorganizer to generate, explore and identify the tangent function.

In tandem with Henry's exploration mentioned above in which he used his prior knowledge to identify the graph of a tangent function, and similar to the events in task 1, there are scenarios of CMR that depended on the students having prior knowledge about multiple stances of trigonometry as well as understanding of geometry. The students were able to draw on their knowledge of tangent as opposite over adjacent (i.e., TOA from ratio trigonometry), and the knowledge of tangent geometrically. Case in point is Jackie's submission while identifying the tangent segment when she explained that "the green line is the tangent because when referring to the angle at the origin, this is 1 (pointing to the radius) the adjacent, and this is the opposite (pointing to the "green line" ($\overline{CJ}$) so the green line is the tangent since with opposite over adjacent, it is ($\overline{CJ}$) over 1".

The progress into CMR also depended on the support of working in a group. Students in their groups and in the general class discussion had opportunities to build feasible arguments and critique the reasoning of their peers. Also, this arrangement provided students with platforms to convey their thinking and opinions, which facilitated the rest to understand and construct their arguments and reasoning around the same ideas. For example, Eron was able to reason in a creative mathematical way about the tangent function partly by developing her argument on the ideas of Ben and Henry (Excerpt 9). After Henry opined that the secant ($\overline{AJ}$) was the tangent line, the instructor had the students discuss in their groups to substantiate this claim. As an update to Henry's initial conjecture Ben proposed that the exsecant ($\overline{BJ}$) was the tangent. These two claims seemed to have confused Eron until Henry correctly revised his conjecture and demonstrated on the board (see Figure 15) and identified $\overline{CJ}$ as the tangent. To this Eron stated, "Oh! Now I am convinced. Because if there are to be defined that way, they are not anymore, and that makes sense it is undefined, but they are not meeting anymore either, I am confused. Oh, wait, you need to go the opposite way and then be parallel like this (*gesturing with her hands horizontally)* (see Figure 14). Her reasoning was not only novel but also plausible. Furthermore, when the instructor probed further, Eron and Jackie stated as below.

Dr. Kay: Is that piece the tangent? What things are you checking?

Eron: I think if you get to the perpendicular part, the lines are going to be parallel, so it is undefined (*gesturing both the perpendicular and parallel positions*) (Figure 14).

Jackie: That makes sense. If you think of the length of the green line as it gets closer to the point at the top of the circle is going to be increasing to infinity, which is what a tangent line does.

Both students were able to provide arguments that were based on the mathematical properties of the tangent function and the back and forth process of engaging with other students in their respective groups and whole class discussion contributed to their transition to CMR.

Task 3: Investigating the Secant Function

Following the extensive discussion in Task 2 on the details of the tangent function and a concise definition of the segment, students were also exposed to identifying when the function was positive and when it was negative in relation to the orientation of the segment. Then, during the same session, students were tasked to extend their geometric understanding of the chord and tangent functions to the secant function. Students were exposed to the same routine as described in the task investigating the tangent function for addressing the secant function. The instructor thus generated a similar pattern for investigating the secant function as summarized in Table 6. Also presented are percentages of each reasoning type adopted, and the mathematical content registered from the students as they completed the task at hand.

Table 6

Resolving Task 3: Investigating the Secant Function

Percentages		Task Part	Math Content from students
CMR	Imitative		
0%	100 %	Identify the secant function in the diagram (With limited guidance)	• SOH-CAH-TOA • Secant to a circle

(Table Continues)

Percentages		Task Part	Math Content from students
CMR	Imitative		
11%	89%	Attributes of the secant function	• Cosine is zero • Undefined function • ASTC (All Students Take Calculus)
30%	70%	• What constitutes a secant • Identify the secant function in the diagram (Instructor-guided)	• 1/cosine • exsecant • Secant to a circle

Like in previous tasks, the students were working in their assigned groups. The instructor shared a pre-made file with all students so that the class could move forward from the same place. The instructor displayed that same file on the projector in *GeoGebra* (see Figure 20). As before, the constructed objects (arc, chord, and tangent) varied as point C was being translated on the circle.

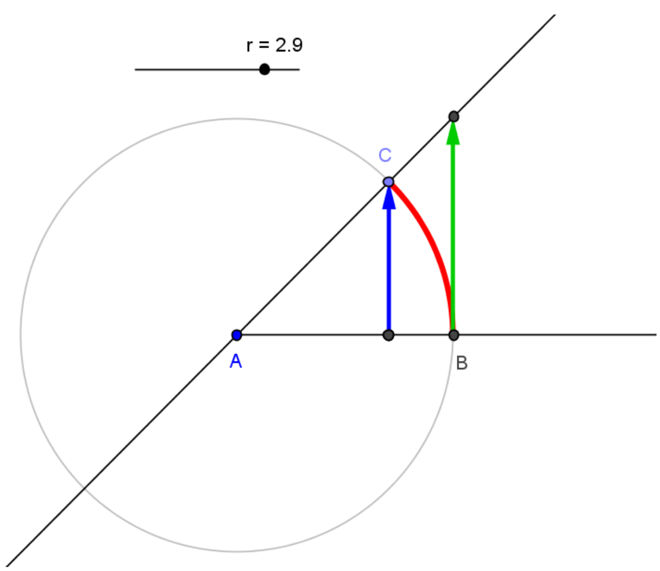


Figure 20. Variable point C and the associated objects

Having been exposed to the basic object-to-graph relations addressing the positive and negative forms of the chord (sine), and tangent functions during previous tasks, students were tasked by the instructor to discuss in their groups "How you know when the secant function is positive and negative, when is it zero and when is it undefined?" Students in Groups A (Excerpt 10) and B (Excerpt 11) had a brief deliberation on the questions at hand as presented below.

Excerpt 10

1	Eddie:	Secant is undefined at π/2 and 3π/2 and then first quadrant and last quadrant when x and y are … oh no!	MR
2	Eron:	When y is positive. Secant is cosine?	MR
3	Suzie:	Uhm!	
4	Eron:	Alright! Uh! In the first quadrant and fourth quadrant, yeah you are right.	AR-PG
6	Suzie:	All Students Take Calculus (*in a mocking voice*). That is not true.	
7	Eddie	Yeah! I learned it (*the mnemonic ASTC*) in pre-calculus	

Excerpt 11

1	Andrea:	(*Appearing to refer to the diagram on the board*) Based on that, it is never zero. I mean, if we look at secant as 1/cosine no matter what cosine is, it will never equal zero.	CMR1
2	Ben:	Yeah, since it is 1/cosine it can be undefined but cannot be zero.	CMR2
3	Andrea:	Yeah	
4	Ben:	I can't even remember what secant looks like	
6	Andrea:	You mean what a secant looks like, or the graph?	
7	Ben:	No, the secant function. My guess is it is the stationary point and the point we are moving.	AR-D
8	Andrea:	What?	
9	Ben:	I am now thinking about the secant on the circle. You know it crosses the circle twice, right?	
10	Group	Yeah	
11	Ben:	I think it is going from the stationary point (*point B*) to the point we are moving (*point C*) – (see Figure 20), that is all we have left.	AR-D
12	Andrea:	Ohh!	
13	Ben:	And so, if we talk about that, it does have a value of zero. Or would that just be considered undefined?	AR-D
		I think if we are talking about the cosine, I am going to go to that triangle again (*referring to* Δ*ABC,* see Figure 20). Our x-value at zero, is one, which means our cosine value is one, making secant one there,	CMR3

	which means you flip over (*gesturing a flip with his hand to make 180 degrees*) you get one. So, what happens at 90? That is when it is undefined again. [Instructor interrupts at this point to pause the group discussion].

Group A's attempt to briefly discuss the questions posed by the instructor did not generate much debate as the whole group seemed to go along with Eddie's initial ideas about the attributes of the secant function, except for Mark who was not partaking in the discussion. In this situation, it was notable that the students again elected to reason about the secant function using their prior knowledge of ratio and unit circle trigonometry without any reference to line-segment trigonometry. Similarly, those in Group B reasoned about the secant function based on their prior knowledge of viewing secant as a quotient ($\sec(\theta) = 1/\cos(\theta)$). However, in their reasoning, they also incorporated an approach that used the arc and associated segment to evaluate the quantity (i.e., secant). For example, Ben stated "I think if we are talking about the cosine, I am going to go to that triangle again (*referring to* ΔABC, see Figure 20). Our x-value at zero, is one, which means our cosine value is one, making secant one there, which means you flip over (*gesturing a flip with his hand to make 180 degrees*) you get one."

With the group discussions temporarily halted, class reconvened, and the instructor encouraged the students to briefly think about how they came up with their reasoning about the attributes of the secant function. This general class interaction is presented below (Excerpt 12).

Excerpt 12

1	Dr. Kay:	Remembering your experience of this, I want to get a rough sketch, an idea of how you know some of these important things. So, when is it (*referring to the secant function*) zero and when is it undefined?	
2	Student:	Never zero	AR-F
3	Dr. Kay:	How do you know?	
4	Student:	Because of the graph.	AR-F
5	Smith:	By definition the secant is one over cosine.	MR
6	Dr. Kay:	How do you know it is one over cosine? "By definition" is a good statement in my book but we need to recall how we dealt with	

		defining the tangent. It seems to me that we memorized that secant goes with cosine. That is conceptual understanding, right? (*Speaking sarcastically*).	
7	Class	*Laughter*	
8	Dr. Kay:	I feel that is not a good way to learn or "know" trigonometry. But if this is the definition (*pointing to sec(x) = 1/cos(x) on the board*), fine. Smith says since we have a fraction, we just look at the denominator and see that we shall never get zero. Okay! What about when is the secant undefined and how do you know?	
9	Eddie	When the cosine is zero.	MR
10	Dr. Kay:	I suspect most of us looked at secant as one over cosine and remembered that cosine is the x-value in the unit circle and the x is zero and that is at 90 degrees and?	
11	Class:	270 degrees	
12	Dr. Kay:	If we can get something here in the diagram, we might be able to talk about that in a different way. I am guessing we will reason the same for "when is it positive or negative". Has anyone thought of how to find and add a secant to the diagram? Jackie what do you think?	
13	Jackie:	I think I have an idea basing on when it, the secant, is undefined (*explains what lines she considers to be secant lines and the instructor adds them to the circle on board*) – Figure 21.	AR-F
14	Dr. Kay:	Just like we had chord and tangent, not the whole line but just a section of it, are you saying it is this section? And if yes, does it match everything else know about the secant? It hits some of the points like intersecting the circle twice and ceasing to exist at 90 degrees. How about its size? How big should it be?	
15	Ben:	A diameter, 2 radians.	MR
16	Mark:	[*On the board*] I think the length of this thing right here is the secant (*draws a segment from the origin to intersect the vertical tangent at an incline*).	CMR4
		I don't know why. I don't have a conceptual reason why but from looking at it, if we take this point (*referring to variable point C*) and move it to zero degrees, the length [of the secant] will be 1 and it will never be lower than 1 unless you wanna go negative.	CMR5
		And it is the same idea as tangent when you are approaching this point right here *(points to 90-degree position)*, it is also gonna become undefined as well. (see Figure 23)	

Whereas the majority of the students were convinced that the secant function is never zero, hardly anyone was able to reason why that is the case without resorting to the popular but “unproven” fact that sec(x) = 1/cos(x). Apart from one student who posited that secant was never zero “because of the graph”, all others described secant in terms its “definition”, indication that they were involved in MR and in some cases AR-F. For example, when Jackie was identifying what lines may be considered secant lines to the given circle, she clearly stated that she had “an idea basing on when the secant is undefined” (Line 13, Excerpt 12). When probed further for the special case when the secant line turns into a tangent line, she posited that it ceases to exist at 90 degrees. Jackie and other students came up with such values like 90 degrees, and 270 degrees from the idea that it is when the x-value that represents the cosine is equal to zero. Again, “a strategy choice founded on recalling a complete answer” (Lithner, 2008. p. 258), and/or the remaining attributes of the secant “regarded as trivial for the reasoner” (Lithner, 2008. p. 259).

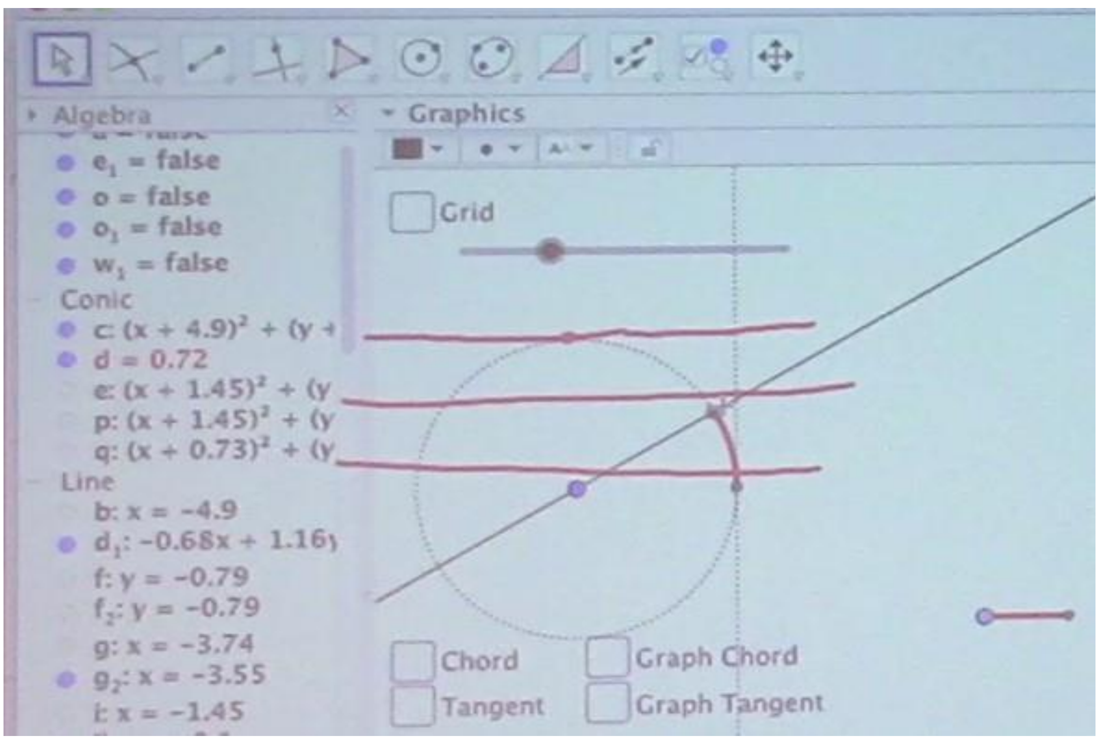


Figure 21. Jackie’s idea of the secant

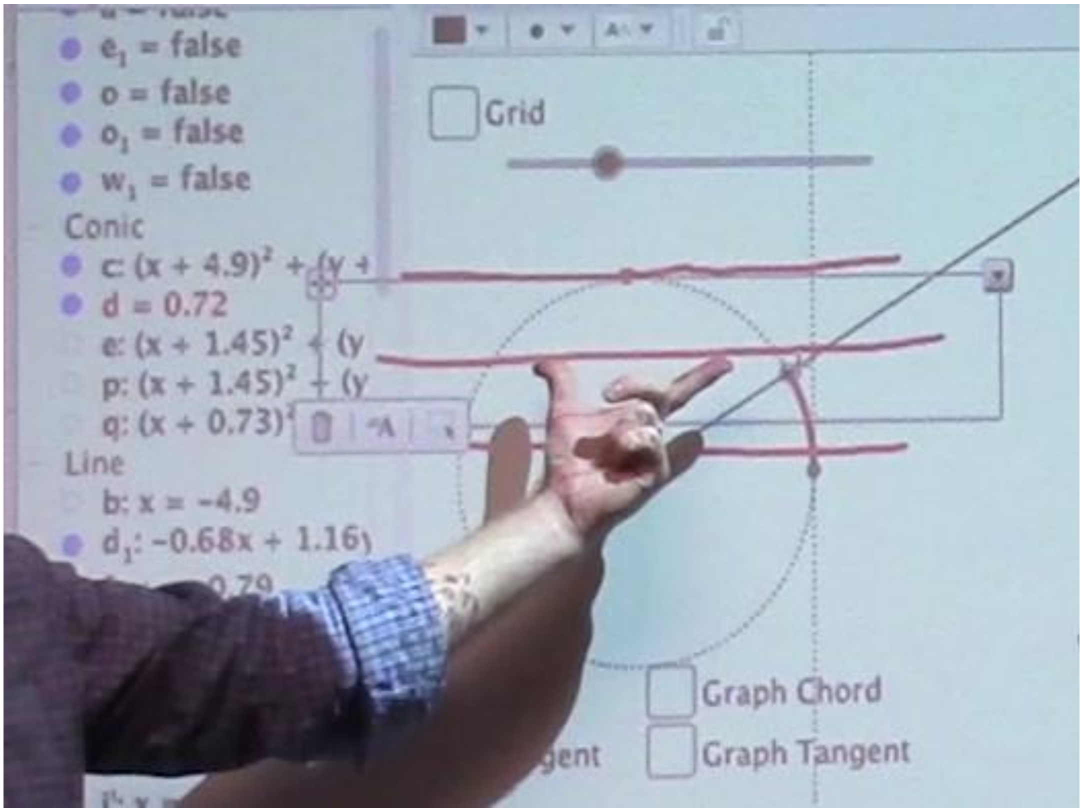


Figure 22. Is this the secant function?

Mark's submission shows traces of a student who is developing ways to reason about trigonometric functions using an approach—connecting the directed length and the graph of the function—"which is useful in addressing many of the difficulties students have with trigonometry concepts" (Hertel & Cullen, 2011, p. 1406). While Mark initially displays MR and AR-D when he stated that he did not have a conceptual reason for the decision he made about which segment represented the secant function (see Figure 23), in the same breath he was able to articulate a few attributes of the secant function, and consequently identifying the correct segment for the secant function, a show of CMR. This reasoning was most probably informed by the fact that while on the previous task (tangent function) he was also able to build the secant function (see Figure 23)—when he built a graphical representations for different segments and zeroed in on one that produced a familiar secant function graph— and made a "connection between [the] directed length and the graph of [the secant] function with a DGE" (Hertel & Cullen, 2011, p. 1406).

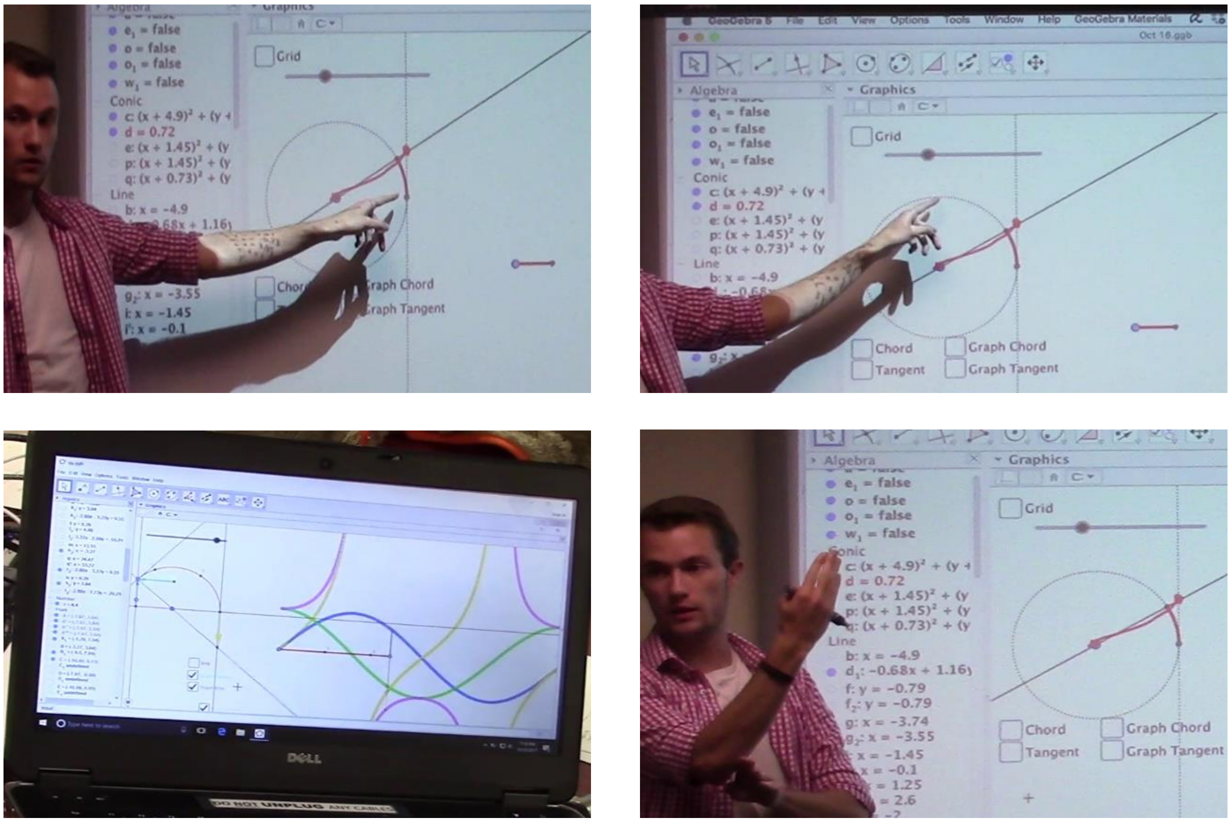


Figure 23. Mark explaining his reason for the choosing the red segment as the secant and his prior work on graphing the secant function

After Mark's explanation, the instructor acknowledged the direction the discussion had taken and encouraged further deliberations on the same (Excerpt 13).

Excerpt 13

1	Dr. Kay:	Any reactions? [Mark's] "piece" seems to fit the behavior of the secant. Like for the tangent function before, is this segment lying on a secant?	
2	Mark:	*(Moves to the board and adds a horizontal radius in the diagram).* The idea is the opposite over hypotenuse, and that is the length of whatever happens to be the chord over 1, so we want it to be the	AR-F AR-D

		other way. We want 1 over the length of what our secant is going to be. And here we are looking at … (*looks seemingly lost*)	
3	Victor:	Can we say hypotenuse over adjacent?	MR
4	Joshua:	From before we identified this as the tangent, and this is a unit circle, so radius is 1, and from Mark's idea this is the secant because $\tan^2(\theta) + 1 = \sec^2(\theta)$. And so, this is the secant (*Writing* $\sec(\theta)$ *on the correct segment).* See Figure 24	CMR6
5	Class:	(*As Joshua is writing the identity*) Oh!	
6	Ben:	(*Talking to fellow group members*) Is he teaching this? He has to be teaching this to remember it.	
7	Ben:	That was really good. You must be teaching this right now.	
8	Dr. Kay:	I cannot help but add this chord back. And looking at these triangles, I bet we can get some interesting stuff here. What do you think? We shall pick it up from there [next week] and also deal with when the secant is positive and negative.	

Whereas Mark had earlier used the DGE to have a unique approach to the directed length interpretation of the secant function by drawing the correct figure and exploring the connection between the segment and the arc length and probing the resulting graph, when he labored to justify his initial claim after no other student volunteered to critique it, he seemed to be retreating into AR-F and ratio trigonometry. In fact, he posited that all that was needed was to figure out the opposite side and the hypotenuse, it gave Joshua a queue to work with a familiar identity, $\tan^2(\theta) + 1 = \sec^2(\theta)$, (see Figure 24) to confirm that the segment Mark had identified was indeed the secant function. Although, all students were amazed at Joshua's idea, those in Ben's group insisted that the only way he could have been able to remember that identity was from his then current work with an area 'cooperating teacher.'

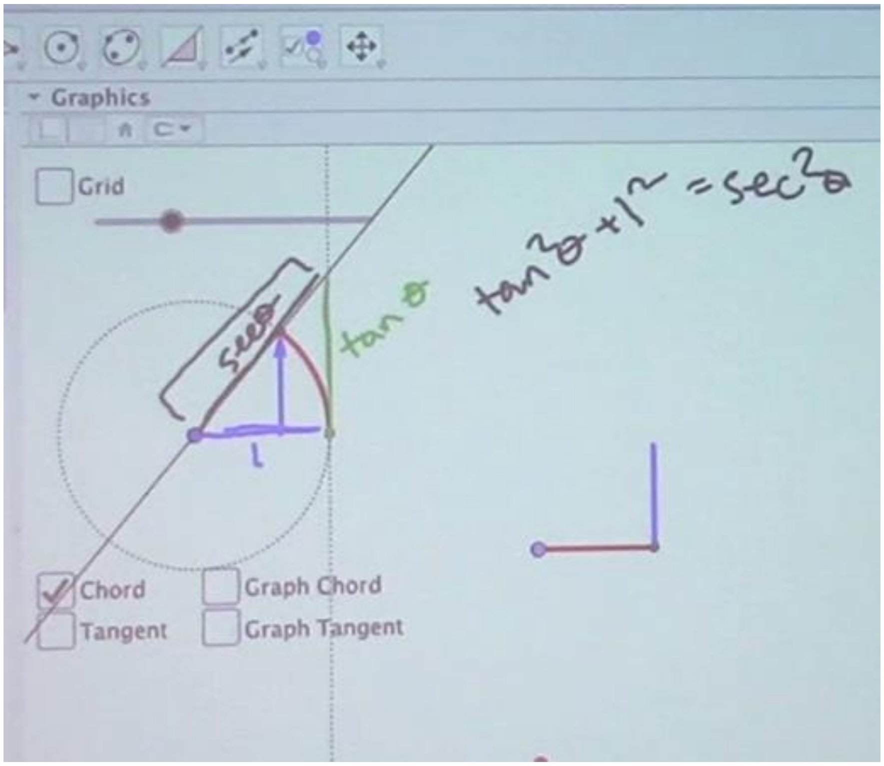


Figure 24. Joshua's strategy for verifying the segment that represents the secant function

The session was concluded with the instructor challenging the students to figure out how to update their drawings by including the directed length and graph for the secant function. Note that in the current version suggested by Mark, the segment representing the secant function is not vertical like the chord (sine) and tangent functions were. In the first two functions the sign of the functions was determined by noting when the vector was pointing up or down. This does not translate well to this version of the secant function because the vector is neither vertical nor horizontal. In the instructional session that followed a week later, the instructor carried on with having the students finalize the updating of the DGE file to include a version of the secant that was either vertical or horizontal. This process involved identifying when the secant function is positive and negative.

After agreeing with Meghan's submission on when the secant function is positive and negative, when she stated that, "it is positive in the first and fourth quadrants, because of the secant line and how it faces for any angle, and negative in the second and third quadrants", and coming to a realization that it was challenging to construct the graphical representation of the secant function using the proposed segment $\overline{AO_1}$, the students led by Andrea constructed a two-column proof to show that another segment $\overline{AQ_1}$, was congruent to the earlier proposed segment, $\overline{\mathrm{AO}_1}$ (see Figure 26 and Figure 27).

In opting to use this alternative segment that was oriented horizontally (i.e., $\overline{AQ_1}$) as the directed length of the secant function, the students were not only able to reason with mathematical creativity about when the secant function is negative and positive, but they also accomplished the construction challenge they had earlier struggled with.

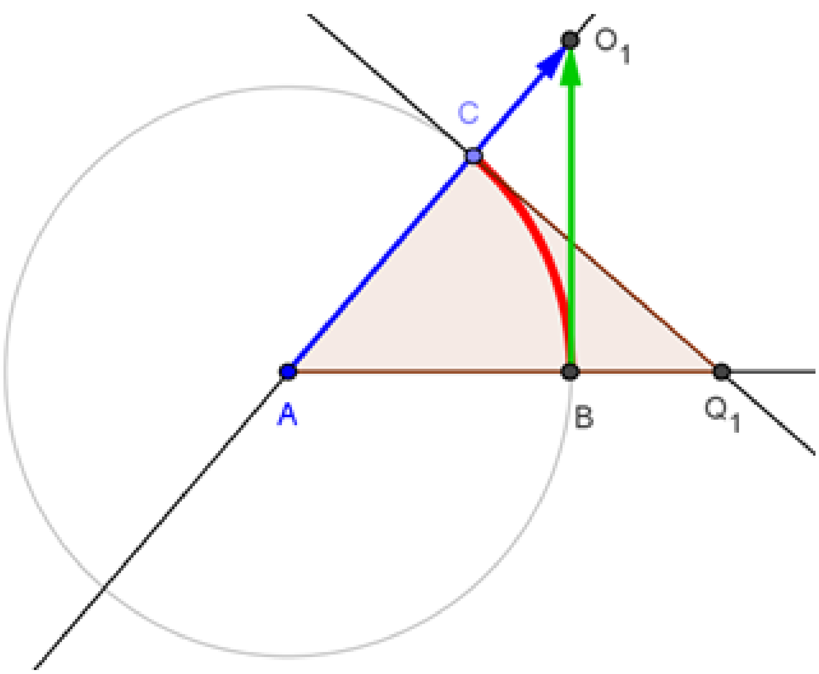


Figure 25. Investigating when the secant is positive and negative

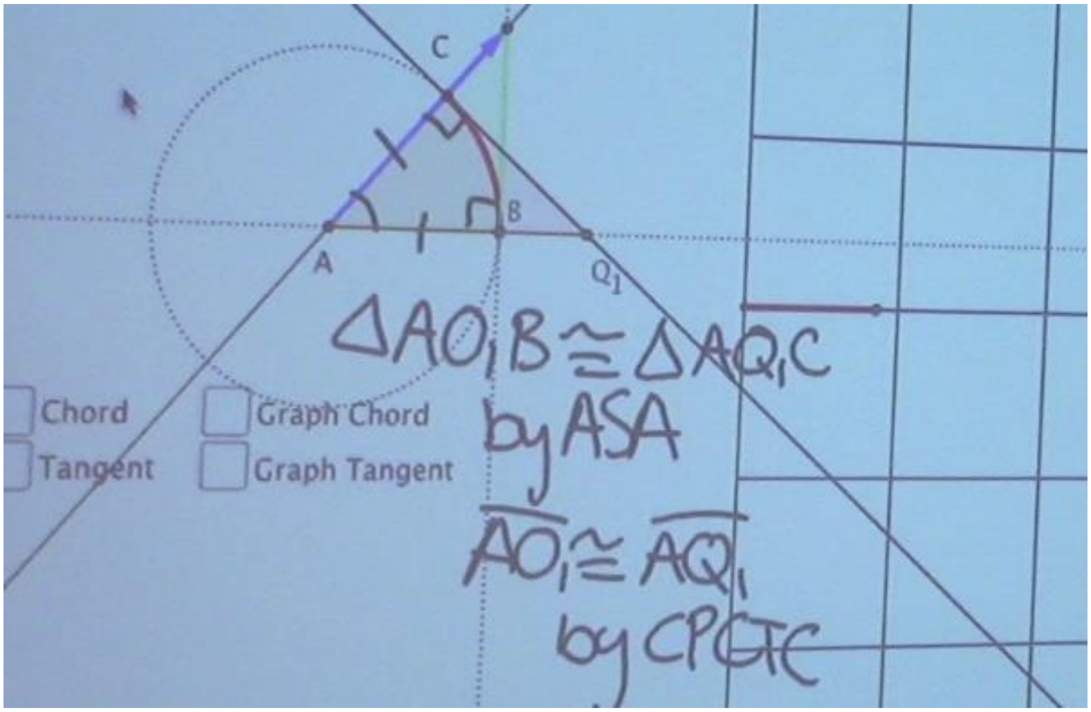


Figure 26. Andrea's proof for congruency of segments

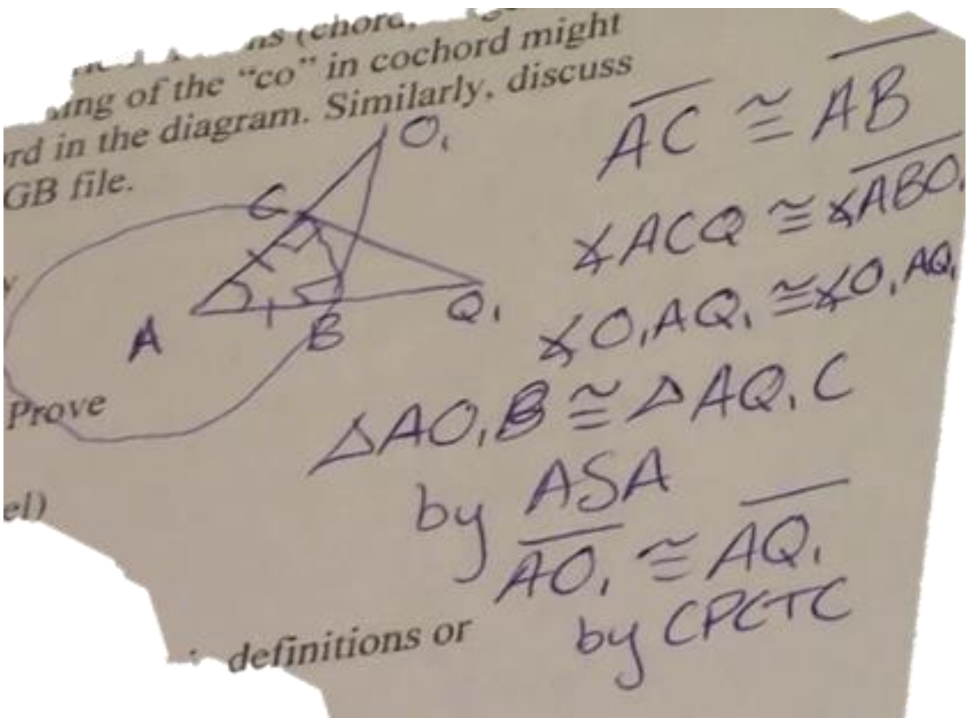


Figure 27. Andrea's initial proof for congruency of $\overline{AO_1}$ and $\overline{AQ_1}$

In response to Andrea's two-column proof about the congruency of $\overline{AQ_1}$ and $\overline{AO_1}$, the instructor once again prompted the students to consider when the secant is "positive and negative and how do you know?" However, this time he was varying the position of point C on the circle while the students responded to the same question at various positions of C (see Figure 25). At all different positions of point C, the students correctly identified when the secant function was positive and negative by using the orientation of $\overline{AQ_1}$, neither relying on the mnemonic ACTS (All Students Take Calculus) nor on the fact that secant is a reciprocal of cosine as before.

After making these discoveries about the secant function and deciding upon which directed segment to use in the construction of the graphical representation of the secant function, the instructor requested the students add the secant function to the files they were building. The rest of the instruction session was used partly for students to update their files and the other part for the instructor to introduce the concept of cofunctions.

Counts of the different reasoning strategies adopted by the students while working on this task are provided in Figure 28.

<table>
<tr><th>C (6)</th><th colspan="5">I (15)</th></tr>
<tr><td>6</td><td>MR (6)</td><td colspan="4">AR (9)</td></tr>
<tr><td></td><td>6</td><td>AR–F</td><td>AR–D</td><td colspan="2">AR–G (1)</td></tr>
<tr><td></td><td></td><td>4</td><td>4</td><td>AR–TG</td><td>AR–PG</td></tr>
<tr><td></td><td></td><td></td><td></td><td>0</td><td>1</td></tr>
</table>

Figure 28. Counts of students' reasoning while investigating the secant function

Although it is apparent that initially there was only one instance in which the students' reasoning was characterized as being mathematically creative, as the instructional session was progressing towards its conclusion, students exhibited this type of reasoning, albeit coupled with some instructor's guidance. For example, after the instructor prompted the students to consider using a different segment than $\overline{AO_1}$ as a definition of the secant function, and students elected to use $\overline{AQ_1}$ after proving that it was congruent to $\overline{AO_1}$, the students were able to correctly identify when the secant function was positive and negative.

In nine cases, students' reasoning was recorded as being algorithmic, while students used MR in only six scenarios. Furthermore, like in prior tasks, the only form of guided AR that was exhibited by the students while investigating the secant function was the AR-PG (Figure 28). The students' actions and statements during this task were consistent with those exhibited in Task 2 sessions. Relative to the definition of a secant, the students continued to reason about what constitutes a secant function as reciprocal of the cosine function. In fact, in four of the six cases in which the students started off with such a premise, their strategy choices were founded on MR. Similar to the students' submission in the previous task, the CMR (6) and some of the AR instances (3) were borne out of their ability to reason about the secant function as a quantitative relationship.

Consistent with the observations from the earlier tasks, students did not only reason by relying on memorized concepts or by carrying out routine procedures but there were also several components of the milieu that contributed to their CMR. In the six instances of CMR in Task 3, students leveraged or were supported by (a) the use of technology, (b) having prior knowledge about multiple stances of trigonometry as well as understanding of geometry, and (c) the teacher moves.

There were four instances of CMR in Task 3 (CMR1, CMR3, CMR4, CMR5) that depended on the students having access to technology. By working in the DGE, students were able to generate and measure dynamic and interactive representations which facilitated them to be able to focus on looking for patterns and making and testing conjectures rather than on drawing and measuring the different line segments and other variables. For example, when tasked by the instructor to investigate the attributes of the secant function, the students were presented with a pre-made file from the previous session (see Figure 20). While explaining to other members in the group about which segment is the secant function, Ben transitioned from, "I can't even remember what secant looks like" to "And so, if we talk about that, it does have a value of zero. Or would that just be considered undefined? I think if we are talking about the cosine, I am going to go to that triangle again (*referring to* ΔABC, see Figure 20). Our x-value at zero, is one, which means our cosine value is one, making secant one there, which means you flip over (*gesturing a flip with his hand to make 180 degrees*) you get one. So, what happens at 90? That is when it is undefined again". Ben's latter stance about the secant function was only arrived at after he focused on the already generated file. This perhaps unintentional use of technology offered Ben and other group members not only an array of affordances to investigate the concepts related to the secant function; but also presented them an opening to question their thinking and revise their conjectures. Ben was able to re-create a forgotten reasoning sequence, by using arguments that were based on basic mathematical properties of the secant function. Even though Ben did not necessarily get to the correct answers, his submissions indicated a transition from MR to CMR with the aid of the technology.

Furthermore, the use of the dynamic technology afforded students an interactive environment in which they investigated and visualized several representations of functions while dynamically covarying and linking line segments and arcs to graphs, which aided them in reasoning creatively. For example, when the instructor asked if anyone had "thought of how to find and add a secant to the diagram?", Mark went to the front of the class and using the displayed figure on the board (see Figure 23) was able to identify the correct segment for the secant function (Line 16 Excerpt 12). Like was the case for the tangent function, Mark while working ahead of the rest of the members of his group was to investigate different line segments and how they covaried with the given arc until he zeroed in on one that produced a graph familiar to what he knew was the graph of the secant function. Without utilizing this interaction and the back and forth feedback from the dynamic technology at play, Mark would probably not have been able to identify the correct segment for the secant function. Beyond the construction of the correct segment, he stated,

> I think the length of this thing right here is the secant (*draws a segment from the origin to intersect the vertical tangent at an incline*). I don't know why. I don't have a conceptual reason why but from looking at it, if we take this point (*referring to variable point C*) and move it to zero degrees, the length [of the secant] will be 1 and it will never be lower than 1 unless you wanna go negative. And it is the same idea as tangent when you are approaching this point right here *(points to 90-degree position)*, it is also gonna become undefined as well. (see Figure 23).

He was able to create a new (to Mark) reasoning sequence that had supporting arguments from unit circle trigonometry and were founded on correct mathematical properties of the secant function.

Secondly students' CMR also depended on having prior knowledge about multiple stances of trigonometry as well as understanding of geometry (CMR1, CMR2, CMR5, CMR6). The students were able to draw on their prior knowledge of the secant function from unit circle and right triangle trigonometry to reason about the task, without which they would not be able to proceed in a mathematically creative way. For example, in the case of CMR1, without any guidance from the instructor, Andrea used a combination of her prior knowledge of the secant function from right triangle trigonometry, and also referred to the diagram (see Figure 25) to make a plausible argument about some attributes of the secant function. When the instructor tasked the students to determine when the secant function is positive, negative, zero, and when it is undefined, Andrea opined to the other members in the group that "based on that (*appearing to refer to the diagram on the board*), it is never zero. I mean, if we look at secant as 1/cosine no matter what cosine is, it will never equal zero." Andrea's anchored her acceptable reasoning on her prior knowledge of the reciprocal identity from right triangle or unit circle trigonometry. Moreover, it is on the basis of her submission that the rest of the group (see Excerpt 11) was able to critique her and construct other viable arguments about the task at hand in a mathematically creative way.

Furthermore, as stated earlier in the case of CMR 5, Mark would not have been able to identify the correct segment that represents the secant function without having prior knowledge of unit circle trigonometry and an understanding of the geometry of the graph of the secant function. In fact, in the process of explaining his strategy choice, Mark also hinted at having used some of his prior knowledge from right triangle trigonometry when (after adding a horizontal radius in the diagram) he argued that "the idea is the opposite over hypotenuse, and that is the length of whatever happens to be the chord over 1, so we want it to be the other way. We want 1 over the length of what our secant is going to be." It is on the basis of this explanation that Joshua (from a different group) picked a queue and presented a divergent reasoning founded in right triangle trigonometry but concurring with the segment Mark had selected. Joshua's submission that "from before we identified this as the tangent, and this is a unit circle, so radius is 1, and from Mark's idea this is the secant because $\tan^2(\theta) + 1 = \sec^2(\theta)$. And so, this is the secant (*Writing* $\sec(\theta)$ *on the correct segment).*" (See Figure 24). Evidently, Joshua used his prior knowledge to re-create plausible reasoning that was forgotten to the rest of the class (judging by their reaction, see Lines 4 – 5, Excerpt 13), and combining a series of familiar AR with Mark's initial strategy was sufficient to create mathematically founded reasoning about the secant function.

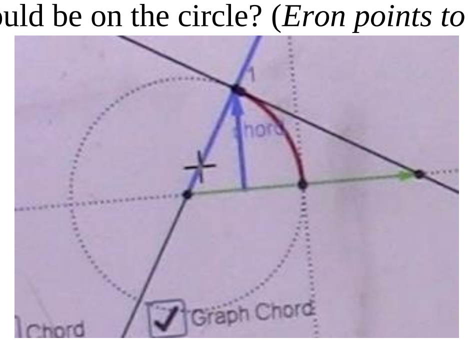
A further study of the students' collaborative reasoning on this concept of the secant function revealed instances of CMR that were a result of the instructor's teaching moves (CMR2, CMR 4, CMR6). The instructor's guidance and different teaching moves were measured and focused on aiding students to not be drawn towards person guided IR. The instructor implemented planned and situational teaching strategies that promoted students' CMR. For example, statements like "how do you know when the secant function is positive or negative, zero, and when it is undefined?", not only gave students an opportunity to choose how they individually and as a group reasoned about the task by engaging in a productive struggle to figure out several ways of solving the it, but also afforded the instructor an outlet to observe the class dynamics and used the information about what students were thinking to guide the instruction. Indeed, even after noticing that the discussion was skewed towards MR (see Excerpt 12), the instructor continued probing students with questions such as: "How do you know it [the secant] is one over cosine?", "Just like we had chord and tangent, not the whole line but just a section of it, are you saying this whole line or a section of it is the secant?". Such questions encompassed more than the immediate intervention, as they encouraged students to further questions their current line of thinking and also make connection with what was done in the previous tasks with the chord and tangent functions. Consequently, as a group, the students continuously refined their conjecture and eventually were able to re-create forgotten reasoning sequences that was founded on plausible mathematical properties of the secant function. (e.g., Mark, Line 6, Excerpt 12, & Joshua, Line 4, Excerpt 13).

Co-functions

After creating the display of the three main trigonometric functions (chord, tangent, and secant), the students worked in their groups and then as a whole class embarked on discussing what the geometric meaning of the "co" in cofunction (cochord, cotangent, and cosecant) might mean and how this can be used to identify the cofunctions in the diagram in the DGE file. At the end of the session in which the students created the secant function, there was a brief discussion about the cochord function. This is not considered as one of the tasks I analyzed because the information derived from these tasks (investigating the cosine, cosecant, and cotangent) was deemed similar to that obtained from the first three tasks (i.e., investigating the sine, tangent, and secant functions).   However, the information presented here was still necessary to offer continuity and context of how the remaining functions were introduced. Responses from members of Group A (excluding Mark who was working alone) are presented below.

Excerpt 14

Eron:	Okay! So why cochord is cochord?
Eddie:	Yeah. It is kind like the opposite of sine
Suzie:	I have no idea.
Eron:	I wish we would be graphing it at the same time.
Me:	I mean my point is where do you think it would be on the circle? (*Eron points to the secant – green directed segment*)



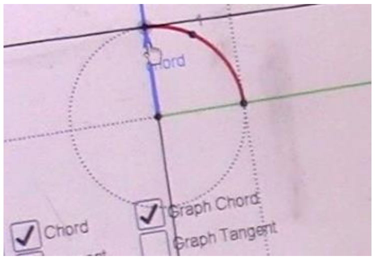
Me:	Which one? The green one?
Eron & Eddie:	Yeah, the green one.

Me: How about the secant?

Eddie: That is the secant.

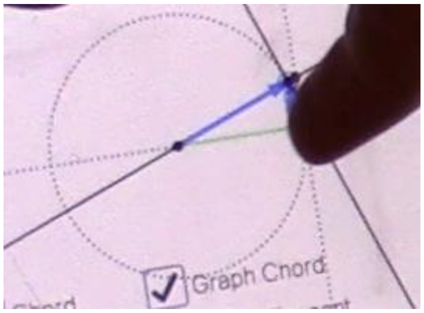
Eron: Maybe it is the distance from the center to the chord.

Eddie: I don’t know. I was gonna say, for 0, cochord will be one (as he moves the variable point C to the 90° mark), and the chord will be zero and they like kind of flip, right?



Me: So, Eddie, by pointing, which one do you think is the cochord?

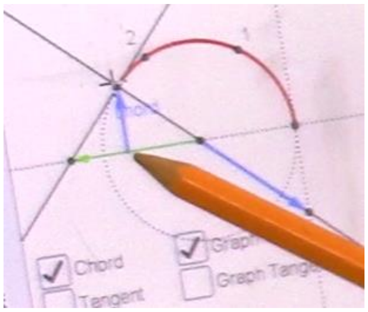
Eddie: (*Instead Eron points to the negative secant*)

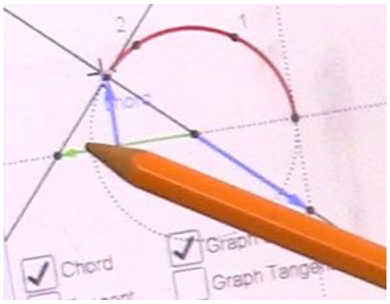


Suzie: I agree with Eron

Eddie: Yeah, I feel like it will be this way

Me: Like from the center to here or there?





Suzie & Eron: Yeah (*Referring to the distance from the center to the foot of the chord function)*

Me: Or from here to here?

Suzie: No, it cannot be the radius all the time.

Me: So why do you think that is the case?

Eron: Because of my previous knowledge, not because of anything else.

Me: So, when will it be undefined?

Eddie: Right here (*moving the variable point C to the 90-degree mark*).

Eron: Wait, cochord is never undefined.

Me: When is it 0?

Suzie: When it is 0 it is zero.

Eron: Duh!

This dialogue is typical of what also happened when the students discussed the cotangent and cosecant functions. It is notable that initially the students did not leverage the “co” (complementary) part in the cofunctions. For example, while speaking about the cochord function, Eddie acknowledged that “Yeah. It is kind of like the opposite of sine” (Line 2, Excerpt 14) but went ahead to identify the directed segment representing the secant as the cochord. When probed further he stated “I don’t know. I was gonna say, for 0, cochord will be one (*as he moves the variable point C to the* 90^o *mark*), and the chord will be zero and they like kind of flip, right?”

However, the students were also notably wishing they could compare their chosen segment with a graphical representation, which is consistent with (Hertel & Cullen, 2011) when they suggest that the “connection between [the] directed length and the graph of a trigonometric function … easily made within a DGE … is useful in addressing many of the difficulties students have with trigonometry concepts” p. 1406). Indeed, in the subsequent session when presented with the opportunity to further their discussion about the cofunctions while at the same time working with the DGE (Figure 30) to construct the graphical representations of the cofunctions, the students identified and created the required line segments correctly.

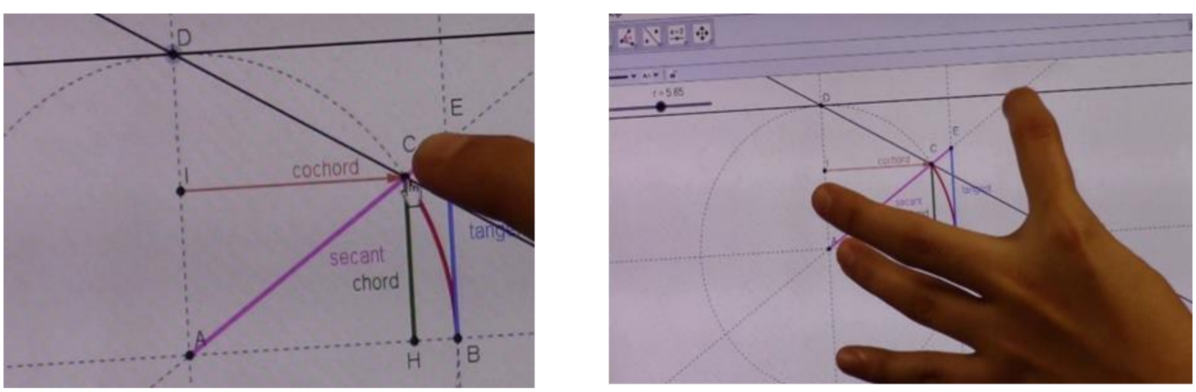


Figure 29. Suzie identifying the cofunctions

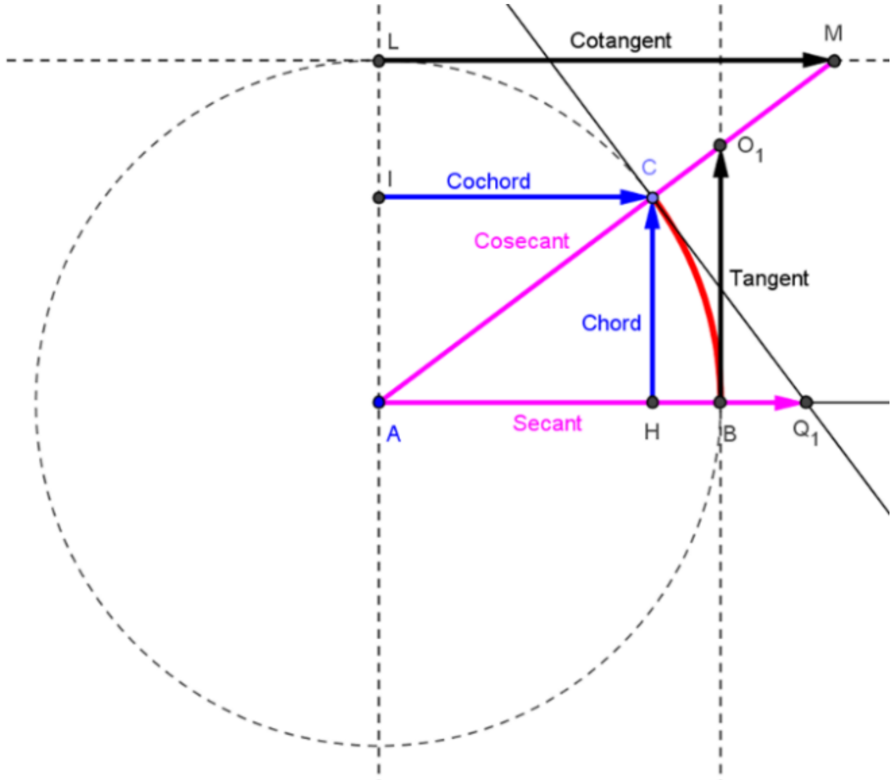


Figure 30. The geometric definitions for the trigonometric functions and their complements

Task 4: Exploring Trigonometric Relationships

With the three main trigonometric functions and their complements now identified and added to the same unit circle, the ground was set to explore the relationships between and among the different functions. While investigating the cofunctions, students were also tasked to develop two different diagram versions with the geometric definitions of three functions and their complements (Figure 31)

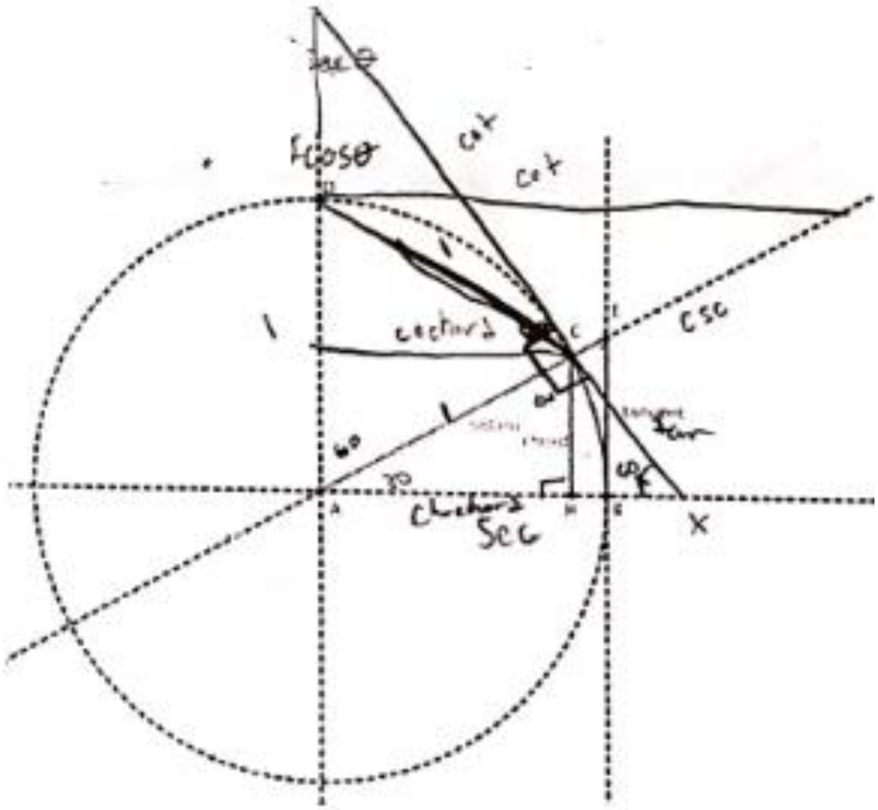


Figure 31. An example of the two diagrams embedded in one with the geometric definitions of trigonometric functions

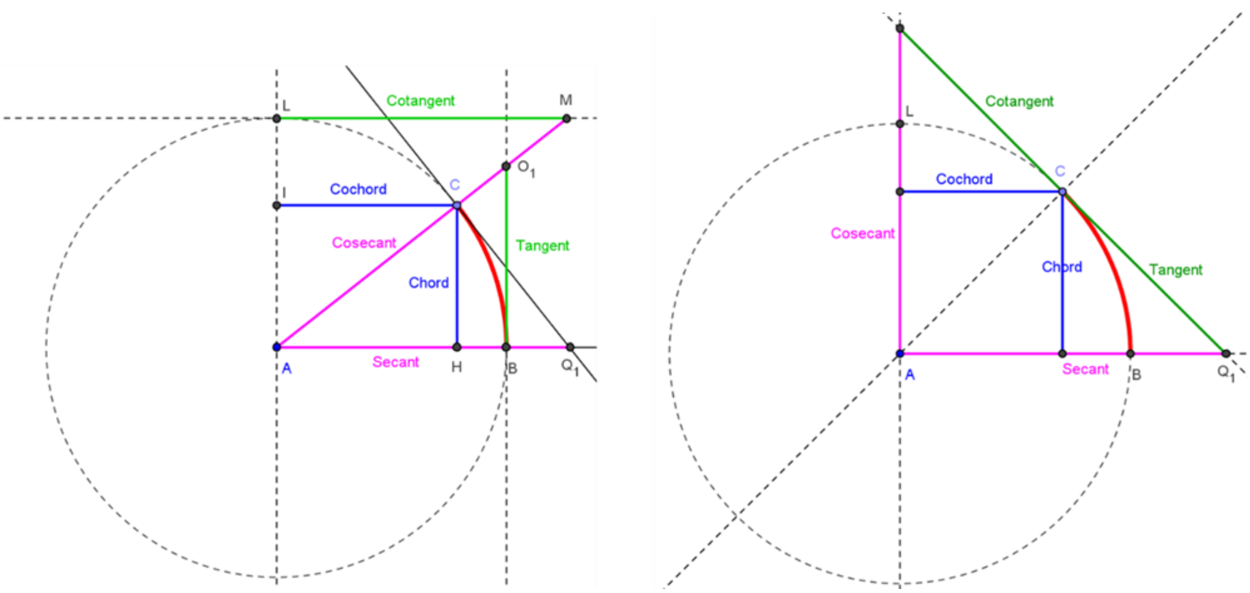


Figure 32. Construction of the two versions with the geometric definitions of three functions and their complements as highlighted by the instructor

The instructor then challenged the students to use "the two different diagram versions created above (Figure 32), the paper diagrams (e.g., Figure 31) the geometric definitions of the six functions, and your vast geometry knowledge, [to] make conjectures about relationships and prove them. Be prepared to share your proof(s) with the class."

Unlike in the previous tasks, students spent little time on building the functions in Geogebra. The instructor after letting the students draw all the functions on a paper diagram in two different ways, developed the two diagrams in the DGE and displayed them on the board for students to use as a reference when forming their conjectures. As it was an open-ended task, the students did not necessarily have a prescribed pattern for identifying the relationships, but nonetheless, a summary of the different classifications (i.e., relationships due to similarity of figures, or due to using the Pythagorean theorem) is provided in Table 7 along with percentages of each reasoning type exhibited, and the mathematical content registered from the students as they completed the task at hand.

Table 7

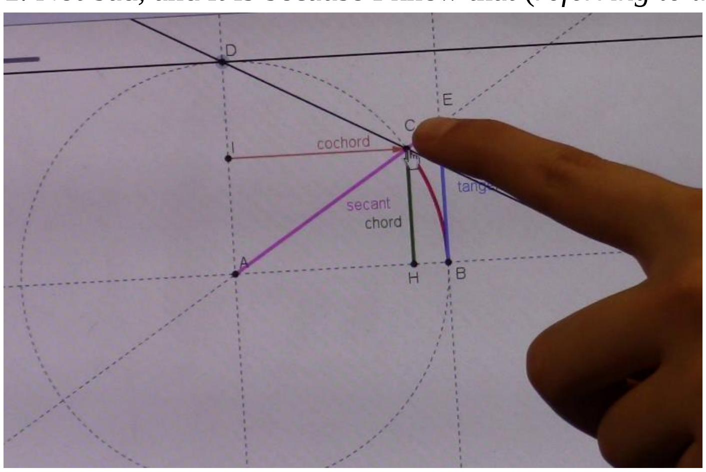
Resolving Task 4: Exploring Trigonometric Relationships

Percentages		Task Part	Math Content from students
CMR	Imitative		
57%	43%	• Identify the relationships using Pythagorean theorem.	• $1 + \tan^2\theta = \sec^2\theta$ • $(\cot x + \tan x)^2 = csc^2x + sec^2x$
46%	54%	• Identify the relationships due to similarity of figures (triangles)	• $\tan(x) = sec(x)\sin(x)$ • $[1 - \cot(x)]^2 + [\tan(x) - 1]^2 = [\sec(x) - \csc(x)]^2$

For this task, the students were working individually but they were also encouraged to discuss, share their findings in their assigned groups, and eventually with the whole class. In the following excerpts, I present the dialogue among the students in Group A (Eddie, Eron, Suzie, and Mark), followed by a general classroom discussion with some selected relationships between the trigonometric functions as presented by the students.

Excerpt 15

1	Suzie:	What is the end game of this?	
2	Steve:	How many do you think you can find?	
3	Eddie:	All of them.	
4	Steve:	Give me a number. I know there are three Pythagorean …	
6	Eddie:	Oh, I thought there were only six of them. I thought they're like 15.	
7	Suzie:	There is a lot. There is a ton.	
8	Eron:	I only know a few. I only know $1 + \tan^2\theta = \sec^2 \theta$, a.k.a $\sin^2 \theta+\cos^2 \theta$. I also know $1/\sin(\theta)$ and similar ones.	MR
9	Eddie:	I don't know anything. How is it $\sin^2 \theta + \cos^2 \theta = 1$?	
10	Suzie:	Because the radius of the circle is 1. In a unit circle, it is 1.	AR-F
11	Eron:	You know you can think about it as a square. radius x radius = r^2	
12	Suzie:	(*Pointing to the segments on the screen*). We look at this triangle and this is the chord, and cochord and there is the radius, so $\sin^2 \theta + \cos^2\theta = 1$. Not bad, and it is because I knew that (*referring to the identity*).	CMR1



13	Eron:	One down.	
14	Suzie:	I think you've an advantage if you already knew them. And that is all I knew. Uh, tangent squared minus one is …	AR-F

15	Eron:	You could go with the same thing. We know that this is perpendicular to this line, we know that this is going to be a right angle, and that this a cotangent, and that this blue including this purple is going to cosecant, so $1 + \cot^2 \theta = \text{cosecant}^2 \theta$.	CMR2
16	Eddie:	Or one squared.	
17	Suzie:	Heck yeah. And that's the same 1 for secant and tangent. Or there is …	AR-F
18	Eddie:	Yeah, because this one is $1 + \tan^2 \theta = \sec^2 \theta$. *(pointing to the sides of one of the triangles in the diagram,)*	CMR3
19	Eron:	Okay, now that we have what we knew …	

Student responses thus far, were mostly founded on the Pythagorean theorem and their prior knowledge of trigonometric identities. However, it is notable that Eron was also thinking of relationships beyond what one would usually characterize as an identity. For example, she listed that secant and tangent are parallel at the 90^0 mark (see Figure 33) as one of the relationships.

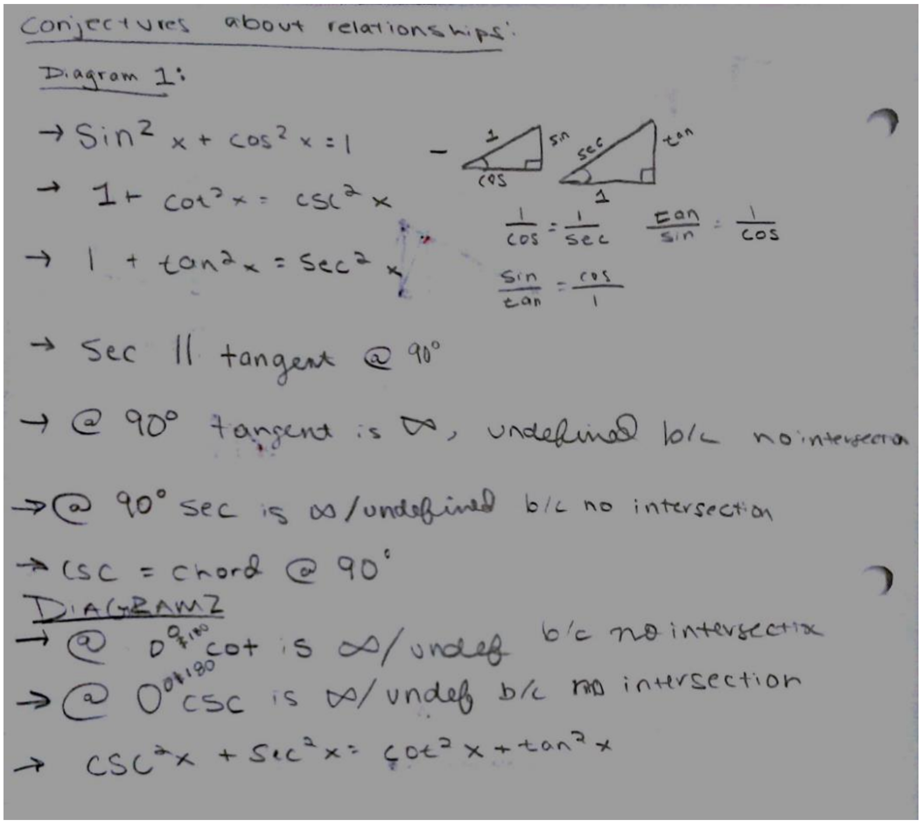


Figure 33. Eron's conjectures

While clarifying one of the relationships to another student, Suzie was able to reason creatively by using the segments and how they are related using the Pythagorean theorem (Line 12, Excerpt 15) and she backed that up with clearly pointing to the segments in the diagram. However, in the same breath she surprisingly states that “and it is because I knew that (*referring to the identity*)”. Perhaps not yet coming to terms with the fact that she just stated an identity without necessarily recalling in from memory but by using the directed length interpretation to derive it.

Almost all the students responded to the task of making conjectures about relationships among different function by only identifying the “commonly” taught identities. For example, after finding some of those “common” identities, Mark stopped working. “I give up. I only got seven", he said. A little later though, and not prompted, when he investigated his diagrams further, he was able to generate more relationships (see Figure 34)

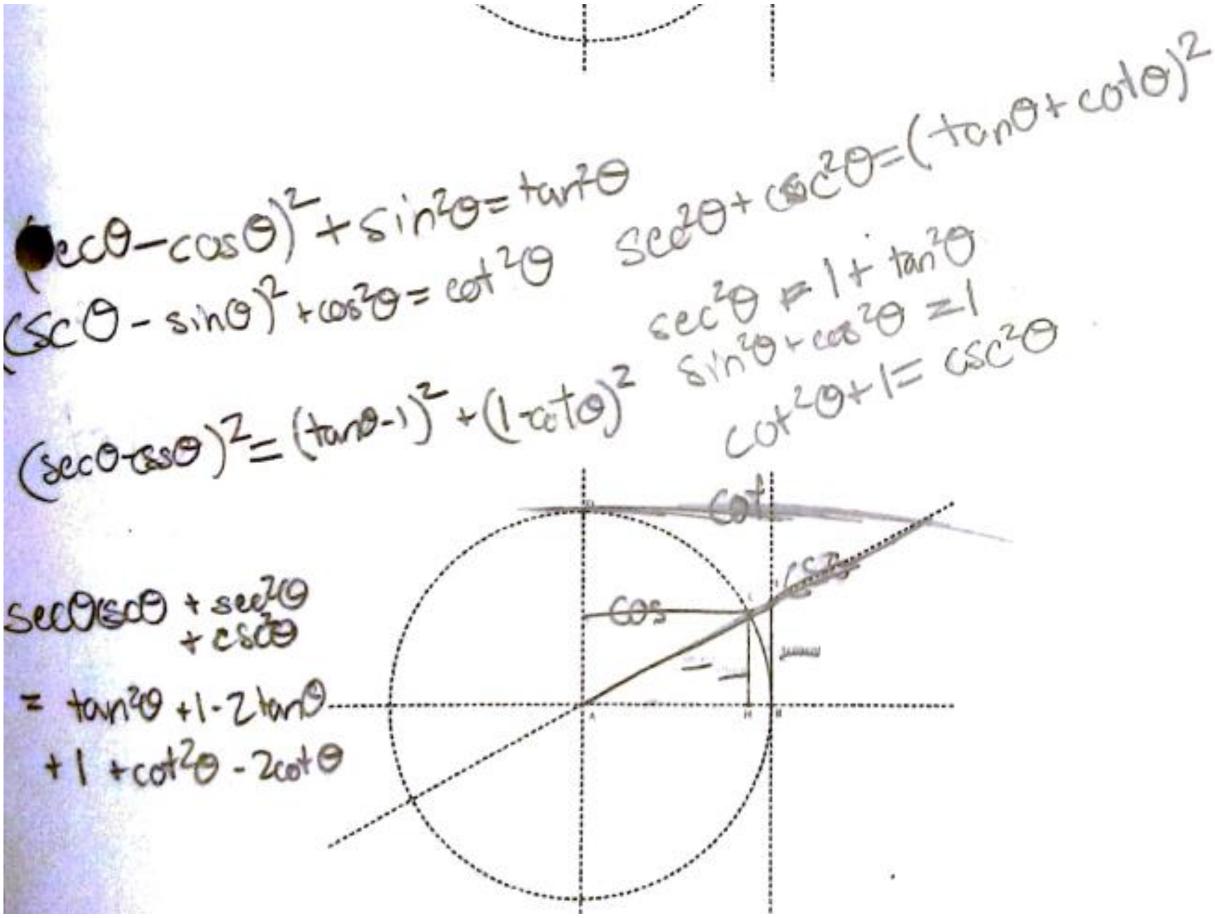


Figure 34. Mark's trigonometric relationships

Having explored and noted all the familiar relationships, the students struggled to develop more conjectures from the two diagrams. For instance, in Group A, I heard statements such as, "Okay, now we have what we knew …", "Mark is throwing in the towel", and "I give up. I only got seven". This prompted me to suggest to them that "Whatever relationship you can come up with from the diagram [is okay]. You don't have to know it [from before]". This perhaps egged the students on, as they immediately started to explore their diagrams noting more conjectures. Amidst all this, some relationships were not instantly noticeable by all the students and in some cases, it took the help of their peers to explain the reasoning (Excerpt 16).

Excerpt 16

1	Eddie:	(*Talking to Eron)* Go to version 2 and see what happens. Go back to like the first quadrant.	
2	Suzie:	What if we did $(\cot x + \tan x)^2 = csc^2 x + sec^2 x$? (*pointing at the computer screen while tracing out the different triangle parts she used)*. (see Figure 35)	CMR4
3	Eddie:	I like it.	
4	Me:	Suzie, can you talk to me about that?	
6	Suzie:	Which one? The one I just wrote?	
7	Me:	Yes	
8	Suzie:	This is 90^o, right? This is our chord which is sine, and this is … oh! you got me. This is true if ... but this bottom is not secant.	CMR5
9	Me:	But the whole piece is secant.	
10	Suzie:	Yeah, oh which is $\sec^2 x – \csc^2 x$	CMR6
11	Eron:	Why did you subtract?	
12	Suzie:	This is $(\sec x – \cos x)^2$ plus this thing squared.	CMR7
13	Me:	That thing is?	
14	Suzie:	Sine, uhm chord squared is equal to tangent squared. Okay, that was pretty good, that was exciting.	AR-PG
15	Eddie:	So, every single radius is 1?	
16	Eron:	We are in radians, so we can call it 1	AR-PG

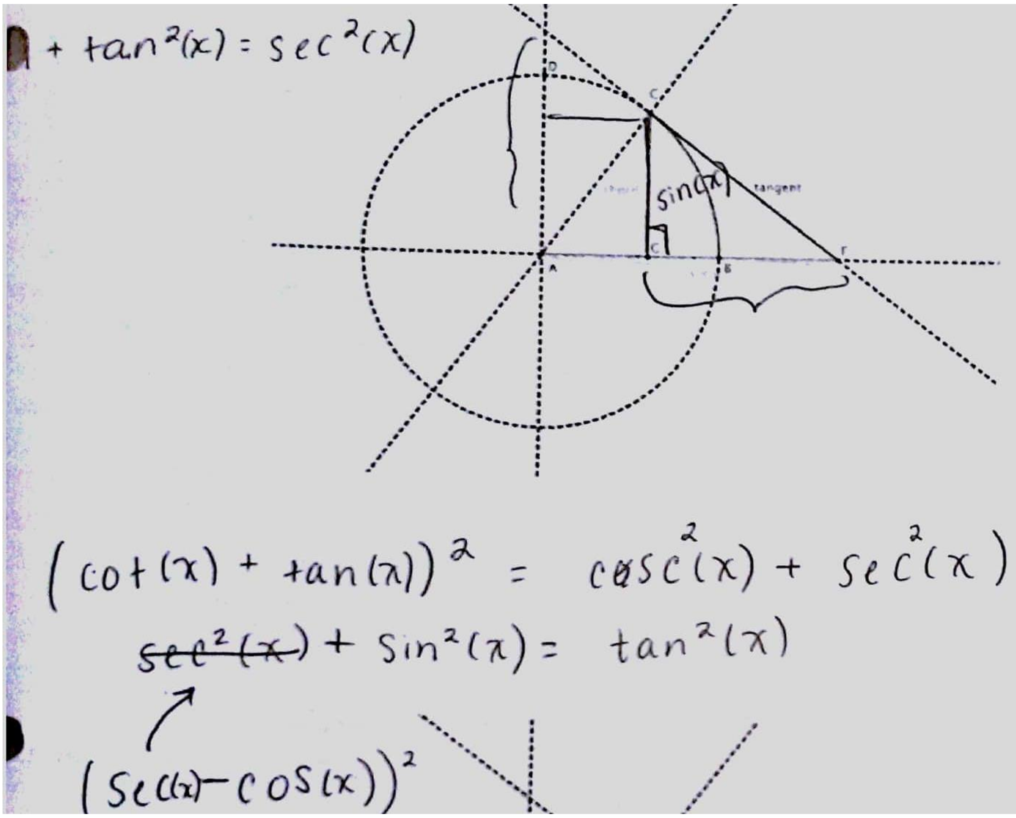


Figure 35. Suzie's conjectures using the Pythagorean Theorem

Like before, identifying the relationships and creating conjectures was entirely based on the Pythagorean theorem. However, in this case, unlike before, the students moved beyond using just prior knowledge of identities to facilitate their exploration of the new identities, but they ably justified their conjectures, and when probed further, they were able to notice their mistakes if any in the individual conjectures. These conjectures were novel in the sense that they involved positive and negative forms of the six trigonometry functions but not in the form of the relationships typically taught in trigonometry course at any level. Unlike in earlier tasks, there were several cases in which students were using CMR with almost no sign of MR. In fact, as shown in Excerpt 16, this part of the task made it nearly impossible for students to use strategies that were founded on recalling complete answers as they had not seen anything similar.

As shown above, nearly all the discussion was centered around identifying the right triangles in the two diagrams and using the Pythagorean theorem to make conjectures dealing with the relationships between and among the different trigonometric functions. I now present below (Excerpt 17) their response when I prompted them to consider using the concept of similarity of triangles to make more conjectures.

Excerpt 17

1	Me:	In all scenarios you've used Pythagorean theorem, can you use something else, maybe you will get something.	
2	Mark:	But there is another one you can get, I don't remember how I got it exactly now, but I know it is there.	AR-D
3	Suzie:	Now you are doing which triangle?	
4	Mark:	I am doing 1 minus cotangent, so that you have that little bottom piece, and tangent minus, and then secant minus cosecant squared. That's all I got.	CMR8
5	Me:	How about similar triangles?	
6	Suzie:	If we can make ratios for similar triangles. So, this is 1 (*pointing to the radius*, $\overline{AC}$), $\overline{\text{AE}}$ is secant, right? So, $\frac{\sin(x)}{\tan(x)} = \frac{1}{\sec(x)}$ and $\tan(x) = sec(x)\sin(x)$. Oh! It is the same thing. (see Figure 36)	CMR9
7	Me:	That is the whole idea behind this. You cannot use a statement you know yet.	
8	Suzie:	Can I use the same triangle? (*Noticing one of the intermediate steps she had written earlier*). Oh, look at this, $\frac{1}{\cos(x)} = \frac{\sec(x)}{1}$	CMR10
9	Me:	Now that is the definition.	
10	Suzie:	It is not a definition, it is from these ratios.	
11	Me:	Right, but now you know where that comes from.	
12	Suzie	Yeah	
13	Eron:	You found out how to … explain to me Suzie.	
14	Suzie:	Now you see these triangles are similar (*Referring to triangle AEB and triangle ACH*) – see Figure 36.	
15	Eddie:	Are they all similar triangles?	
16	Suzie:	Not all of them.	
17	Eron:	Oh! This little one to the bigger one? Okay!	AR-PG
18	Eron:	Maybe it will help me when I draw them out. Okay, where are we at?	
19	Mark:	They are all similar. I am putting down my chip 10 out of 10 they are all similar.	

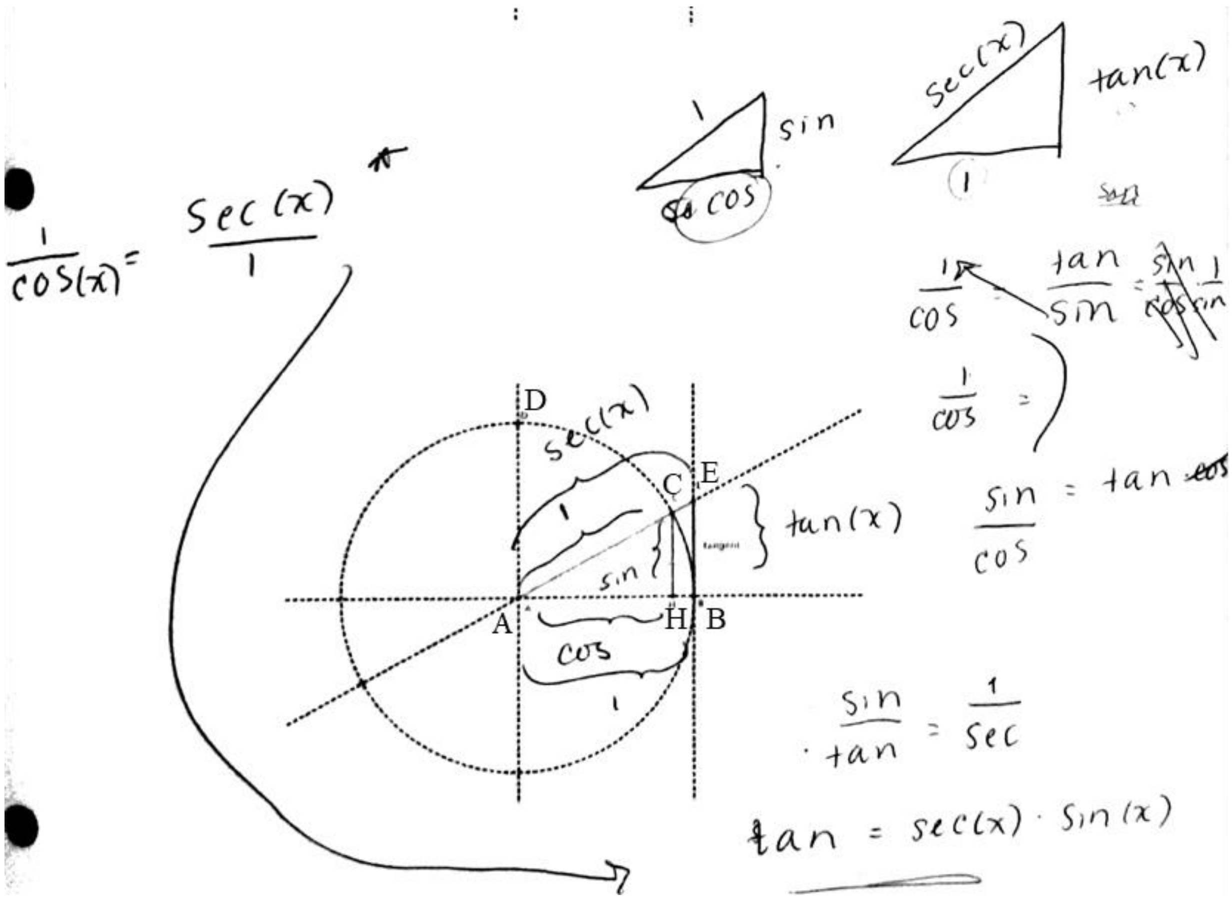


Figure 36. Suzie's relationships due to similarity of figures

Following the suggestion to consider similarity of triangles the students made more discoveries that involved reciprocal relationships. They also spotted congruent relationships that were not initially evident. For example, as Suzie was exploring that $\frac{1}{\cos(x)} = \frac{\sec(x)}{1}$, she was also able to come to the realization that it was the same identity as $\tan(x) = sec(x)\sin(x)$. This is captured in her drawing an arrow to link the two (see Figure 36) as well as in her explanation to Eron below.

Eron: So, you did sine over tangent?

Suzie: It is the same. I did tangent over sine and secant over 1.

Eron: So, how did you know they are the same?

Suzie: Oh, because I did this one (*referring to* ΔACH), 1 over cosine, and then secant over 1 (*referring to* ΔAEH). So that tells you that secant is 1 over cosine.

However, as much as there was a notable number of cases with CMR as evidenced by the creation of new reasoning sequences with supporting arguments and plausible conclusions, there were also cases in which some students seemed not to make sense of what was going on. For example, even after Eron discovered particular relationships from the diagram, she did not quite register them as new conjectures because they were part of her prior knowledge. In fact, that inhibited her reasoning about subsequent relationships as she did not see a way out of the loop. When she had trouble proceeding, I queried her to explain her inability to reason about the task at hand. “What happened here lady?”, I probed while pointing at her work on the paper (see Figure 37). “I started it but could not proceed. Both sides were equal to cosine. But this alone would not mean anything”, she responded with a dejected look. “So now what?”, Suzie interjected. “You could do more”, I replied. “You could do them all using the same thing with 1 and the cochord”, Suzie shot back. A seemingly confused Eron interposed, “I give up on that one” ($referring\ to\ tan(\theta) = \frac{sin(\theta)}{cos(\theta)}$). This showed that, even at this stage in the instructional sequence, the students were still employing reasoning strategies in which the “outcome is not predicted” (Lithner, 2008, p. 263), evidencing traces of delimiting AR.

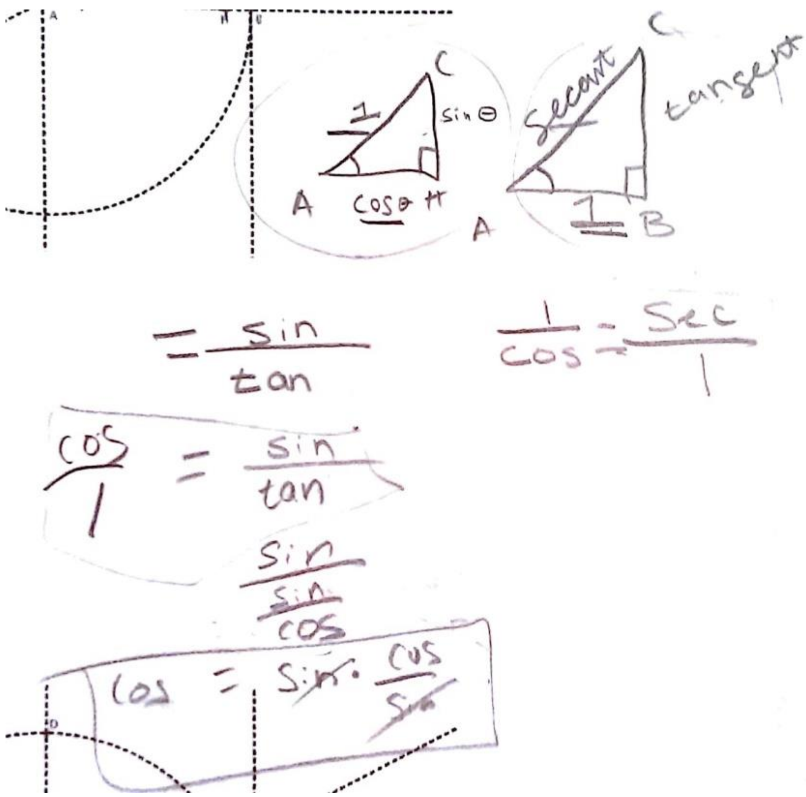


Figure 37. An example of a reasoning strategy with no predicted outcome

As Eron was still pondering on how to proceed, the instructor interrupted the group discussions and students' attention was called to a general class discussion. The instructor inquired about how what the students were working on compared to their prior experience of trigonometry. Smith intimated that he actually got to learn a lot of relationship more than what he had learned [before]. "It was more enlightening", he continued. Eddie pointed to the need to infuse such an experience on the high school trigonometry curriculum. He stated, "I think doing something like this definitely gives it substance, which is something I very much wanted to have in high school, which I think a lot of kids today would appreciate to have because I know when I was learning $\sin^2 \theta + \cos^2 \theta = 1$, I was like awesome. But then just moving on and just having to remember that."

Because of the unfamiliar nature of relationships that were expected from this task, the instructor called for volunteers to share their conjectures with the rest of the students. Andrea offered to present one of her conjectures on the board (see Figure 38). As a conclusion to her presentation, she offered the following explanation along with her written work; "I was focusing on this triangle right here. Triangle FGE and I got $[\cot(x) - 1]^2 + [1 - \tan(x)]^2 = [\csc(x) - \sec(x)]^2$. I was challenged to change the setup, and I came up with almost a similar identity $[1 - \cot(x)]^2 + [\tan(x) - 1]^2 = [\sec(x) - \csc(x)]^2$."

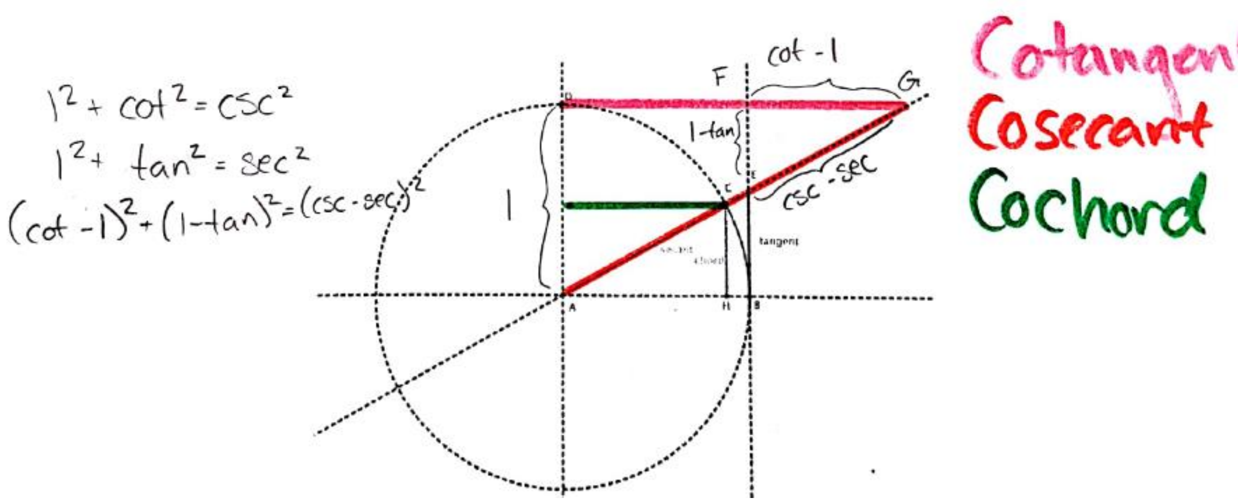


Figure 38. Andrea's diagram for exploring the relationships between the functions

Several other relationships became evident to the students as they investigated the two versions of the geometric representations of trigonometric function. In the following couple of pages, I present some of these students' conjectures that also reflect their CMR, given that each student was able to justify their novel work with "arguments that are anchored in intrinsic mathematical properties" (Lithner, 2008, p. 266) void of simply recalling and writing down complete answers or recalling algorithms.

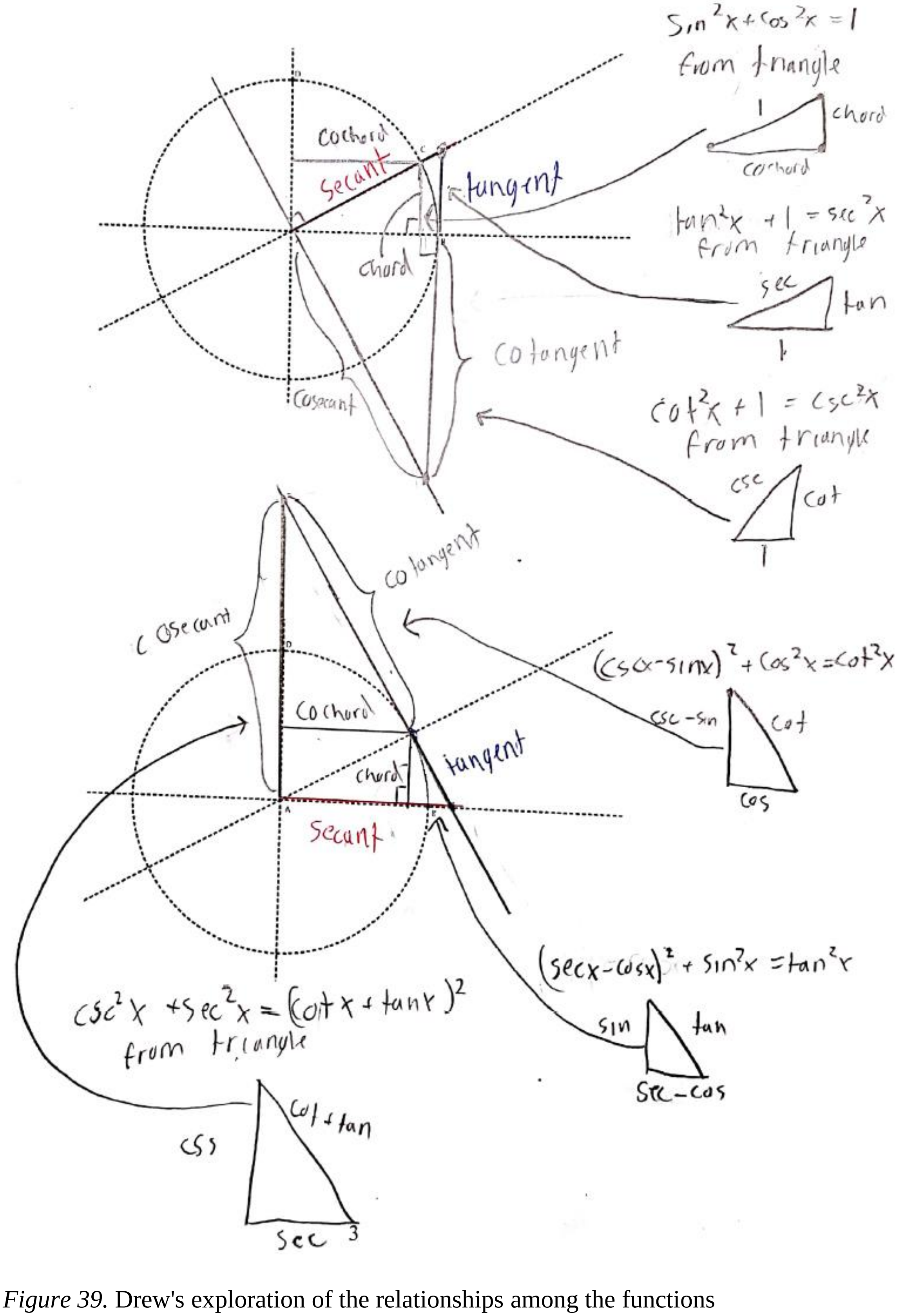


Figure 39. Drew's exploration of the relationships among the functions

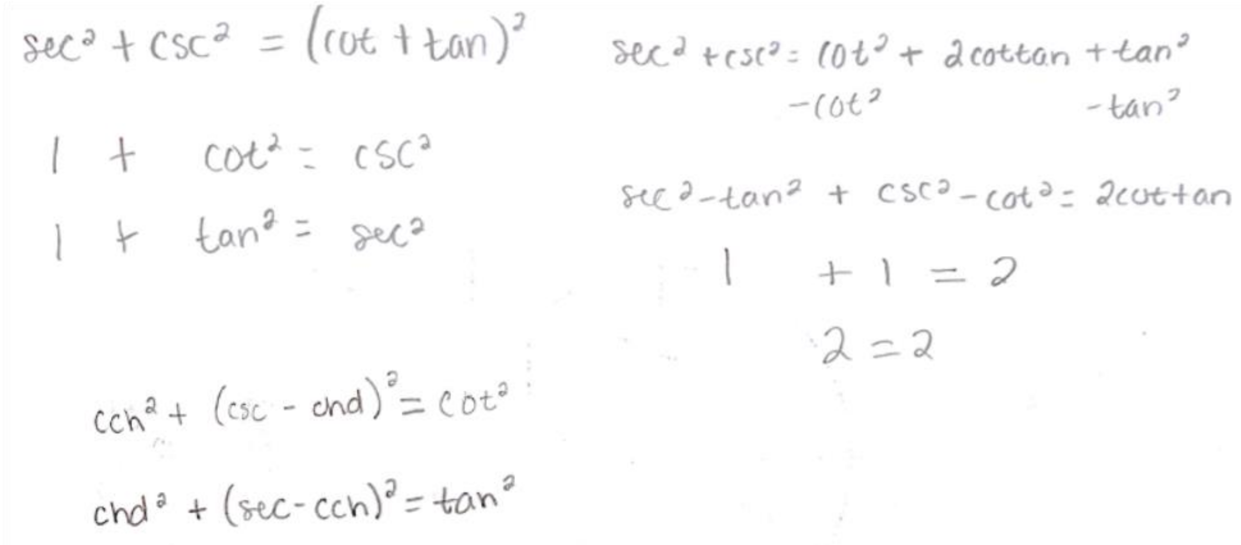


Figure 40. Monica's relationships from using the Pythagorean theorem

Up to this point, the instructor had engaged the students in focusing on the unfamiliar identities some of which are presented above. On a different note, the instructor revealed to the students how, as a high school teacher, his experience teaching trigonometry was pretty different. It was more focused on memorization of the identities than reasoning about them. To proceed when the attention was turned to the “basic” relationships, Jackie gracefully shared quite a number of them with the rest of the class (see Figure 41 & Figure 42). Synonymous with how the majority of the students had navigated the task of exploring the relationships between and among the different functions, Jackie’s strategies as CMR. Jackie’s presentation stretched until the end of the day’s teaching session, and before departing for the day, the instructor promised to continue with the same task in the next session with more focus on the reciprocal identities.

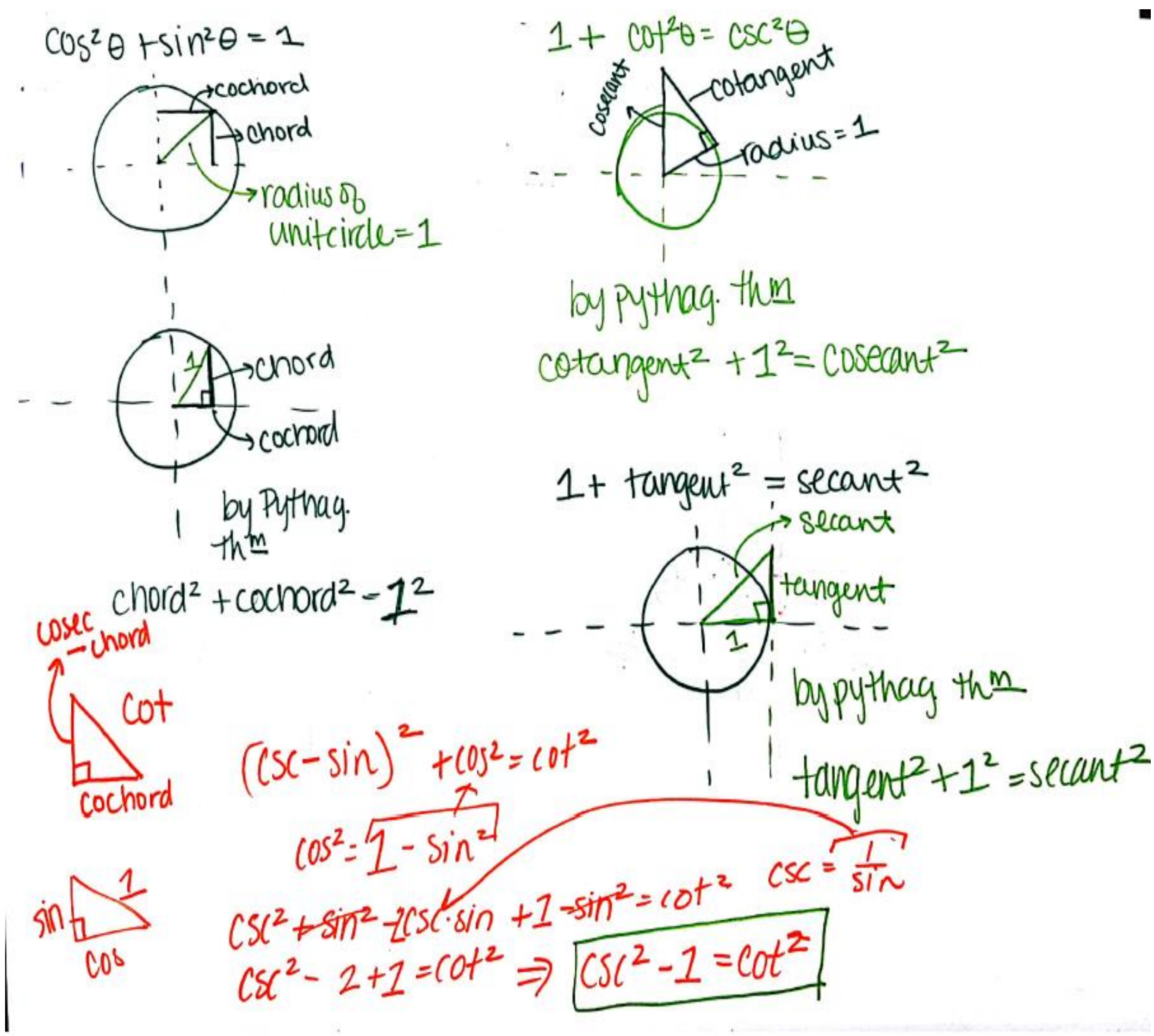


Figure 41. An example of an exploration involving familiar relationships

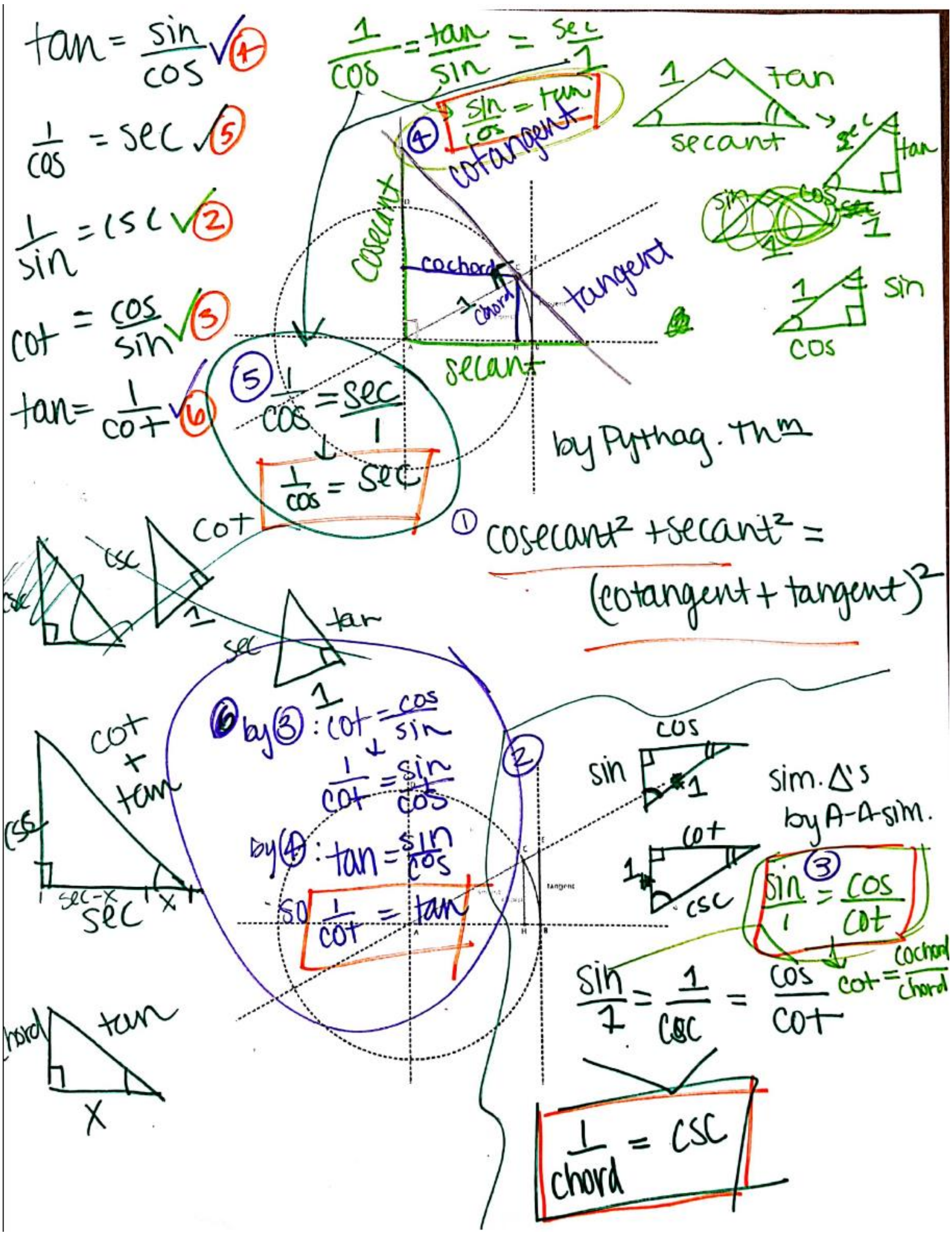


Figure 42. Jackie's colorful relationships which were shared with the whole class

In the previous instruction session, students successfully accomplished the task of identifying both the usual and some less-obvious relationships among and between the trigonometric functions from the geometric representations of these functions (see Figure 32). In the first part of this new instruction session we picked up from where we left off, using a list of identities from the website http://www.themathpage.com/atrig/trigonometric-identities.htm, the instructor invited students "to make nice formal proofs of a selection of identities from the list" using the two versions of the diagrams (Figure 43) that were worked on the previous week. Different groups were assigned to prove one of each of the reciprocal identities and the tangent/cotangent identities. The instructor stressed the need for the proofs to be geometric in nature and "should not include any algebra steps if possible! (Also, it should be possible.)"

Version 1 Version 2

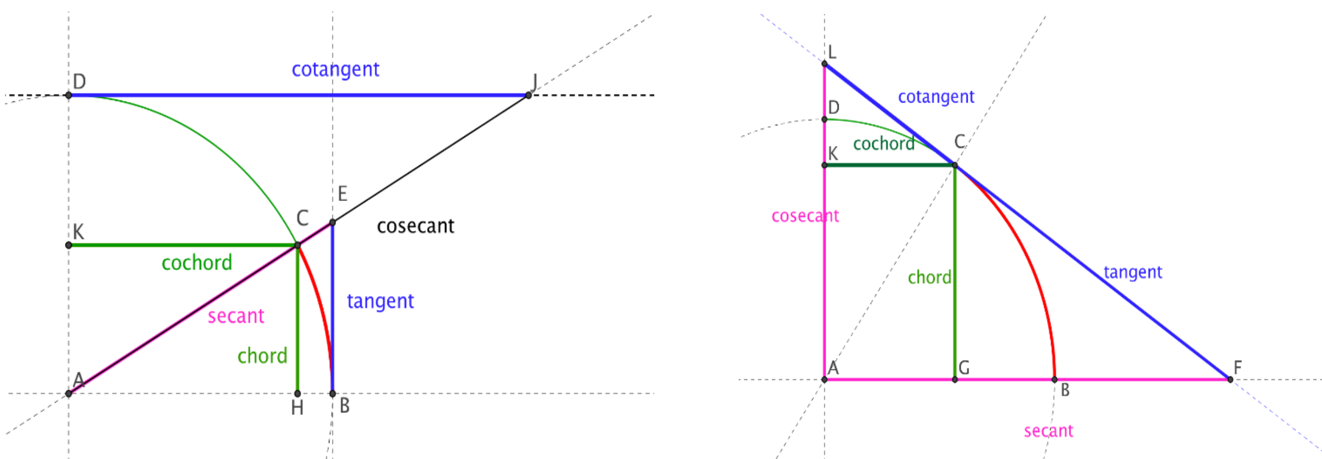


Figure 43. Two definitions for each trigonometric function

Group A members (Eron, Suzie, Eddie and Mark) were assigned to prove the identities $cochord(\theta) = \frac{1}{secant(\theta)}$ and $tangent(\theta) = \frac{chord(\theta)}{cochord(\theta)}$. Their actions and notes offered insights into their justifications and reasoning (Excerpt 18). Initially, Suzie and Mark are each working individually while Eddie is working with Eron. However, midway through the discussion as the students seemed to have trouble progressing, they regrouped and shared ideas as a group.

Excerpt 18

1	Eron:	Okay! So, we wanna show …, so we probably wanna find similar triangles. So, we do $cosecant(\theta) - chord(\theta)$ (*referring to version 2*).	CMR11
2	Eddie:	So, we can't use that secant = 1/cosine?	
3	Eron:	We probably have to prove it but for now I am going to assume it is. What were you guys going to do to prove they are? And using Angle-Angle property. Okay! Are we allowed to use what cosecant is?	CMR12
4	Eddie:	No	
5	Mark:	What version are you guys using?	
6	Eron:	Why? Are you looking at a better one?	
7	Mark:	Yeah. Version 1 is straightforward.	
8	Eron:	What triangles are you looking at?	
9	Mark:	I am looking at ΔAKC and ΔADJ	
10	Eron:	The cosecant is confusing me. What is cosecant? Do you think it is $\overline{AJ}$? And $\overline{AE}$ is secant?	AR-PG
11	Eddie:	Yeah, I agree.	
12	Mark:	Then change of plans.	AR-PG
13	Eron:	So, we can't use that secant = 1/cosine?	
14	Eddie:	No. So we can't use these triangles anymore?	

Whereas the students in Group A were headed in the right direction, for some reason they appeared to be held back by fact they could not easily distinguish between the secant and cosecant as they were overlapping on the same segment $\overline{AJ}$ in version 1. It is worth recalling that identifying two different versions of each function was part of the task from the previous week's session. In the meantime, the instructor had displayed the same diagrams on the interactive white board, which I leveraged and asked them to identify where the segment representing the secant function starts and ends. Looking at the version on the smartboard, Eron exclaimed, "Oh, oh! I think it stops at E. So, we can use triangle ΔAHC and ΔABE." With the functions identified, the students then continued to work on the task (Excerpt 19).

Excerpt 19

1	Eron:	And $\overline{AC}$ is the cochord then?	
2	Eddie:	That is 1	
3	Mark:	That looks … it turned out pretty well.	
4	Eron:	So, we have $\overline{AC}/\overline{AE}$ which is $1/secant$, and so saying cochord = $\overline{AH}$?	CMR13
5	Suzie:	Yes	
6	Eron:	How do we get that?	AR-D
7	Suzie:	Because it obvious to me	AR-F
8	Eron:	Will you say that in a proof? (*She proceeds to prove that* ΔAKC *is congruent to* ΔAHC*.)*. (See Figure 44)	
9	Suzie:	I don't think it's the current position of the cochord that matters. What matters is how we define the cochord.	AR-F

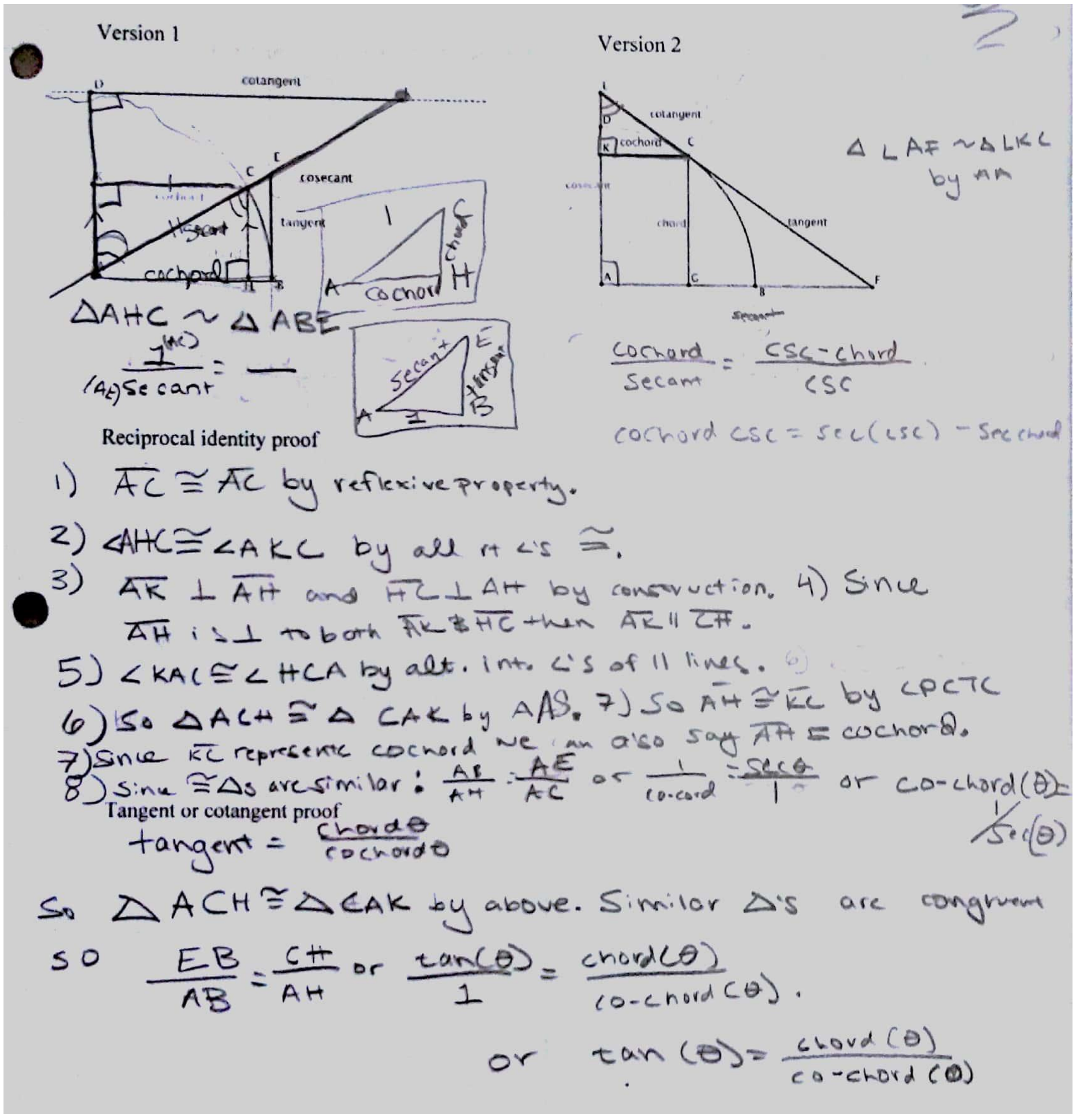


Figure 44. Eron's work on the reciprocal identities

As revealed above (Excerpt 19 and Figure 44), Eron with the help of her peers in the group was able to reason creatively about the reciprocal relationship between the cosine and secant. However, even though her explanations and writings were dominated by CMR while proving that the tangent function is equivalent to the ratio of sine to cosine, “the initial strategy implementation followed guidance” (Lithner, 2008, p. 264) from Eddie. After executing the remaining routine steps to conclusion, Eron later acknowledged, “Okay! I am just used to similarity using different triangles not on the same triangle.” After, as a class we quickly chatted through the three main Pythagorean identities as well as the two additional identities included in the collection on the website.

Concerning the task of exploring the relationships between and among the trigonometric functions a few cases of IR arose but unlike in the previous tasks there was convincing evidence that students were not only able to provide new reasoning sequences but also used strategies that were rooted in fundamental mathematical properties. While engaging in this activity, novel relationships were discovered. For example, it was a common occurrence to see students with relationships like the ones in Figure 45 below

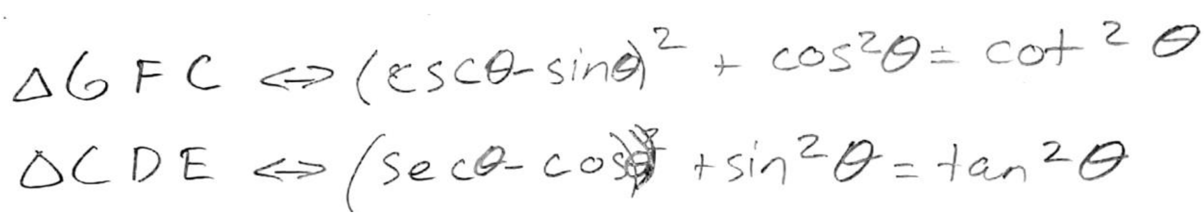


Figure 45. Some of Justin's relationships from the exploration.

As a whole, Task 4 offered students several avenues to exhibit their reasoning. Of the 30 registered cases of reasoning, 16 were categorized as strands of IR. However, in only one of these cases the strategy choice was founded on recalling memorized answers. This pales in comparison to the other tasks in which quite a number of MR cases were registered (Figure 46).

C (14)	**I (14)**				
14	MR (1)	AR (13)			
	1	AR–F	AR–D	AR–G (5)	
		4	4	AR–TG	AR–PG
				0	5

Figure 46. Counts of students' reasoning when exploring the relationships between functions

Notably also, there was an increase in the instances of person guided AR. This is significant because in all the recorded five cases, the students were seeking clarifications that eventually led them to justification and reasoning that was mathematically creative. Furthermore, unlike in prior tasks, in which students exhibited CMR in few instances, half of the strategies were identified as CMR. Most of the cases of CMR in this task were borne out of guidance from an external source. Also, the almost nonexistence of MR is attributed to both the familiarity students had with quantitative reasoning (i.e., line-segment trigonometry) at this stage in the instruction, and perhaps to the nature of the task, as it did not afford them a chance to use strategies that were dependent on recalling complete answers.

Analysis of the data builds a picture of the different mechanisms and workings of the instructional setting that aided students towards CMR. The main influence in almost all the recorded case of CMR was the established prior knowledge of geometry, including the Pythagorean theorem, similarity and congruence of triangles. The remaining instances of CMR were a result of the instructor's teaching techniques and students' understanding of geometry.

All but one (CMR 5) of the fourteen total instances of CMR registered in this task were supported by students having prior knowledge and an understanding of geometry. In six instances (CMR1, CMR3, CMR4, CMR6, CMR 7, CMR10) the students were able to draw on their knowledge of the Pythagorean theorem and realized how to connect the idea that the sides of right triangles in a unit circle correspond with line segments that represent different trigonometric function. For example, after Eron stated from memory the identities, "$1 + \tan^2\theta = \sec^2 \theta$, a.k.a $\sin^2 \theta+\cos^2 \theta$", and Eddie sought for guidance on "how is it $\sin^2 \theta + \cos^2 \theta = 1$?", Suzie while pointing to the different segments of the circle on her computer screen offered the following explanation; "we look at this triangle and this is the chord, and cochord and there is the radius, so $\sin^2 \theta + \cos^2\theta = 1$ (see Figure 47) below.

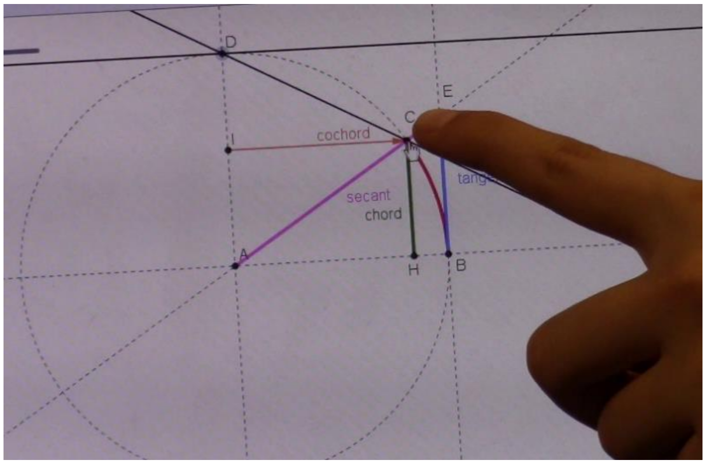


Figure 47. Suzie's demonstration of a trigonometric relationship

Notably, perhaps due to the deliberate immersion into quantitative reasoning (considering the three tasks prior to this) and the use of geometry, Suzie found herself subconsciously reasoning creatively about the identity. This is evidenced in her concluding remarks, in which she stated that, "Not bad, and it is because I knew that (referring to the identity). However, she could not have been further from the truth. Suzie did not just recite the relationship from memory like Eron before her. Rather she leveraged her knowledge of geometry (even if it was not apparent to her) to generate a fundamental trigonometric relationship. Suzie re-constructed a reasoning sequence from her prior knowledge of geometry by identify the specific segments that are related by a particular right triangle, used the identified segments and attached the meanings from different trigonometric functions, which provided reasons for verification of the plausible validity of her reasoning. Suzie's reasoning was also mathematically founded and provided cogency by anchoring in central properties of the components in the reasoning: the relation between the legs of a right triangle and its hypotenuse.

This breakthrough set the foundation for other group members to identify other relationships using the Pythagorean theorem. For example, Eddie (pointing to the sides of one of the triangles in the diagram) submitted that "Yeah, because this one is $1 + \tan^2 \theta = \sec^2 \theta$."

Although still adamant that knowing the identities beforehand plays a role in being able reason about them creatively (see Line 14, Excerpt15), using the Pythagorean theorem, Suzie was able to create a new identity, which is safe to say hitherto had not been known to any of the students. Suzie pointing at the computer screen while tracing out the different triangle parts argued that. "What if we did $(\cot x + \tan x)^2 = csc^2x + sec^2x$?" She also argued that $sec^2x - sin^2x = tan^2x$. With some probing from the instructor, she revised the second relationship to replace the term sec^2x with $(\sec x – \cos x)^2$ in the identity (see Figure 35). In the general class discourse, Andrea (from a different group) also depended on her prior knowledge of the geometry to create a couple of novel (to her and perhaps all her peers) and plausible conjecture; $[\cot(x) - 1]^2 + [1 - \tan(x)]^2 = [\csc(x) - \sec(x)]^2$, and $[1 - \cot(x)]^2 + [\tan(x) - 1]^2 = [\sec(x) - \csc(x)]^2$, which were justified with supporting logical and valid reasoning based on the Pythagorean theorem. Without any prior knowledge of geometry, none of the stated trigonometric relationships could have been generated without the students engaging in IR.

Similarly, albeit with initial egging from the instructor, students were able to creatively reason about more trigonometric relationships. In some instances (e.g., CMR8, CMR11, CMR13) basing their CMR on the similarity relationship between particular triangles, whereas in others (e.g., CMR 9, CMR14) the congruence of triangles provided the foundation that supported the students in realizing how to relate and connect different line segments from the different triangles. For example, Eron was able to reason creatively about the reciprocal relationship between the cosine and secant and also $tangent(\theta) = \frac{chord(\theta)}{cochord(\theta)}$ by leveraging her prior knowledge of similarity of triangles as shown in Figure 48 below.

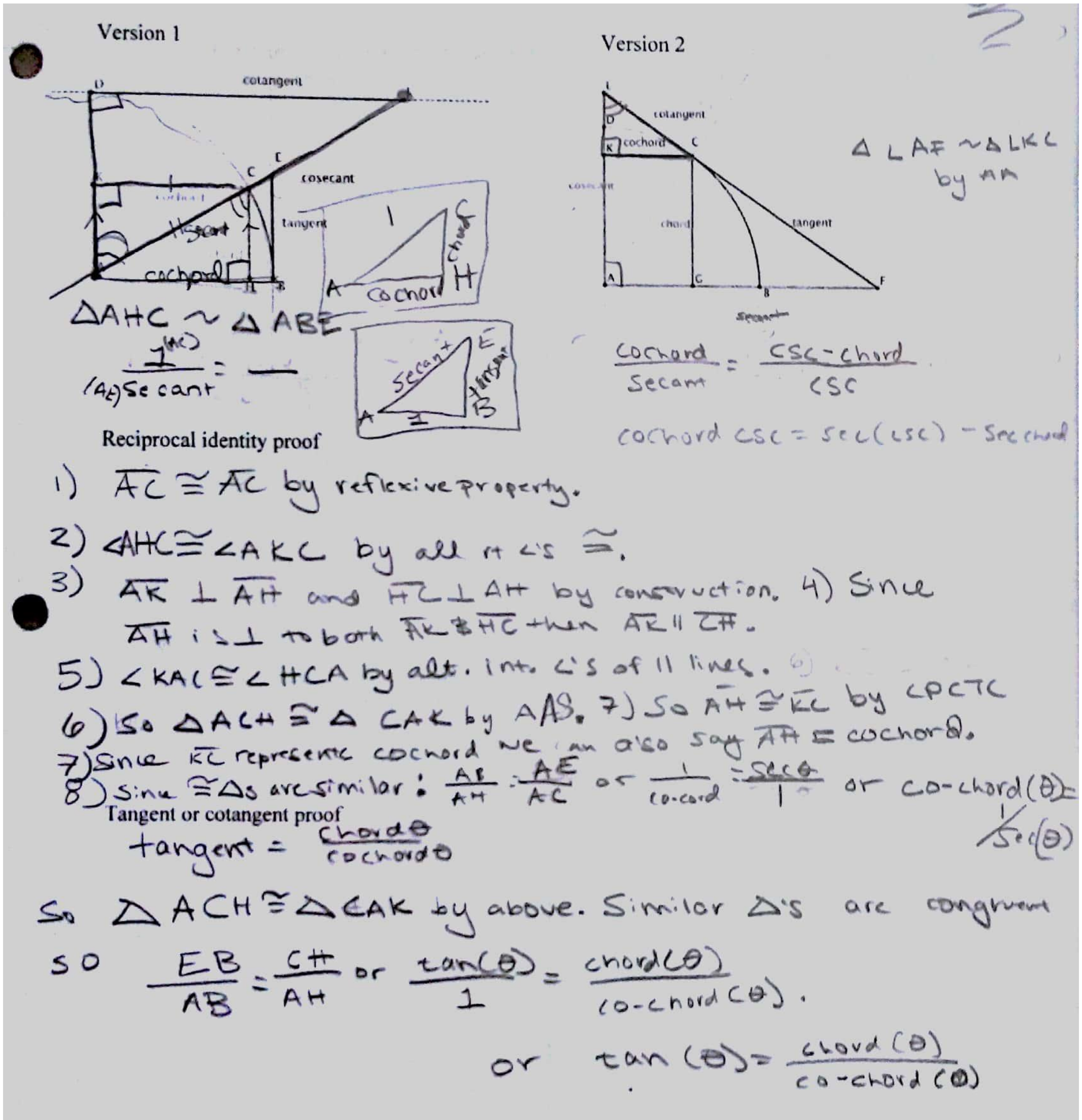


Figure 48. Eron's work on the reciprocal identities

Secondly, students' CMR also depended on the particular teaching moves adopted by the instructor. Students leveraged the peer interaction in their small groups to achieve various plausible outcomes. The deliberate setting up students for small-scale group work and following it up with whole class dialogue, generated a platform for students to communicate their thinking (as conjectures) and which gave access to others not only critique but also build on them. Furthermore, in this collaborative discussion, the students resolved contrasting viewpoints and consequently created new or re-created forgotten reasoning sequences. Such avenues seldomly occur in a traditional lecture setting. For example, at the beginning of the task Eron stated, “I only know $1 + \tan^2\theta = \sec^2 \theta$, a.k.a $\sin^2 \theta+\cos^2 \theta$. I also know $1/\sin(\theta)$ and similar ones.” This was a case of IR, as she was only recalling complete answers from memory. However, after she gained access to Suzie’s idea of reasoning creatively about the same trigonometric function (see Line 12, Excerpt 15), Eron stated, “you could go with the same thing. We know that this is perpendicular to this line, we know that this is going to be a right angle, and that this a cotangent, and that this blue including this purple is going to cosecant, so $1 + \cot^2 \theta = \text{cosecant}^2 \theta$.” This instruction technique in which the students were not subjected to traditional tutorial or lecture setup, played a fundamental role in facilitating Eron in being able to build on her peer’s idea to create a novel (to her) reasoning sequence that was mathematically founded on the properties of the Pythagorean theorem to eventually transition from IR and engage in CMR.

Acting as an intermediary also helped the students enhance their understanding of the concepts and they were able to reason creatively. Aware of the fact that technology is just a tool, the instructor effectively leveraged the DGE to support student engagement, and reasoning but not as an end onto itself. The students at times pointed to the segments on their computer screens while making arguments to support their strategy choices. Also, the instructor intentionally interjected in various group interactions to point the discourses in the direction of CMR. Some examples include, “In all scenarios you’ve used Pythagorean theorem, can you use something else, maybe you will get something”, and “How about similar triangles?” These subtle interjections led students to explorations in which the formulated and creatively reasoned about sets of conjectures which they otherwise would never have been able to without such teacher moves. Another glaring example where instructor use of technology and intentional interjection led to CMR is the case Andrea in CMR10. She stated: “I was focusing on this triangle right here. Triangle FGE and I got $[\cot(x) - 1]^2 + [1 - \tan(x)]^2 = [\csc(x) - \sec(x)]^2$. I was challenged [by the instructor] to change the setup, and I came up with a similar identity $[1 - \cot(x)]^2 + [\tan(x) - 1]^2 = [\sec(x) - \csc(x)]^2$.”

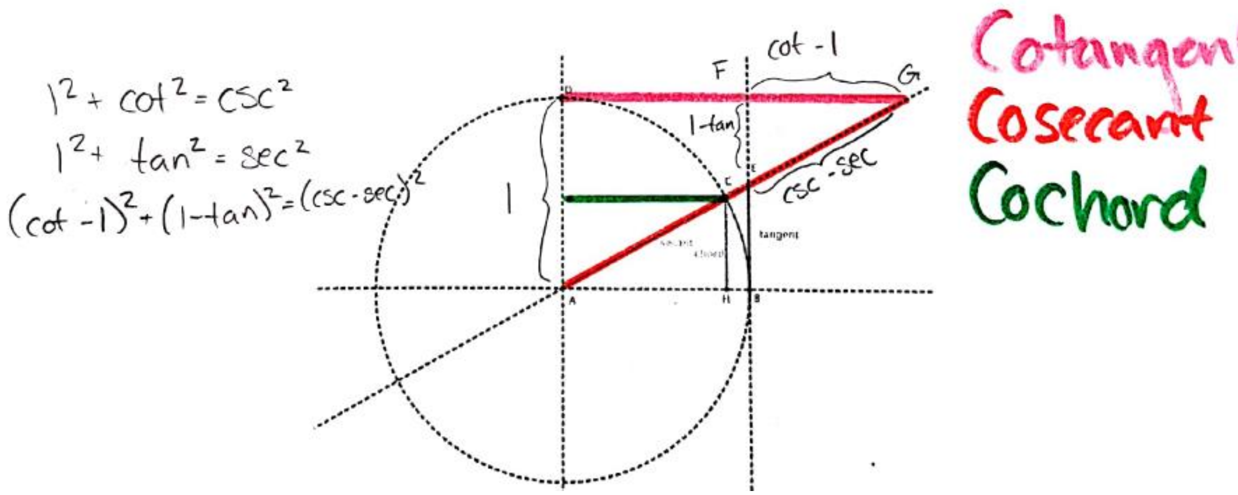


Figure 49. Andrea’s diagram for exploring the relationships between the functions

Andrea constructed novel reasoning sequences when determining the functions for each leg of the different right triangle used. She had not practiced any tasks that are analogous to the one before her, and though her basic knowledge of length of a line segment is used it does not furnish her with the complete answer or a solution algorithm as in the case of MR or AR. Also, Andrea had arguments to back up the plausible validity of her reasoning. The arguments were not just amplifying something known beforehand (the Pythagorean theorem in this case) but influenced the strategy choices and the approach implemented and resulting in new knowledge (to Andrea). Finally, Andrea’s arguments were mathematically founded and took into account the core properties of the components in the reasoning: the trigonometric functions representing each segment, and the relation between the legs of a right triangle and its hypotenuse (see Figure 49)

CHAPTER V: CONCLUSIONS

In this study I explored the types of reasoning that preservice secondary mathematics teachers use in the process of learning trigonometry with instructional emphasis on quantitative reasoning through a line-segment definition of trigonometry. In this chapter I summarize my findings, provide a general description of the strands of reasoning (Lithner, 2008) through a discussion of plausible answers to the research question that guided the study. This question was:

- What types of reasoning do preservice secondary mathematics teachers engage in while participating in an instructional sequence on trigonometry that focuses on a quantitative reasoning approach?

Types of Reasoning

Students' Reasoning Strategies

In Table 8, I present an overview of the counts of students' reasoning strategies from all four tasks, indicating the number of times the students engaged in IR and CMR. The data presented in Table 8 does not differentiate the different strands of IR that the students engaged in. A more detailed consideration of the several forms of IR adopted by the students in the different tasks is provided in the next subsection. There were 65 instances of IR throughout the four tasks, and 10 of those instances occurred during Task 1. Similarly, there were 32 instances of CMR throughout the four tasks, and 4 of those instances occurred during Task 1.

Table 8

Students Reasoning Strategies

Task	Students' Reasoning Strategies		Total
	Imitative Reasoning (IR) (N=65)	Creative Mathematical Reasoning (CMR) (N=32)	
Task 1: Chord function	10 (16%)	4 (13%)	14 (15%)
Task 2: Tangent function	26 (40%)	8 (25%)	33 (34%)
Task 3: Secant function	15 (23%)	6 (19%)	21 (22%)
Task 4: Trigonometric relationships.	14 (22%)	14 (44%)	28 (29%)
Total	65 (67%)	32 (33%)	96 (100%)

Results of the qualitative analysis of the reasoning strategies from the instructional sequence as a whole indicate that there were twice as many instances of IR than instances of CMR during the study. In general, Task 1 (Chord function) elicited the smallest percentage of instances of reasoning (15%), whereas Task 2 (Tangent function) produced the greatest percentage of instances of reasoning (34%) from the whole instructional sequence.

Imitative reasoning. The further analysis presented in Table 8 indicates that in about 67% of the reasoning instances, students' reasoning strategies were either based on recalling and writing down complete answers (MR) or on a solution algorithm (AR). As we progressed through the tasks, there was an initial increase in instances of IR (Task 1 to Task 2) but from Task 2 (Tangent functions) to Task 4 (Trigonometric relationship), the instances of IR declined (Table 8).

A more in-depth look at the distribution of the different strands of IR (see Table 9) shows that students engaged in MR only one in every four instances in which they reasoned imitatively.

The qualitative analysis indicates 14 out of 16 (88%) of all coded instances of MR to have occurred when students were reasoning about the tangent (Task 2) and secant (Task 3) functions (Tangent function – 8 [50%], and Secant function – 6 [38%]). In comparison, AR represented 48 out of 64 (75%) of the coded instances of IR, with 18 of the 49 (37%) being in Task 2 (Tangent function), 15 of the 48 (27%) coming from Task 4 (Trigonometric relationships), Task 1 (Chord function), and Task 3 (Secant function) each having nine instances (18%). These results show that in utilizing their prior knowledge of trigonometry to navigate the tasks, students heavily depend on recalling solution algorithms instead of memorized complete solutions. In the particular case of trigonometric relationships, these numbers indicate that in all but one of the 14 instances of IR, students could only reason using pre-specified procedures (AR).

Table 9

Students' Imitative Reasoning Strategies

Task	Students' Imitative Reasoning Strategies	
	Memorized Reasoning (MR) (N=16)	Algorithmic Reasoning (AR) (N=48)
Task 1: Chord function	1 (6%)	9 (18%)
Task 2: Tangent function	8 (50%)	18 (37%)
Task 3: Secant function	6 (38%)	9 (18%)
Task 4: Trigonometric relationships.	1 (6%)	13 (27%)

A further scrutiny of the findings reveals that in all the instances when students used reasoning schemes that were coded as IR, their strategies had emanated from the ratio and unit circle definitions of trigonometry. This is was perhaps due to the fact that at the beginning of instruction unit, the line-segment idea was still a new concept to the students. This observation is consistent with previous research (e.g., Kendal & Stacey, 1998; Moore, 2013, 2014, Thompson et al., 2007, Thompson 2008) about the impact of ratio and unit circle approaches of instruction have on the learning of trigonometry and the difficulties students encounter while dealing with trigonometry and trigonometric functions and. That is not to say that the ratio and unit circle approaches do not provide opportunities to reason creatively. I posit that students' ability to reason is curtailed by how the ratio and unit circle concepts are initially introduced to them. A similar call was made by Weber (2005) and earlier by Kendal and Stacey (1997). Weber (2005) stated that even though students in his experimental instruction group who were taught using the unit circle approach performed better than their counterparts in the lecture-based instruction group, emphasis should be on having students understand the unit circle procedures as *processes.*

Creative mathematical reasoning. Of all the coded responses from all four tasks, students engaged in CMR one in every three instances of the whole instructional sequence. Furthermore, 14 of the 32 instances (44%) of CMR originated during Task 4. As we progressed through the tasks, there was a registered comparable increase in instances of CMR.

Also, important to note is that at the beginning of the instruction unit (i.e., Task 1 & Task 2) when students were still novices at using the technology and the line-segment approach, there are 12 instances of CMR (out of 32 total instances of CMR) that were recorded, despitethe prevalence of the unit circle and ratio trigonometry approaches by the students.. This is an indication, that, not all instances of creative reasoning occurred when the line segment approach was emphasized. When the students used the unit circle and ratio approach appropriately, they were sometime able to reason creatively. In fact, researchers (e.g., Kendal & Stacey, 1997; Schnotz & Bannert, 2003) found that even if students were taught emphasizing procedures and algorithms (versus concepts), they would ably learn a concept as long as the algorithm was accompanied by a representation. It is this use of representations that line-segment approach brought to the fore which the students utilized to transition their reasoning from AR to CMR.

Notably, on specific concepts of trigonometry (e.g., trigonometric identities), Fi (2003) reported that the preservice teachers have misconceptions, and with the exception of the fundamental Pythagorean identity (i.e., $sin^2\theta + \ cos^2\theta = 1$), they labored to derive others, leave alone applying them. However, in Task 4 of this study—which addressed the concept of relationships between and among trigonometric function—students' reasoning was evenly distributed between CMR (14 instances) and IR (14 instances; see Table 8). Furthermore, I contend that CMR was more prevalent in Task 4 (44%) due to: 1) the line-segment approach, which was deliberately and systematically promoted by the instructor was well-established, and 2) the availability and opportunity to use the different geometric definitions for the same trigonometric functions3) the students' vast geometry knowledge to explore several conjectures, and 4) prove them using the different geometric definitions.

Remarks on Lithner's Framework

In the preceding sections I delineated a scientific motivation for this research project. Along with offering added clarity and context to other studies carried out by earlier researchers, I established a rationale for choosing Lithner's (2008) framework to analyze the data in this study. In the following paragraphs, I reflect on the use of Lithner's (2008) framework by providing a critique of it in the specific context of trigonometry in a geometric setup, zooming out to identify which of the critiques apply outside the context of the present study. More explicitly, I discuss the difference between the distinctive forms of reasoning within the line segment approach to trigonometry.

Lithner (2008) alluded that an essential variable in learning mathematics is the reasoning that students actuate in relation to particular tasks. In this study, two main types of reasoning were exhibited: CMR and IR. In general, identifying both CMR and IR was not challenging because there is no ambiguity in defining their differences. CMR included a novel reasoning sequence, which could be justified and was centered on mathematical foundations. The other main category, IR, characterized by rote learning was divided into two strands: MR, in which the student, for instance, solved a problem by recalling a full answer from their prior knowledge, and AR, when a problem is solved by recalling and applying a given algorithm.

Unlike MR, which was routine to recognize, identifying AR was challenging in the context of this study. This was due to several factors unique to this study. For example, the content was not algebraic in nature and thus did not lend itself to following an algebraic algorithm as described in Lithner's examples. In its current state, Lithner's (2008) framework was not well suited to pinpoint which reasoning can be categorized as AR because in this instructional sequence of the line-segment approach there were no particular algorithms for the students to follow in solving the given tasks. In fact, this challenge is not limited to this study but may also extend to other non-algebraic contexts. In its present state, AR can be easily identified if there is a recurring numerical task-solving process that depends on algorithmic support (i.e., a known algorithm is utilized to resolve a given task). If we accept this view of AR, then it becomes clear that there is no sufficient requirement to categorize any reasoning in a geometric context as AR.

Given that much of the uncertainty and confusion regarding AR is a direct consequence of lack of agreed upon and unambiguous definitions in the geometric context, I contend that a more encompassing title for AR is warranted. With hind sight, using this framework in a non-algebraic context may require the AR to be renamed Guided Reasoning (GR) with four strands: (a) Procedure Guided Reasoning (P-GR) – in which a solution strategy choice is made by familiar geometric setup but void of plausible arguments, (b) Text Guided Reasoning, (c) Person Guided Reasoning, both as defined by Lithner (2008), and (d) Delimited Guided Reasoning – when a student cannot reason about a task, regardless of any form of guidance from an external guide.

However, it should also be noted that the ambiguity and challenge in identifying AR as according to Lithner (2008) could be attributed to the fact that in all the tasks observed in this study, the instructor did not provide worked examples (from a person or text) for the students to practice. Thus, they did not get to use any algorithms that they knew, and the students were not likely to be successful with the tasks posed using only IR. It seems as though this is aligned with what Lithner suggested is desirable in a mathematics classroom; that is to say, we need to get students engaged in more CMR. Instances in which the use of IR is suppressed and the use of CMR is promoted, the framework seemed to be lopsided. There are fine categorizations of IR (i.e., MR, AR-F, AR-D, AR–PG, and AR–TG) but only one for CMR. If a goal of mathematics educators is to promote CMR, the framework would be more powerful if it provided finer detail of the types of CMR.

Implications

The findings presented in this dissertation have significant implications for the teaching and learning of trigonometry in that they highlight the different ways students reason while studying trigonometry. In particular, the findings suggest that when the different definitions of trigonometry (ratio, unit circle, and line-segment) are used in conjunction with each when appropriate, it leads to learners creating new (to the reasoner) reasoning sequencings or recreating a forgotten one. The results emphasize that providing such tailored instruction is essential for students to engage in more CMR that builds upon their thinking than IR.

The students intoned concerns about the way they were taught trigonometry in high school. They claimed that they had not learned relationally (Skemp, 1976) and that the struggles they occasionally exhibited during the instructional sessions discussed here may have stemmed from their deficiency in conceptual understanding. Knowing that these future teachers did not get a good understanding of trigonometry when they were in high school, and the setup of the college curriculum is such that this deficiency is not going to be addressed in their standard mathematics curriculum—as it is often presumed that students have the requisite comprehension of those areas —at the undergraduate level, they are then likely to continue the cycle of teaching the topic procedurally to their own high school students.

As is evidenced in this study, in many instances in the first weeks of the study, the students' strategy choices were founded in the line-segment approach mostly when they were alerted to think along that line, or when they could easily imitate what had seen in the previous tasks. Otherwise, they retreated to the ratio and unit circle approaches. However, they also indicated that they "find the line-segment approach both thought-provoking and valuable" (Eddie), which implies that this rarely used approach of teaching and learning trigonometry would be fascinating and engaging for students of trigonometry. In particular, based on differences between the tasks in which reasoning was coded as being creative and the tasks that did not, findings from this study also suggest that in teaching and learning trigonometry students might have more success retaining trig knowledge with more opportunities to reason about relationships.

Zooming out and reflecting on the tasks as a unified activity, over the course of the study, specific questions did arise about CMR. Some of these include: Can the entire collection of activities be recommendable for other populations? Could CMR be achieved without the technology, working in groups, or without specific teacher moves? Could it work with high school students (who might not have the same student competencies that the undergraduate majors do)?

Based on the analysis of the data, I posit that the activity is better suited for a population that has prior knowledge of geometry and other stances of trigonometry. In contrast, this instruction would be an appropriate instructional approach for a group of preservice secondary mathematics teachers (PSTs), as they were found to have the necessary background knowledge of trigonometry and geometry. For these PSTs, beyond offering them an avenue to engage in CMR, this approach allows them an opportunity to review their trigonometry content that they could be teaching the following school year. The activities presented in this study appear to fall in a sort of Goldilocks zone. That is to say, the activities depend on topics within the secondary curriculum but are still challenging enough to provide opportunities for preservice teachers to engage in CMR. For the same reasons, I anticipate that in-service secondary mathematics teachers would also benefit from engaging with the activities based on the line-segment approach. Additionally, in as much as technology was crucial in the initial stages of the instructional sequence, it was not required when investigating the relationships between the trigonometric functions. This suggests that this instructional sequence could be adapted for a methods course or capstone course for the same population of PST.

Beyond these two particular groups, my study provided little support to determine if the instructional sequence would be appropriate. However, based on goals and realities of other populations and contexts I think there are a few other populations to consider. First, I think that this instruction would not be well suited for high school students unless they had the background knowledge of all of the different approaches of trig (e.g., unit circle, ratio), and in other components of the milieu (e.g., geometry) that support CMR. Unfortunately, I think it would be difficult to justify investing the additional time for the majority of high school students. There are far too many topics in the high school curriculum that are already not taught and learned well that will be applicable to more students than trigonometry. Additionally, I am not confident that replacing what students are currently doing with respect to trigonometry in high school with this instruction would be successful because it seemed that much of what the students in this study drew upon to produce CMR was prior knowledge of trigonometry. They were pulling from the ratio and unit circle approaches in many instances. Without that background knowledge, I am not confident that high school students would be equipped to engage with the material in the same way.

There is at least one other group of students with which we could consider using this instruction. Undergraduate students who are not mathematics education majors would in theory have the necessary prerequisite knowledge of geometry as well as the ratio and unit circle approaches to trigonometry. Could this instruction be appropriate for them? My initial reaction is yes it would be appropriate, however, there are some differences in these populations which might cause issues. The undergraduate who would be in a course for which trigonometry is appropriate may not have the same interest and curiosity in mathematics as secondary mathematics PSTs. I think it would be likely that the motivation to invest in this instruction for this population would not be as high as it was for the participants in this study.

Finally, the literature I reviewed did not give a benchmark for how much CMR is considered good enough or how little IR and MR should be tolerated. Without some benchmark it is difficult to evaluate the instructional sequence made up of the four tasks. However, it is worth noting that in all the instructional sessions of this study CMR was required. In my opinion, the students do not necessarily have to have all instances of their reasoning to classified as CMR to be productive. Some of the concepts needed to achieve CMR are indeed forms of AR that students build on to eventually achieve CMR. A few instances of IR that are not the end in themselves but a catalyst to achieving CMR can be tolerated in any given task. This however should be taken with caution to guard the students from yet again falling in the loop of IR as the main or only reasoning strategy. Additionally, one may wonder, was the reasoning elicited through this instructional sequence evidence that can support promoting instruction on trigonometry that focuses on the line-segment definition over other approaches (e.g., ratio, unit circle)? Although this study may not be able to answer that question, it does provide a description of the reasoning elicited, which could be compared to reasoning elicited when using other instructional approached. Studies to establish comparisons between the reasoning elicited from the unit circle approach and ratio approach could be helpful.

Limitations of the Study

This dissertation study has at least three limitations. First, the students were all preservice secondary mathematics teachers in their final semester before student teaching. Apart from those who may have had a chance to teach trigonometry as part of their work during clinical experience, most of the participants had taken years without sitting in a trigonometry class. The students' responses and reasoning were therefore largely influenced by what they could recall from their previous learning experiences. If one uses a different group of participants, for example, students who are being introduced to trigonometry for the first time, the results may be different.

Thompson (2008) reported how right triangle trigonometry instruction in U.S. schools is dominated by the use of the mnemonic, SOH-CAH-TOA, and special right triangles. The students are then presented with numerous exercises that are intended to focus them on solving lengths of segments in a triangle. There is therefore a dearth of tasks to use in studying concepts like trigonometric ratios. With this shortage of extant tasks to explore the students' reasoning, I opted for a course that was focused on teaching mathematics with technology and having trigonometry as one of the areas to investigate. The findings of this study are therefore reported from such tasks that the instructor used to achieve the course objective (i.e., learning to use technology to support the teaching and learning of mathematics). Perhaps, if another study focuses on different tasks regarding trigonometry, students may exhibit different reasoning approaches.

During the data collection process, I focused on students' responses and written notes. Constrained to students' spoken words, gestures while explaining, and written notes during the instruction sessions, it is possible that occasionally students may have concealed their thoughts and not write them down or even verbalize them. My inferences were therefore from the students' utterances, notes, and actions in terms of gestures, and their work on the computers as they were using the dynamic geometry environment.

Recommendation for Future Research

The results of this study provide insight into the reasoning of preservice secondary mathematics teachers in regard to trigonometry. It was not envisioned and should not be read as a study to contribute to the pessimistic findings of previous studies that have reconnoitered the same group. On the contrary, the focus was on college students as a whole and the results of this study should be viewed through the lens of someone investigating post-secondary participants.

As I reflected on what the data from this study told me about the appropriateness of the instructional sequence, I ran into some frustration because the literature I reviewed does not give a benchmark for how much CMR is considered good enough or how much IR can be tolerated. Without some benchmarks it is difficult to evaluate the instructional sequence made up of the four tasks. Therefore, future research related to this will be important. I do contend, as did other researchers (e.g., Hirsch, Weinhold, & Nichols, 1991; Markel, 1982; Weber, 2005), that knowledge of trigonometry is necessary if students are "to *mathematicize* the world around them, problem solve, and develop an appreciation for the relevance and utility of mathematics" (Fi, 2003, p. 214). Therefore, learning how much creative reasoning is needed to achieve this will be an important addition to the field.

I can say, however, that every task required CMR. In my opinion, the students do not necessarily have to have all instances of their reasoning to classified as CMR for their learning experience to be seen as productive. Some of the concepts needed to establish CMR are indeed forms of IR (particularly forms of AR) that students build on to eventually achieve CMR. A few instances of IR that are not the end in themselves but a catalyst to achieving CMR can be tolerated in any given task. This, however, should be taken with caution to guard the students from yet again falling in the trend of IR as the main or only reasoning strategy. Additionally, one may wonder—was the reasoning elicited through this instructional sequence evidence that can support promoting instruction on trigonometry that focuses on the line-segment definition over other approaches (e.g., ratio, unit circle)? Although this study may not be able to answer that question, it does provide a description of the reasoning elicited, which could be compared to reasoning elicited when using other instructional approaches. Studies to establish comparisons between the reasoning elicited from other trigonometry instruction could be helpful.

One final suggestion for further study is to pick up where this study leaves off. I have identified the types of reasoning that occurred when students engaged with the instructional sequence and shown that CMR does in fact appear to be exhibited by the students. What is missing, however, is understanding what in particular in the instruction supported CMR.

Although I did not set out to investigate the question: what aspects of the milieu encouraged the students to engage in CMR? I could not help but be engaged by the question as I analyzed my data. The following discussion explores my cursory thoughts on this question to support those who may choose to investigate the question more deeply in the future.

Components Supporting CMR

In numerous studies, researchers have demonstrated techniques for teaching specific trigonometry topics (e.g., Borba & Confrey, 1996, Clements & Battista, 1989, 1990; Keiser, 2000, 2004), while others have investigated instructional strategies for improving the teaching and learning of trigonometry as a whole (e.g., Hertel & Cullen, 2011, Moore, 2013; Weber, 2005). The recommendations from all these studies are diverse and they advocate for either the ratio approach (e.g., Kendal & Stacey, 1998), or the unit circle approach (e.g., Weber, 2005) and more recently the line-segment approach (e.g., Clements & Burns, 2000; Hertel & Cullen, 2011). Some researchers have recommended an amalgamation of more than one approach (e.g., Moore, 2013). Because researchers have also recommended instructional techniques that promote CMR, as it has been found to be essential in the learning of mathematics (e.g., Lithner, 2015; NCTM, 2000), in the course of this study, I noticed several key components that seemed to spark this CMR, and these included: (a) the support of having access to the Dynamic Geometry

Environment (e.g., representations, feedback), (b) the support of working in a group, (c) the instructor's teaching moves, and (d) the students' competences (e.g., prior knowledge). Below, I give a brief summary of each of these components.

Technology. From the qualitative analysis of students' work I noticed CMR about trigonometric concepts seemed to have been fostered by working in a Dynamic Geometric Environment (DGE). In several instances the technology perhaps provided feedback that supported the students in realizing several concepts and allowed them to reason creatively about them. For example, in Task 2, students utilized their access to technology to connect the idea that parallel lines in the construction correspond with asymptotes in the graphical representation of the tangent. In particular, the students had the opportunity to explore the properties of trigonometric functions and relate them to different graphical representations. This unique access and interaction, which was essential in aiding the students to engage in CMR, not only allowed students to construct segments representing the trigonometric functions, but also to identity and quantify specific attributes of these quantities. This allowed the students to focus on how the corresponding graphs were transformed when the different components were varied in a coordinate plane (Hertel & Cullen, 2011). How much of this technology was required, is still an open question for investigation.

Access to a group. The support of working in a group was another component that appeared to contribute to students' engagement in CMR. With the help of the instructor, the students engaged in mathematical discourse in their small groups as well as with the whole class that helped them reason creatively. This arrangement helped students (a) work with and rely on one another, and (b) work in small groups before sharing in the large groups. In the small groups, students came up with or refined ideas with partners, or the instructor, and discussed ideas as a group to investigate and reason about the tasks. In my opinion, this inspired students to formulate conjectures as well as reason and reach plausible conclusions founded on their own mathematical knowledge and the implicit properties of the trigonometric functions without relying on the authority of the instructor. In the general class discussion, when the students discussed their ideas, they listened to different perspectives, and engaged in sense-making, which resulted in CMR. Moreover, this is consistent with what the Common Core State standards (CCSSM) and NCTM have advocated for. According to the CCSSM, students must not only be able to explain and justify their thinking, but also "construct viable arguments and critique the reasoning of others" (CCSSI, 2010, p. 6). Similarly, the NCTM (2000) Communication standard states that students should be able to:

- organize and consolidate their mathematical thinking through communication,
- communicate their mathematical thinking coherently and clearly to peers, teachers and others,
- analyze and evaluate the mathematical thinking and strategies of others, and
- use the language of mathematics to express ideas precisely. (NCTM, 2000, p. 63).

However, future research can further probe if there is an undeniable connection between group work and promotion of CMR.

Student competences. The students' CMR also depended on their competences mainly in form of their prior knowledge of geometry. In all but a few instances, the prior knowledge students possessed played a key part in their ability to engage in CMR. Analysis of the data points to the need for students to have prior knowledge about multiple approaches of trigonometry (e.g., T2CMR1 - first CMR instance in task 2, T3CMR6), and geometry (e.g., T2CMR2, T3CMR5, T4CMR1, T4CMR2, T4CMR3) to be in position to engage in mathematical reasoning that is novel, plausible and founded in mathematical properties of the trigonometric functions. Even though the basic knowledge from other approaches of trigonometry (e.g., SOH-CAH-TOA from ratio trigonometry) is required, this knowledge does not provide students with complete solutions algorithms as in the case of AR or MR. In fact, students leverage this prior knowledge to construct new (to the reasoner) or re-construct forgotten reasoning sequences that lead to CMR. The students were also able to draw on their knowledge of geometry in form of the Pythagorean theorem, similarity and congruence of triangles to construct plausible arguments.

Consistent with Hertel and Cullen's (2011) observation that the directed length approach can be used to circumvent difficulties students encounter, results from this study indicate that whenever there was no use of geometric figures in the students' strategies, there were limited instances of CMR registered. Although for other approaches of trigonometry this issue can be reconciled by giving students avenues that lead them to make geometric connections, my findings suggest that students were more inclined to make use of geometry under the line-segment approach. For example, Task 2 (Excerpt 7), when investigating the attributes of the tangent function, all strategies were void of any form of geometry and the reasoning was all imitative.

Findings from this study indicate that prior knowledge can promote CMR, not only in helping the students re-create a reasoning sequence from this prior knowledge, but even when creating new (to the reasoner) plausible reasoning sequence. Furthermore, prior knowledge of unit circle and ratio trigonometry was beneficial, not only for learning of procedures, but also for correct transfer of the conceptual structure, which was essential in achieving CMR.

Teacher's moves. Viewing the data from the six sessions as a whole, there was a collection of several teacher moves that I contend did not only lay a platform for students to promote CMR but also to avoid drawing them towards person-guided algorithmic reasoning. Among others, these moves included: (a) encouraging discussions, (b) assigning open-ended tasks, and (c) utilizing technology strategically. The promotion of student collaborations enabled students to bounce ideas off of each other, making it possible to shift from IR to CMR (e.g., T1CMR1). Several researchers (e.g., Balacheff, 1991; Ball & Bass, 2003; Maher, 2005; Yackel & Hanna, 2003) have acknowledged the significance of students constructing arguments to corroborate their solutions to different questions and then justifying these with reasons. In intentionally setting up students to work in groups, having general class discourse, and asking questions that stimulated further justification, I posit contributed to students engaging in CMR. The open-ended tasks supported students to stretch their reasoning as they developed and rationalized their solutions (e.g. T4CMR9).

Technology was crucial in this study, but the instructor deliberately orchestrated moves to assure that the technology did not cloud the trigonometric content. This intentional use of technology kept the mathematics and not the technology as the focus of instruction. And in not always providing pre-made files of the tasks, the students were in some instances able to effectively explore the DGE and result in a case of CMR (e.g., T2CMR4, T3CMR4). Even though not many of the instances of CMR can be directly attributed to a particular teacher move, there is number of them that were an implicit byproduct because the instruction techniques facilitated the other components (e.g., technology, group work) in promoting CMR. However, because this study did not set out to investigate the effect of the teacher moves in particular, it seems to be an area that can be further studied to either confirm or dismiss mu hypothesis.

One final suggestion for further study would be to follow up with PSTs who engaged in this instructional sequence. Would these PSTs teach trigonometry differently from PSTs who had not engaged in the line segment approach to trigonometry? If they did teach differently, would it have any measurable difference in their students understanding of trigonometry? If not, it would be difficult to suggest this instructional sequence for other PSTs as improved student understanding should be the ultimate goal for our research.

REFERENCES

Akkoc, H. (2008). Pre–service mathematics teachers' concept image of radian. *International Journal of Mathematical Education in Science and Technology, 39*(7), pp. 857–878. doi:10.1080/00207390802054458

Allen, H. D. (1977). *The teaching of trigonometry in the United States and Canada: A consideration of elementary course content and approach and factors influencing change, 1890-1970* (Doctoral dissertation). ProQuest Dissertations and Theses database. (Order No. 7713452)

Ball, D.L. (2003). *What mathematical knowledge is needed for teaching mathematics?* Prepared for the Secretary's Summit on Mathematics, U.S. Department of Education, February 6, 2003; Washington, D.C. Available at http://jwilson.coe.uga.edu/situations/Framework%20Folder/Framework.Jan08/articles/Ball2003Math%20Summit.pdf

Ball, D., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). Reston VA: National Council of Teachers of Mathematics.

Barnes, J. A. (1999). Creative writing in trigonometry. *The Mathematics Teacher, 92*(6), 498–503.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London, UK: Hodder & Stoughton.

Balacheff, N. (1991). The benefits and limits of social interaction: The case of a mathematical proof. In A. J. Bishop et al. (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 175-192). Dordrecht, The Netherlands: Kluwer Academic Publishers. doi:10.1007/978-94-017-2195-0_9

Bergqvist, E. (2012). University mathematics teachers' views on the required reasoning in calculus exams. *The Mathematics Enthusiast*, *9*(3), 371–408.

Blackett, N., & Tall, D. O. (1991). Gender and the versatile learning of trigonometry using computer software. In F. Furinghetti (Ed.), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 144–151). Assisi, Italy.

Borba, M. C., & Confrey, J. (1996). A student's construction of transformations of functions in a multiple representational environment. *Educational Studies in Mathematics*, *31*(3), 319–337. doi:10.1007/BF00376325

Bressoud, D. M. (2010). *Historical reflections on teaching trigonometry*. *Mathematics Teacher, 104*(2), 106–112.

Brown, S. A. (2005). *The trigonometric connection: Students' understanding of sine and cosine* (Order No. 3233908). Available from Dissertations & Theses @ Illinois State University; ProQuest Dissertations & Theses Global. (304986316). Retrieved from http://libproxy.lib.ilstu.edu/login?url=https://search.proquest.com/docview/304986316?accountid=11578

Brown, S. A. (2006). The trigonometric connection: Students' understanding of sine and cosine. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 228). Prague, Czech Republic: PME.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 114–162). Washington, DC: Mathematical Association of America. doi:10.1090/cbmath/007/04

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. doi:10.2307/4149958

Carlson, M., & Oehrtman, M. (2004). Key aspects of knowing and learning the concept of function. *Research Sampler Series, MAA Notes Online*(invited).

Cajori, F. (1890). *The teaching and history of mathematics in the United States.* Washington, DC: Government Printing Office.

Clements, D. H. & Battista, M. T. (1989). Learning of geometric concepts in a Logo environment. *Journal for Research in Mathematics Education, 20,* 450–467. doi:10.2307/749420

Clements, D. H. & Battista, M. T. (1990). The effects of Logo on children’s conceptualizations of angle and polygons. *Journal for Research in Mathematics Education, 21,* 356–371. doi:10.2307/749394

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York, NY: Macmillan.

Clements, D. H. & Burns, B. A. (2000). Students' development of strategies for turn and angle measure. *Educational Studies in Mathematics, 41*(1), 31–45. doi:10.1023/A:1003938415559

Common Core State Standards Initiative (CCSSI). (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org /wp-content/uploads/Math_Standards.pdf

Cullen, C. J., Hertel, J. T., & Nickels, M. (in press). The roles of technology in mathematics education. *The Educational Forum.*

DeJarnette, A. F. (2014). *Students' conceptions of trigonometric functions and positioning practices during pair work with Etoys* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (Order No. 3646483)

Demir, Ö. & Heck, A. (2013). A new learning trajectory for trigonometric functions. In E. Faggiano & A. Montone (Eds.), *Proceedings of ICTMT, 11* (pp. 119-124). Bari, Italy: University of Bari.

Doyle, W. (1983). Academic work. *Review of Educational Research*, 53(2), 159–199. doi:10.3102/00346543053002159

Duval, R. (2002). Proof understanding in mathematics: What ways for students? In F. L. Lin (Ed.), *Proceedings of the international conference on mathematics– "Understanding proving and proving to understand"* (pp. 61–77). Taipei, Taiwan: National Science Council and National Taiwan Normal University.

Elgin, R. L. (2007, May). The demise of basic surveying mathematics. *The American Surveyor, 4*(4). Retrieved from http://www.amerisurv.com/PDF/TheAmericanSurveyor_Elgin-DemiseOfMath_May2007.pdf

Farmaki, V., & Paschos, T. (2007). Employing genetic 'moments' in the history of mathematics in classroom activities. *Educational Studies in Mathematics*, *66*(1), 83–106. doi:10.1007/s10649-006-9056-y

Fi, C. (2003). *Preservice secondary school mathematics teachers' knowledge of trigonometry: Subject matter content knowledge, pedagogical content knowledge and envisioned pedagogy.* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (Order No. 3097526)

Fi, C. (2006). Preservice secondary school mathematics teachers' knowledge of trigonometry: Cofunctions. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 833–834). Mérida, México.

Francisco, J. M., & Hähkiöniemi, M. (2012). Students' ways of reasoning about nonlinear functions in guess-my-rule games. *International Journal of Science and Mathematics Education, 10,* 1001–1021. doi:10.1007/s10763-011-9310-3

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A proceptual view of elementary arithmetic. *Journal for Research in Mathematics Education*, *26*(2), 114–141. doi:10.2307/749505

Guralnik, D. B. (Ed.). (1982). Webster's new world dictionary (2nd ed.). New York, NY: Simon & Schuster.

Gordy, G. A. (1933). *The development of trigonometry in the secondary schools of the United States* (Doctoral dissertation). ProQuest Dissertations and Theses database. (Order No. TM18640)

Hardison, H. L. (2017, April). *A ninth-grade student's conception of angle measure*. Paper presented at the 2017 National Council of Teachers of Mathematics Research Conference. San Antonio, TX.

Harel, G. (2006). Mathematics education research, its nature, and its purpose: A discussion of Lester's paper. *Zentralblatt für Didaktik der Mathematik*, *38*(1), 58–62.

Harel, G., & Dubinsky, E. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85–106). Washington, DC: Mathematical Association of America. doi:10.1007/BF02655906

Hirsch, C. R., Weinhold, M., & Nichols, C. (1991). Trigonometry today. *Mathematics Teacher, 84*(2), 98–106.

Hertel, J. T. (2013). *Investigating the purpose of trigonometry in the modern sciences.* (Doctoral dissertation). ProQuest Dissertations and Theses database. (Order No. 3592298)

Hertel, J., & Cullen, C. (2011). Teaching trigonometry: A directed length approach. In L. R. Wiest, & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1400–1407). Reno, NV.

Hiebert, J. (2003). What research says about the NCTM standards. In J. Kilpatrick, G. Ben, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 5–26). Reston, Va.: National Council of Teachers of Mathematics.

Hirsch, C. R., Weinhold, M., & Nichols, C. (1991). Trigonometry today. *The Mathematics Teacher*, *84*, 98–106.

Hoachlander, G. (1997). Organizing mathematics education around work. In L. A. Steen (Ed.). *Why Numbers Count: Quantitative Literacy for Tomorrow's America* (pp. 122–36). New York, NY: College Entrance Examination Board.

Jonsson, B., Norqvist, M., Liljekvist, Y., & Lithner, J. (2014). Learning mathematics through algorithmic and creative reasoning. *The Journal of Mathematical Behavior*, *36*, 20–32. doi:10.1016/j.jmathb.2014.08.003

Keiser, J. (2000). The role of definition. *Mathematics Teaching in the Middle School, 5*, 506–511.

Keiser, J. (2004). Struggles with developing the concept of angle: Comparing 6th grade students' discourse to the history of the angle concept. *Mathematical Thinking and Learning, 6(3)*, 285–306. doi:10.1207/s15327833mtl0603_2

Kendal, M., & Stacey, K. (1997). Teaching trigonometry. *Vinculum, 34*(1), 4–8.

Kendal, M., & Stacey, K. (1998). Teaching Trigonometry. *Australian Mathematics Teacher*, *54*(1), 34–39.

Kennedy, E. S. (1969). The history of trigonometry. In J. K. Baumgart, D. E. Deal, B. R. Vogeli, & A. E. Hallerbert (Eds.), *Historical topics for the mathematics classroom* (pp. 333–358). Reston, VA: National Council of Teachers of Mathematics.

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *The Journal of Mathematical Behavior, 26*(1), 60–82. doi:10.1016/j.jmathb.2007.02.001

Lithner, J. (2000). Mathematical reasoning in task solving. *Educational Studies in Mathematics*, *41*(2), 165–190. doi:10.1023/A:1003956417456

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67*(3), 255–276. doi:10.1007/s10649-007-9104-2

Lithner, J. (2015). Learning mathematics by creative or imitative reasoning. In *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 487–506). Springer, Cham. New York, NY: doi:10.1007/978-3-319-17187-6_28

Maher, C. A. (2005). How students structure their investigations and learn mathematics: Insights from a long-term study. *Journal of Mathematical Behavior*, *24*(1), 1–14. doi:10.1016/j.jmathb.2004.12.006

Maher, C. A., & Davis, R. B. (1995). Children's explorations leading to proof. In C. Hoyles & L. Healy (Eds.), *Justifying and proving in school mathematic*s. London, UK: Mathematical Sciences Group, Institute of Education, University of London.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27*(2), 194– 214. doi:10.2307/749600

Maher, C. A., & Martino, A. M. (2000). From patterns to theories: Conditions for conceptual change. *Journal of Mathematical Behavior*, *19*(2), 247–271. doi:10.1016/S0732-3123(00)00047-X

Mansi, K. E. (2003). *Reasoning and geometric proof in mathematics education: A review of the literature* (Master's Thesis, North Carolina State University). Retrieved from http://repository.lib.ncsu.edu/ir/bitstream/1840.16/2692/1/etd.pdf

Maor, E. (1998). *Trigonometric delights.* Princeton, NJ: Princeton University Press.

Markel, W. D. (1982). Trigonometry - Forgotten and abused? *School Science and Mathematics,* 82, 548–551. doi:10.1111/j.1949-8594.1982.tb10055.x

Martin, W. G., Carter, J., Forster, S., Howe, R., Kader, G., Kepner, H., & Valdez, P. (2009). *Focus in high school mathematics: Reasoning and sense making.* Reston, VA: National Council of Teachers of Mathematics.

Mason, M. M., & Moore, S. D. (1997). Assessing readiness for geometry in mathematically talented middle school students. *Journal of Secondary Gifted Education, 8*(3), 105–111. doi:10.1177/1932202X9700800302

Mayes, R. L., Peterson, F., & Bonilla, R. (2013). Quantitative reasoning learning progressions for environmental science: Developing a framework. *Numeracy, 6*(1), 4. doi:10.5038/1936-4660.6.1.4

Moore, K. C. (2010). *The role of quantitative reasoning in precalculus students learning central concepts of trigonometry* (Doctoral dissertation). ProQuest Dissertations and Theses database. (Order No. 3425753)

Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225–245. doi:10.1007/s10649-012-9450-6

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, *45*(1), 102–138. doi:10.5951/jresematheduc.45.1.0102

Moore, K. C., LaForest, K. R., & Kim, H. J. (2012). The unit circle and unit conversions. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 16–31). Portland, OR. Retrieved from http:// sigmaa.maa.org/rume/crume2012/RUME_Home/RUME_Conference_Papers_files/RUME_XV_ Conference_Papers.pdf

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Niss, M. (2003, January). Mathematical competencies and the learning of mathematics: The Danish KOM project. In *3rd Mediterranean Conference on Mathematical Education* (pp. 115-124). Athens, Hellas: Hellenic Mathematical Society.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 27-42). Washington, DC: Mathematical Association of America. doi:10.5948/UPO9780883859759.004

Pea, R. D. (1987). Cognitive technologies for mathematics education. *Cognitive Science and Mathematics Education*, 89–122.

Piaget, J. (1970). *Genetic epistemology.* New York, NY: Columbia University Press. doi:10.7312/piag91272

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*(2-3), 165-190. doi:10.1007/bf01273662

Polya, G. (1954). *Mathematics and plausible reasoning* (Vols. I and II). Princeton, NJ: Princeton University Press.

Rowan, B., Chiang, F., & Miller, R. J. (1997). Using research on employees' performance to study the effects of teachers on students' achievement. *Sociology of Education, 70,* 256–284. doi:10.2307/2673267

Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298-303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Schnotz, W., & Bannert, M. (2003). Construction and interference in learning from multiple representation. *Learning and Instruction, 13*(2), 141–156. doi:10.1016/S0959-4752(02)00017-8

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press Inc.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1–36. doi:10.1007/BF00302715

Sfard, A. (2000). Steering (dis) course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education*, 296–327.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teacher*, *77*(1), 20–26.

Sumpter, L. (2009). *On aspects of mathematical reasoning: Affect and gender*. Umeå: Umeå University Retrieved November 2, 2018, from http://umu.diva-portal.org/smash/get/diva2:216376/FULLTEXT01

Tall, D. (1996). Functions and calculus. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 289–325). Dordrecht, The Netherlands: Kluwer.

Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Progress report to the National Science Foundation. *Center for Research in Mathematics & Science Education:* San Diego State University, San Diego, CA.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel, & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 181–234). Albany, NY: State University of New York Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOMe* (pp. 33–57). Laramie, WY: University of Wyoming

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415–432. doi:10.1007/s10857-007-9054-8

Topçu, T., Kertil, M., Akkoç, H., Kamil, Y., & Osman, Ö. (2006). Pre–service and in–service mathematics teachers' concept images of radian. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 281–288). Prague, Czech Republic: Psychology of Mathematics Education.

van Brummelen, G. (2009). *The mathematics of the heavens and the Earth: The early history of trigonometry*. Princeton, NJ: Princeton University Press.

van Sickle, J. (2011). *A history of trigonometry education in the United States: 1776–1900* (Doctoral dissertation). ProQuest Dissertations and Theses database. (Order No. 3453860)

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, *34*(2), 97–129. doi:10.1023/A:1002998529016

Voigt, J. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational Studies in Mathematics*, *26*(2-3), 275–298. doi:10.1007/BF01273665

Wyndhamn, J., & Säljö, R. (1997). Word problems and mathematical reasoning—a study of children's mastery of reference and meaning in textual realities. *Learning and Instruction*, *7*(4), 361–382. doi:10.1016/S0959-4752(97)00009-1

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, *17*(3), 91–112. doi:10.1007/BF03217423

Weber, K., Maher, C. A., Powell, A. B., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics*, *68*, 247–261. doi:10.1007/s10649-008-9114-8

Wilson, M. (1992, August). *Preservice secondary teachers' images of mathematics, mathematics teaching, and mathematical functions.* Paper presented at the 7th international congress on mathematical education, Quebec, Canada.

Wood, W. E. (2011). Squigonometry. *Mathematics Magazine*, *84*(4), 257–265. doi:10.4169/math.mag.84.4.257

Wu, H.-H. (1996). The role of Euclidean geometry in high school. *Journal of Mathematical Behavior, 15*(1), 221–237. doi:10.1016/S0732-3123(96)90002-4

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 227–236). Reston, VA: National Council of Teachers of Mathematics.

APPENDIX A: WEEKLY STUDY OUTLINE

Weekly Study Outline

	In-class focus group	Whole class
Week 1		
• Functions: 1-D geometric objects→1-D geometric objects • Chord, tangent, and secant functions	✓	✓
Week 2		
• Chord, tangent, and secant functions • Measuring an angle	✓	✓
Week 3		
• Measuring an angle • Secant function • Co-functions	✓	✓
Week 4		
• Radians vs degrees • Trigonometric identities	✓	✓
Week 5		
• Basic identities • Transcendental functions	✓	✓
Week 6		
• Sum and difference formulas	✓	✓

APPENDIX B: IDENTIFIED TASKS

Week	Task Chosen
Week 1 • Functions: 1-D geometric objects→1-D geometric objects • Chord, tangent, and secant functions	Function: Identifying the function (sine)- Task 1
Week 2 • Chord, tangent, and secant functions • Measuring an angle	Properties of the tangent and secant functions – Task 2
Week 3 • Measuring an angle • Secant function • Co-functions	Task 2 – Secant Function
Week 4 • Radians vs degrees • Trigonometric identities	Task 3 – Exploring Trig Identities
Week 5 • Basic identities • Transcendental functions	Task 4 – Comparing sine and tangent
Week 6 • Sum and difference formulas	Task 5 – Sum formulas for sine, and cosine Half angle for tangent

APPENDIX C: TIMETABLE

Timetable for Research Completion to Graduate in August of 2019

Degree Requirements	Date
Proposal	September 2017
IRB approval	September or October 2017
Collect Data	October to December 2017
Degree Audit	May 18, 2018
Proposal Approval Form	May 18, 2018
Right to Defend	October 12, 2018
Last Day for Oral Defense	July 15, 2019
Final Deposit Filing	July 22, 2019
Degree Completion Date	August 10, 2019