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Lie Groups And Euler-Bernoulli Beam Equation

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LIE GROUPS AND EULER-BERNOULLI BEAM EQUATION

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55 pages

Lie groups approach in differential equations was a breakthrough subject in the late nineteenth century. Sophus Lie, a Norwegian mathematician, introduced the systematic approach to study the solutions of differential equations. The main goal of this thesis is to study, using Lie's approach, the Euler-Bernoulli beam equation subject to swelling force, the fourth-order nonlinear differential equation used to describe the beam deflection under the swelling force. In particular, we will classify the symmetry groups of this equation, obtain several reductions, and demonstrate both analytical and numerical solutions.

KEYWORDS: Lie Groups, Differential Equations, Euler-Bernoulli Beam, Analytical Solution

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M. A.

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CHAPTER I: INTRODUCTION

In the nineteenth century, Norwegian mathematician Sophus Lie proposed the systematic way (Lie groups method) of studying the properties of the solutions of differential equations [Li]. He introduced the possibility of finding these solutions based on the invariance of a system of differential equations under the continuous groups of transformations.

Our motivation is to exploit Lie's technique for studying the solution properties, and obtain possible reductions of the Euler-Bernoulli beam equation subject to swelling force. In particular, we are interested in studying the fourth order nonlinear differential equation with the exponential term $u_{xxxx} = Ke^{-Bu}$ that is used to describe the deflections of retaining wall under the soil pressure. This differential equation has no known analytical solution. All details about this equation are introduced from the beginning of Chapter II.

Nowadays scholars from different scientific fields exploit the Lie groups method to analyze the properties of solutions of differential equations from various important research questions, including the Euler-Bernoulli beam equation. For instance, Bokhari et.al. [BoZM] classified the symmetry groups of Euler-Bernoulli beam equation with different forms of the force term including the exponential but raised by the positive u. They reduced the equation to the second-order ODE, however the analytical solution was left for further study. Their reduction differs from our reduction because they used different invariants for the canonical coordinates that resulted from using the solvability of algebra. Wei and Liu [WeL] provided the analytic solutions of the power-law Euler-Bernoulli beam equation with means of general integration, but not for an exponential force term. Wei et.al. [WeLDMJ] designed and implemented two numerical procedures for calculating the deflection of the retaining walls. This was the first attempt to design and implement numerical procedures for such a mathematical model. Wei et.al. [WeZhK] proved the well-posedness, and showed the existence and uniqueness of the solution of the model for the dynamic behavior of a cantilever Euler beam subject to a nonlinear swelling load. Ruiz et.al. [RuMR] obtained the exact general solution to a static fourth-order Euler-Bernoulli beam equation with the force term to be $u^{-5/3}$. They expressed the solution in parametric form in terms of Weierstress elliptic functions. However, they did not consider the case with exponential force term. Bidisha and Ranjan [BiR] classified the Lie groups of the dynamic Euler-Bernoulli beam equation with the axial load and found an exact solution, but again not for the case of exponential term. Shakipov [Sh] provided the numerical solution of the Euler-Bernoulli beam equation with swelling force by using the finite element method.

In this thesis we will give our overview of Lie's approach to classify the symmetry groups, perform some reductions, and analyze the properties of solutions of the Euler-Bernoulli beam equation subject to swelling force.

CHAPTER II: BACKGROUND

The following fourth order nonlinear differential equation is called the Euler-Bernoulli beam equation [Ti], which describes the relationship between the deflection of a beam subject to certain applied force

$$EIu_{xxxx} = f(x) \tag{1}$$

where E is the modulus of elasticity, I is the area moment of inertia, $u_{xxxx} = \frac{d^4u}{dx^4}$, and $f(x)$ is the function that represents some applied load depending on the physical phenomena of interest. Figure 1. demonstrates the deflection of the beam under some applied force.

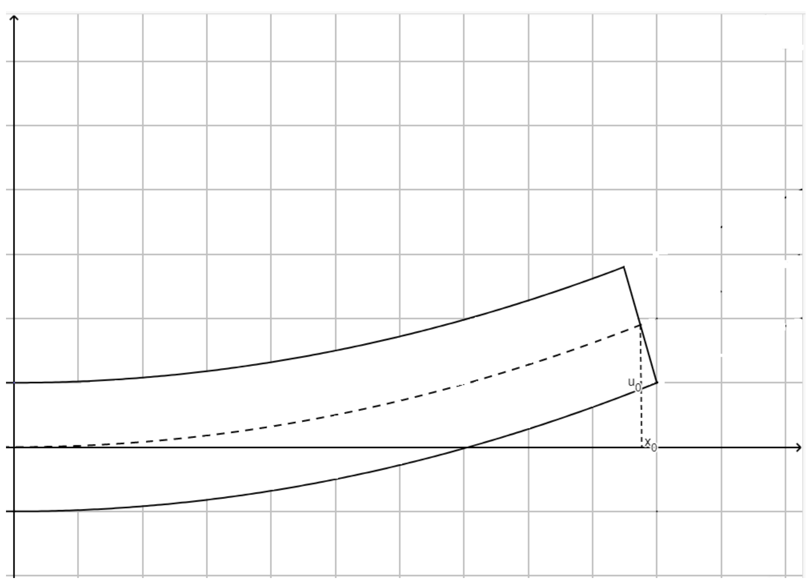


Figure 1: Beam deflection after $f(x)$ is applied.

II.1 Euler-Bernoulli Beam Equation With Swelling Force

One of the interesting examples of the applied load is when $f(x) = e^{-u}$. This function is used to explain the physical phenomena when the retaining wall is subject to a swelling force coming from the expansive soils. For instance, the walls, that are used to protect a highway or hillside farms from the soil shifts due to heavy rains [JaN], can be modeled using this differential equation.

The swelling pressure to the wall is defined by the function

$$\sigma = \sigma_0 10^{-\frac{u}{cd}} \tag{2}$$

where σ_0 is the maximum pressure, and c, d are certain constants. For the distributed load, we need the height of the wall, s , which gives us the applied force

$$f(x) = s\sigma_0 10^{-\frac{u}{cd}}. \tag{3}$$

Eventually, simplifying the expression above gives us the **Euler-Bernoulli beam** equation with swelling force as

$$u_{xxxx} = Ke^{-Bu} \tag{4}$$

where $K = \frac{s\sigma_0}{EI}$, $B = \frac{ln(10)}{cd}$, and with the boundary conditions

$$\begin{aligned} u(0) = u_x(0) = 0 \\ u_{xx}(L) = u_{xxx}(L) = 0 \end{aligned} \tag{5}$$

where 0 and L are the endpoints of the wall.

(**Note**: for the sake of convenience, we will use $K = B = 1$ in our future analysis.)

II.2 Modified Euler-Bernoulli Beam Equation

We find it helpful to modify the Euler-Bernoulli beam equation by multiplying both sides by $-u_x$ as follows:

$$\begin{aligned} -u_x u_{xxxx} &= -u_x e^{-u} \\ u_{xx} u_{xxx} - (u_x u_{xxx})_x &= (e^{-u})_x \\ \left[\frac{1}{2} u_{xx}^2\right]_x - (u_x u_{xxx})_x &= (e^{-u})_x \end{aligned} \tag{6}$$

then we integrate both sides to get the **modified Euler-Bernoulli beam** equation of order 3

$$\frac{1}{2} u_{xx}^2 - u_x u_{xxx} = e^{-u} + C \tag{7}$$

for some constant C. This equation is useful only if $u_x \neq 0$, otherwise, it will lead to singularity. But for the analysis purpose, we can approximate u_x to be a very small value, if needed, and use it, for instance, in our numerical approximations.

CHAPTER III: LIE SYMMETRY GROUPS

In this chapter, we will overview the brief history of Lie symmetry groups of differential equations followed by the concepts that are necessary to understand the study done in this thesis. We will recall the concepts of manifolds in III.2, Lie groups in III.3, tangent space and vector fields in III.4, Lie Algebras in III.5, and finally symmetry groups in III.6.

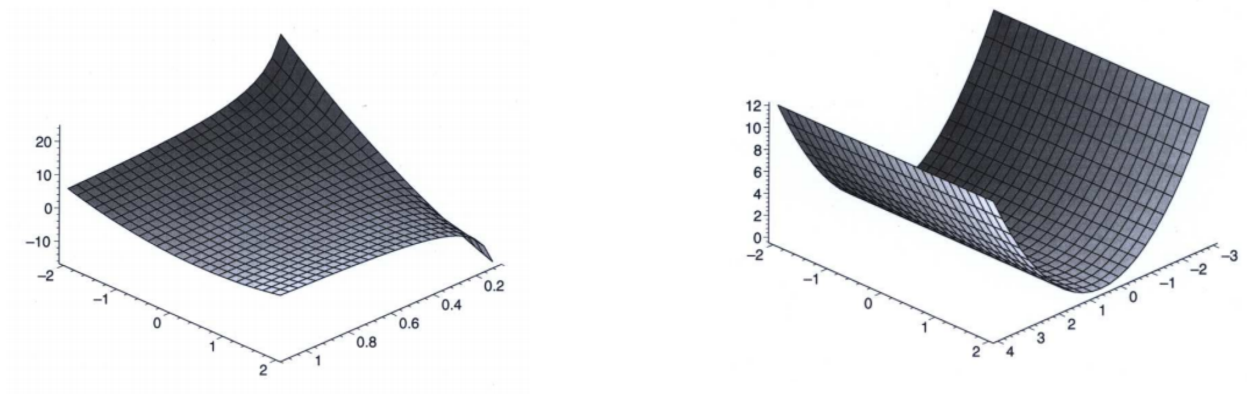
III.1 Brief History

Sophus Lie was particularly inspired by Galois's theory which is related to symmetries of the roots of polynomial [BeB]. In general, he discovered that if the discrete invariance group of a system of algebraic equations can be used to solve this system by "radicals", then continuous invariance groups of systems of differential equations can be used for solving these systems by "quadratures" (i.e. by integration). In general, the idea is to exploit the invariance of differential equations under the transformation of independent and dependents variables. The following figures (2a,2b) demonstrates the transformation of some differential equation

III.2 Manifolds

It is important to know the concept of "manifolds" as we want to view the set of solutions to our system of differential equations as submanifold or a larger manifold.

Definition 0.1. *[[Ol], Def. 1.1]* An *m-dimensional manifold* is a topological space M, together with a countable collection of open subsets $U_\alpha \subset M$, called *coordinate charts*, and homeomorphisms $\chi_\alpha : U_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$, called *local*



(a) The surface of $x\frac{dy}{dx} + yxy^2 = 0$. $\frac{dy}{dx}$ is plotted over (x, y)

(b) $x\frac{dy}{dx} + yxy^2 = 0$. is transformed to $T/R + 1 - R = 0$. T is plotted over (S, R), where $S = ln(x)$, $R = xy$, and $T = x^2\frac{dy}{dx}$.

Figure 2: An example of transformation [Gi].

coordinate maps, which satisfy the following properties:

- The coordinate charts cover M:

$$\bigcup_{\alpha} U_\alpha = M \tag{8}$$

- On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the composite map

$$\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \to \chi_\beta(U_\alpha \cap U_\beta) \tag{9}$$

  is a smooth (infinitely differentiable) function.

- If $x \in U_\alpha$, $\tilde{x} \in U_\beta$ are distinct points of M, then there exist open subsets $W \subset V_\alpha$, $\tilde{W} \subset V_\beta$, with $\chi_\alpha(x) \in W$, $\chi_\beta(\tilde{x}) \in \tilde{W}$, satisfying

$$\chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\tilde{W}) = \emptyset. \tag{10}$$

III.3 Lie Groups

To get invariant solutions in the sense of Lie, we need an appropriate concept of a group acting smoothly on a manifold. Towards this end, we have the concept of a Lie group.

Definition 0.2. *[[Ol], Def. 1.16]* An *r-parameter Lie group* is a group G which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operation

$$m : G \to G, \qquad m(g,h) = g \cdot h, \qquad g,h \in G, \tag{11}$$

and the inversion

$$i : G \to G, \qquad i(g) = g^{-1}, \qquad g \in G, \tag{12}$$

are smooth maps between manifolds.

Our primary concern is with local solutions, so we adapt the above to a local context. In fact, the core concept of Lie's approach to differential equations is the one-parameter Lie groups which is associated with the transformations of manifolds.

Definition 0.3. *[[Ol], Def. 1.20]* An *r-parameter local Lie group* consists of connected open subsets $V_0 \subset V \subset \mathbb{R}^r$ containing the origin 0, and smooth maps

$$m : V \times V \to \mathbb{R}^r, \tag{13}$$

defining the group operation, and

$$i : V_0 \to V, \tag{14}$$

defining the group inversion, with the following properties.

(a) *Associativity.* If $x, y, z \in V$, and also $m(x, y)$ and $m(y, z)$ are in V, then

$$m(x, m(y, z)) = m(m(x, y), z). \tag{15}$$

(b) *Identity Element.* For all x in V, $m(0, x) = x = m(x, 0)$.

(c) *Inverses.* For each x in V_0, $m(x, i(x)) = 0 = m(i(x), x)$.

Now we consider a local group action on a manifold.

Definition 0.4. *[[Ol], Def. 1.25]* Let M be a smooth manifold. A *local group of transformations* acting on M is given by a (local) Lie group G, an open subset U, with

$$\{e\} \times M \subset U \subset G \times M, \tag{16}$$

and a smooth map $\Psi : U \to M$ with the following properties:

(a) If $(h, x) \in U$, $(g, \Psi(h, x)) \in U$, and also $(g \cdot h, x) \in U$, then

$$\Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x). \tag{17}$$

(b) For all $x \in M$,

$$\Psi(e, x) = x. \tag{18}$$

(c) If $(g, x) \in U$, then $(g^{-1}, \Psi(g, x)) \in U$ and

$$\Psi(g^{-1}, \Psi(g, x)) = x. \tag{19}$$

III.4 Tangent Space and Vector Fields

It is usually hard to work solely with local Lie groups without using the concepts of tangent spaces and vector fields for linearization in the group action.

Definition 0.5. *[[AbSG], Def. 9.1]* A *tangent vector* $\mathbf{v}|_{\mathbf{x}}$ to $\mathbb{R}^n$ consists of a pair of elements $\mathbf{v}$, $\mathbf{x}$ of $\mathbb{R}^n$; $\mathbf{v}$ is called the vector part and $\mathbf{x}$ is called the point of application of $\mathbf{v}|_{\mathbf{x}}$.

Definition 0.6. *[[Ol], p. 25]* The collection of all tangent vectors to all possible curves passing through a given point x in M is called the *tangent space* to M at x, and is denoted by $TM|_x$.

The fundamental tool in the Lie's theory and transformation groups is the "infinitesimal transformation" which is based on using the concept of vector field on a manifold.

Definition 0.7. *[[Ol], p. 26]* A *vector field* $\mathbf{v}$ on M assigns a tangent vector $\mathbf{v}|_x \in TM|_x$ to each point $x \in M$, with $\mathbf{v}|_x$ varying smoothly from point to point. In local coordinates

$(x^1, \ldots, x^m)$, a vector field has the form

$$\mathbf{v}|_x = \xi^1(x)\frac{\partial}{\partial x^1} + \xi^2(x)\frac{\partial}{\partial x^2} + \ldots + \xi^m(x)\frac{\partial}{\partial x^m} \tag{20}$$

where each $\xi^i(x)$ is a smooth function of x.

Definition 0.8. *[[Ol], Def. 1.44]* The Lie algebra of a Lie group G, denoted by $\mathfrak{g}$, is the vector space of all right-invariant vector fields on G. That is, the vector space of transformations $h \to hg$ for $\forall h \in G$, and fixed $g \in G$.

III.5 Symmetry Groups

Consider the system Δ of m partial differential equations of order n

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, m \tag{21}$$

with p independent variables, $x = (x^1, \ldots, x^p)$, and q dependent variables, $u = (u^1, \ldots, u^q)$, where we use $u^{(n)}$ to denote the derivatives of u with respect to x up to order n.

Definition 0.9. *[[Ol], Def. 2.23]* A *symmetry group* of the system (21) is a local group of transformations G acting on an open subset $M = X \times U$ of the space of independent variables X and the space of dependent variables U for the system Δ_ν with the property that whenever $u = f(x)$ is a solution of (21), and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

Let

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{22}$$

be a vector field where ξ^i and φ^α are the coefficient functions of x and u. Continuous transformations groups are assumed to be connected, thus, we can work with infinitesimal generators that form a Lie algebra of vector fields (22) on the space of independent and dependent variables. The group transformations in G are obtained by the process of exponentiation of the infinitesimal generators of the corresponding Lie algebra. Therefore, the one-parameter group $G = \{g_\varepsilon | \varepsilon \in \mathbb{R}\}$ is the solution $g_\varepsilon \cdot (x_0, u_0) = (x(\varepsilon), u(\varepsilon))$ to the first order system of differential equations

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u), \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x,u) \tag{23}$$

with initial conditions (x_0, u_0) at $\varepsilon = 0$. The transformations in G act not only on functions $u = f(x)$ but also on their derivatives, hence, induces "prolonged transformations". The prolonged infinitesimal generators are vector fields on the space of independent and dependent variables and their derivatives which are $u^\alpha_J = \frac{\partial^J u^\alpha}{\partial x^J}$, where $J = (j_1, \ldots, j_k)$ represents a k-tuple of integers with $1 \le j_\nu \le p$ showing which derivatives are taken, and $1 \le \alpha \le q$ are indices that distinguish the dependent variables (Notice that, in our case $p = q = 1$). We define the prolongation of the vector field $\mathbf{v}$ to be

$$\mathrm{pr}^{(n)}\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha_J(x,u^{(n)})\frac{\partial}{\partial u^\alpha_J}. \tag{24}$$

The coefficients in (24) are expressed as

$$\varphi_J^\alpha(x, u^{(n)}) = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha \tag{25}$$

where D_J is the total derivative with respect to independent variable(s) $x = (x^1, \dots, x^p)$, showing exactly which derivatives are taken, and Q^α is known as characteristic of the vector field (22). We express Q as following

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x, u) - \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial u^\alpha}{\partial x^i}. \tag{26}$$

Theorem 1. *[[Ol], Thm. 2.31] A connected group of transformations G is a symmetry group of the system of differential equations (21) if and only if the classical infinitesimal symmetry criterion*

$$\mathrm{pr}^{(n)} \boldsymbol{v}[\Delta_\nu(x, u^{(n)})] = 0, \quad \nu = 1, \dots, r \tag{27}$$

holds for every infinitesimal generator $\boldsymbol{v}$ of G.

CHAPTER IV: ALGORITHM FOR FINDING THE LIE SYMMETRY GROUPS

Below we summarize the Lie's technique, using Olver's textbook [Ol], of finding the symmetry groups of differential equations and then apply it to our Euler-bernoulli beam equation. The step-by-step algorithm is described as following

1. Obtain the vector field for the differential equation of order n with p independent and q dependent variables.
2. Obtain the n-th prolongation of that vector field.
3. Apply this prolonged vector field over the differential equation in order to get the form satisfying the infinitesimal symmetry criterion.
4. Find the expressions of the terms in the equation above using the formula (25).
5. Construct the system of determining equations obtained by equating the coefficients of the monomials to zero.
6. Solve the system.
7. Substitute solutions into the vector field in step 1.
8. Obtain the span of vector field and recover the symmetry groups by exponentiation method as shown in (23).

IV. 1 Lie Symmetries Of Euler-Bernoulli Beam Equation

Theorem 2. *For Euler-Bernoulli beam equation with swelling force (4), the Lie algebra of infinitesimal symmetries is spanned by*

$$\boldsymbol{v}_1 = \partial_x, \qquad \boldsymbol{v}_2 = x\partial_x + 4\partial_u, \tag{28}$$

and the one-parameter groups are

$$G_1 : (x+\varepsilon, u), \qquad G_2 : (e^{\varepsilon}x, u+4\varepsilon) \tag{29}$$

Therefore, if $u = f(x)$ is a solution of Euler-Bernoulli beam equation (4), then the following transformed forms are also the solutions.

$$u_1 = f(x-\varepsilon), \qquad u_2 = f(e^{-\varepsilon}x) - 4\varepsilon \tag{30}$$

for any real number ε.

Proof of Theorem 2. We will follow the algorithm stated in Chapter IV.

Step 1. Since the Euler-Bernoulli beam equation has 1 independent and 1 dependent variables, that is $p = q = 1$, the vector field is

$$\mathbf{v} = \xi\partial_x + \varphi\partial_u \tag{31}$$

Step 2. The order of the equation is 4, so we get the fourth prolongation which is

$$\mathrm{pr}^{(4)}\mathbf{v} = \mathbf{v} + \varphi^{x}\frac{\partial}{\partial u_x} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \varphi^{xxx}\frac{\partial}{\partial u_{xxx}} + \varphi^{xxxx}\frac{\partial}{\partial u_{xxxx}} \tag{32}$$

Steps 3 and 4. Applying the prolonged vector field over the equation (4) yields

$$\varphi^{xxxx} + \varphi e^{-u} = 0 \tag{33}$$

whenever $u_{xxxx} = e^{-u}$. The corresponding computations and Mathematica code for finding the expressions are included in APPENDIX A.

Steps 5 and 6. By equating the coefficients of monomials to zero, we obtain the system of determining equations as shown in Table 1 below. According to (i), ξ does not depend on u. (m) and (j) suggest that φ depends only on x. From (a), we reveal that ξ is linear in x, so

$$\xi = c_2 x + c_1 \tag{34}$$

for arbitrary constants c_1 and c_2. By looking at (j) we can say that φ is defined by the form

$$\varphi = 4c_2. \tag{35}$$

Step 7. Substituting (34) and (35) into (31) yields

$$\mathbf{v} = (c_2 x + c_1)\partial_x + 4c_2\partial_u, \tag{36}$$

Monomial	Coefficient	
u_{xxx}	$-\xi_{xx} + 4\varphi_{xu} = 0$	(a)
u_{xx}^2	$-12\xi_{xu} + 3\varphi_{uu} = 0$	(b)
u_{xx}	$-4\xi_{xxx} + 6\varphi_{xxu} = 0$	(c)
u_x^5	$-\xi_{uuuu} = 0$	(d)
u_x^4	$-\xi_{xuuu} + \varphi_{uuuu} = 0$	(e)
u_x^3	$-6\xi_{xxuu} + 4\varphi_{xuuu} = 0$	(f)
u_x^2	$-4\xi_{xxxu} + 6\varphi_{xxuu} = 0$	(g)
u_x	$-\xi_{xxxx} + 4\varphi_{xxxu} = 0$	(h)
$u_x u_{xxxx}$	$-5\xi_u = 0$	(i)
u_{xxxx}	$-4\xi_x + \varphi_u + \varphi = 0$	(j)
$u_{xx}u_{xxx}$	$-10\xi_u = 0$	(k)
$u_x^2 u_{xxx}$	$-10\xi_{uu} = 0$	(l)
$u_x u_{xxx}$	$-16\xi_{xu} + 4\varphi_{uu} = 0$	(m)
$u_x^3 u_{xx}^2$	$-10\xi_{uuu} = 0$	(n)
$u_x^2 u_{xx}$	$-24\xi_{xuu} + 6\varphi_{uuu} = 0$	(o)
$u_x u_{xx}^2$	$-15\xi_{uu} = 0$	(q)
$u_x u_{xx}$	$-18\xi_{xxu} + 12\varphi_{xuu} = 0$	(p)
1	$\varphi_{xxxx} = 0$	(r)

Table 1: Equations for $\varphi^{xxxx} + \varphi u_x e^{-u} = 0$.

which is spanned by $\mathbf{v}_1 = \partial_x$ and $\mathbf{v}_2 = x\partial_x + 4\partial_u$.

Step 8. By the construction shown in (23), we get the following characteristics

$$\begin{aligned} \mathbf{v}_1 \quad &: \quad \tfrac{dx}{d\varepsilon} = 1 \\ \mathbf{v}_2 \quad &: \quad \tfrac{dx}{d\varepsilon} = x, \qquad \tfrac{du}{d\varepsilon} = 4. \end{aligned} \tag{37}$$

Solving the above with the initial conditions at $\varepsilon = 0$, we get the following transformations

$$\begin{aligned} \mathbf{v}_1 \quad &: \quad x(\varepsilon) = x_0 + \varepsilon \\ \mathbf{v}_2 \quad &: \quad x(\varepsilon) = x_0 e^{\varepsilon}, \qquad u(\varepsilon) = u_0 + 4\varepsilon. \end{aligned} \tag{38}$$

Then, the one parameter groups are

$$G_1 : (x + \varepsilon, u), \qquad G_2 : (e^{\varepsilon} x, u + 4\varepsilon) \tag{39}$$

for any real number ε. Then, by Definition 0.9, we complete the proof of the results shown in Theorem 2. □

In general, the main conclusion of this theorem says that by knowing one analytical solution, we can obtain other solutions using the transformations presented in the theorem. In particular, if $u = f(x)$ is a solution to the Euler-Bernoulli beam equation, then the transformations, for any real ε, $u_1 = f(x - \varepsilon)$ and $u_2 = f(e^{-\varepsilon}) - 4\varepsilon$ are also the solutions of the Euler-Bernoulli beam equation. Later, we will link this result to the analytical solution in Chapter VII and see more details.

IV. 2 Lie Symmetries Of Modified Euler-Bernoulli Beam Equation

Theorem 3. *For the modified Euler-Bernoulli beam equation (7), the Lie algebra of infinitesimal symmetries is spanned by*

$$\boldsymbol{v}_1 = \partial_x, \qquad \boldsymbol{v}_2 = \frac{1}{4} x \partial_x + \partial_u. \tag{40}$$

The one-parameter groups are

$$G_1 : (x + \varepsilon, u), \qquad G_2 : (e^{\varepsilon/4} x, u + \varepsilon). \tag{41}$$

Therefore, if $u = f(x)$ is a solution of modified Euler-Bernoulli beam equation (7), then the following transformed forms are also the solutions.

$$u_1 = f(x - \varepsilon), \qquad u_2 = f(e^{-\varepsilon/4}x) - \varepsilon \tag{42}$$

for any real number ε.

Proof of Theorem 3. We will follow the algorithm stated in Chapter IV.

Step 1. Since the modified Euler-Bernoulli beam equation has 1 independent and 1 dependent variables, that is $p = q = 1$, the vector field is

$$\mathbf{v} = \xi\partial_x + \varphi\partial_u. \tag{43}$$

Step 2. The order of the equation is 3, so we get the third prolongation which is

$$\mathrm{pr}^{(3)}\mathbf{v} = \mathbf{v} + \varphi^x \frac{\partial}{\partial u_x} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \varphi^{xxx}\frac{\partial}{\partial u_{xxx}}. \tag{44}$$

Steps 3 and 4. Applying the prolonged vector field over the equation (7) yields

$$\varphi^{xx}u_{xx} - \varphi^x u_{xxx} - \varphi^{xxx}u_x = -\varphi e^{-u} \tag{45}$$

whenever $\frac{1}{2}u_{xx}^2 - u_x u_{xxx} - e^{-u} - C = 0$ holds. The corresponding computations and Mathematica code are included in APPENDIX B.

Steps 5 and 6. Then we obtain the system of determining equations which is shown in Table 2. From the equations (a),(b) and (c) we can easily see that

Monomial	Coefficient	
u_{xxx}	$-\varphi_x = 0$	(a)
$u_x u_{xxx}$	$-2\varphi_u + 4\xi_x - \varphi = 0$	(b)
$u_x^2 u_{xxx}$	$5\xi_u = 0$	(c)
u_x^5	$\xi_{uuu} = 0$	(d)
u_x^4	$3\xi_{xuu} - \varphi_{uuu} = 0$	(e)
u_x^3	$3\xi_{xxu} - 3\varphi_{xuu} = 0$	(f)
u_x^2	$\xi_{xxx} - 3\varphi_{xxu} = 0$	(g)
u_x	$-\varphi_{xxx} = 0$	(h)
$u_x^3 u_{xx}$	$5\xi_{uu} = 0$	(i)
$u_x^2 u_{xx}$	$7\xi_{xu} - 2\varphi_{uu} = 0$	(j)
$u_x u_{xx}$	$2\xi_{xx} - \varphi_{xu} = 0$	(k)
u_{xx}	$\varphi_{xx} = 0$	(l)

Table 2: Equations for $\varphi^{xx}u_{xx} - \varphi^{x}u_{xxx} - \varphi^{xxx}u_x = -\varphi e^{-u}$.

$$\varphi = c_2, \qquad \xi = \frac{1}{4}c_2 x + c_1 \tag{46}$$

for some arbitrary constants c_1 and c_2.

Step 7. Substituting (46) into (43) yields

$$\mathbf{v} = (\frac{1}{4}c_2 x + c_1)\partial_x + c_2\partial_u \tag{47}$$

which is spanned by $\mathbf{v}_1 = \partial_x$ and $\mathbf{v}_2 = \frac{1}{4}x\partial_x + \partial_u$.

Step 8. By the construction shown in (23), we get the following characteristics

$$\begin{aligned} \mathbf{v}_1 &: \tfrac{dx}{d\varepsilon} = 1 \\ \mathbf{v}_2 &: \tfrac{dx}{d\varepsilon} = \tfrac{1}{4}x, \qquad \tfrac{du}{d\varepsilon} = 1. \end{aligned} \tag{48}$$

Solving the above with the initial conditions at $\varepsilon = 0$, we get the following transformation

$$\begin{aligned} \mathbf{v}_1 &: \quad x(\varepsilon) = x_0 + \varepsilon \\ \mathbf{v}_2 &: \quad x(\varepsilon) = x_0 e^{\varepsilon/4}, \quad u(\varepsilon) = u_0 + \varepsilon. \end{aligned} \tag{49}$$

Then, the one parameter groups are

$$G_1 : (x + \varepsilon, u), \qquad G_2 : (e^{\varepsilon/4} x, u + \varepsilon) \tag{50}$$

for any real number ε. Then, by Definition 0.9, we complete the proof of the results shown in Theorem 3. □

The main conclusion of this theorem is that by knowing one analytical solution, we can obtain other solutions using the transformations presented in the theorem. In particular, if $u = f(x)$ is a solution to the modified Euler-Bernoulli beam equation, then the transformations, for any real ε, $u_1 = f(x - \varepsilon)$ and $u_2 = f(e^{-\varepsilon/4}) - \varepsilon$ are also the solutions of the modified Euler-Bernoulli beam equation.

Also we can see that the symmetry groups of the modified Euler-Bernoulli beam equation is very similar to those of the original equation. That means we did not lose the symmetry properties of the equation by using the modification.

Overall, since we found the Lie symmetry groups of the differential equations (4) and (7), we will overview some reduction methods (Chapters V and VII) and use them to reduce the orders of these equations

CHAPTER V: REDUCTION METHOD 1

This method arises from one of the greatest results from Lie groups, which are canonical coordinates. The main idea is that the symmetry transformation in the actual coordinate system is the one-parameter group translation (see the Definitions 0.2 and 0.3) in the canonical coordinate system. Suppose the differential equation has 1 independent variable and 1 dependent variable, x and u, respectively (as in our case). We are interested in the canonical coordinates

$$y = \eta(x,u) \qquad w = \zeta(x,u) \tag{51}$$

such that in this coordinate system the transformed Euler-Bernoulli beam equation admits the group with infinitesimal generator $\frac{\partial}{\partial w}$. If the infinitesimal generator of the group of the original equation is $\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \varphi(x,u)\frac{\partial}{\partial u}$, then we are interested in the solution of the following characteristic system

$$\begin{aligned} \mathbf{v}(\eta) &= \xi\frac{\partial \eta}{\partial x} + \varphi\frac{\partial \eta}{\partial u} = 0 \\ \mathbf{v}(\zeta) &= \xi\frac{\partial \zeta}{\partial x} + \varphi\frac{\partial \zeta}{\partial u} = 1. \end{aligned} \tag{52}$$

Notice that, since we want the one-parameter group translation in $w = \zeta(x,u)$, only the second equation in the above is equal to 1. The equation of η can be found by solving the characteristic equation $\frac{dx}{\xi} = \frac{du}{\varphi}$. The second equation in (52) is usually solved by observation.

It is important to know that this method of reduction is not always the best choice since the canonical coordinates can be complicated for the reduction process depending on the form of the infinitesimal generator. Moreover, the reduction may not work at all because of the form of the equation itself. We will present this outcome in the following chapter.

V.1 Reduction Method 1 For Euler-Bernoulli Beam Equation

Suppose we apply the reduction method 1 to the Euler-Bernoulli beam equation that admits one of its symmetry groups with $\mathbf{v}_1 = \frac{\partial}{\partial x}$ (as presented in chapter IV.1). Then the following theorem is obtained.

Theorem 4. *The Euler-Bernoulli beam equation (4) that admits the symmetry group with* $\boldsymbol{v}_1 = \frac{\partial}{\partial x}$ *reduces to the following third order differential equation*

$$10z^{-6}z_y z_{yy} - z^{-5}z_{yyy} - 15z^{-7}z_y^3 = e^{-y} \tag{53}$$

where $z = w_y = u_x^{-1}$ *and* $y = u$.

Proof of Theorem 4. We will use the reduction method 1 as explained in the beginning of this chapter.

For the one-parameter group with an infinitesimal generator $\mathbf{v}_1 = \frac{\partial}{\partial x}$, the system

$$\begin{aligned} \mathbf{v}_1(\eta) &= \frac{\partial \eta}{\partial x} = 0 \\ \mathbf{v}_1(\zeta) &= \frac{\partial \zeta}{\partial x} = 1 \end{aligned} \tag{54}$$

has a solution $\eta(x, u) = u$ and $\zeta(x, u) = x$. Hence, we have the change of variables as

$$y = \eta(x, u) = u \qquad w = \zeta(x, u) = x. \tag{55}$$

By expressing the fourth order derivative term of our Euler-Bernoulli beam equation in terms of these coordinates, we obtain

$$u_{xxxx} = \frac{1}{w_y^7}\left(10 w_y w_{yy} w_{yyy} - w_y^2 w_{yyyy} - 15 w_{yy}^3\right). \tag{56}$$

Letting $z = w_y$ reduces the Euler-Bernoulli beam equation to the third order differential equation

$$10 z^{-6} z_y z_{yy} - z^{-5} z_{yyy} - 15 z^{-7} z_y^3 = e^{-y}. \tag{57}$$

□

By using this method, we reduced the order of the Euler-Bernoulli beam equation by one with the infinitesimal generator $\mathbf{v}_1 = \frac{\partial}{\partial x}$. But this reduction method is unsuccessful with $\mathbf{v}_2 = x\partial_x + 4\partial_u$, because the canonical coordinates

$$y = \eta(x,u) = xe^{-u/4} \qquad w = \zeta(x,u) = ln(x), \tag{58}$$

which are obtained from solving the characteristic system as shown in (52), are complicated to work with. For instance, the expression of w_y by using the chain rule is

$$w_y = \frac{dw}{dy} = \frac{dw}{dx}\frac{dx}{dy} + \frac{dw}{du}\frac{du}{dy} = \frac{1}{x}\left[\left(e^{-u/4} - \frac{1}{4} x u_x e^{-u/4}\right)^{-1} - \left(x u_x e^{-u/4}\right)^{-1}\right] \tag{59}$$

and further derivations are even more complicated. Furthermore, the reduction process would not be completed, simply because in this case $u = ln(x/y)$ and x is dependent on w

which eliminates the possibility of doing the replacement $z = w_y$. This demonstrates the disadvantage of the reduction method 1.

V.2 Reduction Method 1 For Modified Euler-Bernoulli Beam Equation

Suppose we apply reduction method 1 to the modified Euler-Bernoulli beam equation that admits one of its symmetry groups with $\mathbf{v}_1 = \frac{\partial}{\partial x}$ (as presented in chapter IV.2). Then the following theorem is obtained.

Theorem 5. *The modified Euler-Bernoulli beam equation (7) that admits the symmetry group with* $\boldsymbol{v}_1 = \frac{\partial}{\partial x}$ *reduces to the following second order differential equation*

$$zz_{yy} - \frac{5}{2}z_y^2 = z^6 e^{-y}, \tag{60}$$

where $z = w_y = u_x^{-1}$ *and* $y = u$.

Proof of Theorem 5. We will use the reduction method 1 as explained in the beginning of this chapter.

As this system has also one-parameter group with an infinitesimal generator $\mathbf{v}_1 = \frac{\partial}{\partial x}$, similarly, we have the same change of variables

$$y = \eta(x, u) = u \qquad w = \zeta(x, u) = x. \tag{61}$$

By using the chain rule, we obtain the following expressions

$$u_x = \frac{du}{dx} = \frac{1}{w_y}, \qquad u_{xx} = \frac{d^2u}{dx^2} = -\frac{w_{yy}}{w_y^3}, \qquad u_{xxx} = \frac{d^3u}{dx^3} = \frac{3w_{yy}^2 - w_y w_{yyy}}{w_y^5}. \tag{62}$$

Substituting (62) into the equation and letting $z = w_y$, we obtain the second order differential equation

$$zz_{yy} - \frac{5}{2}z_y^2 = z^6 e^{-y}. \tag{63}$$

□

Again, with $\mathbf{v}_2 = \frac{1}{4}x\partial_x + \partial_u$, the canonical coordinates obtain similar complicated forms as it was demonstrated for the case of the original Euler-Bernoulli beam equation.

CHAPTER VI: REDUCTION METHOD 2

We will now use the concept called "differential invariants" that helps us to determine other reduced forms of the equation that admit the same symmetry groups. Basically, if $\mathbf{v}_1 = \xi^1 \frac{\partial}{\partial x^1} + \ldots + \xi^m \frac{\partial}{\partial x^m} + \varphi^1 \frac{\partial}{\partial u^1} + \ldots + \varphi^k \frac{\partial}{\partial u^k}$ is the infinitesimal generator from the symmetry group G_1 of the differential equation with m independent and k dependent variables, then we construct the characteristic equation

$$\frac{dx^1}{\xi^1} = \ldots = \frac{dx^m}{\xi^m} = \frac{du^1}{\varphi^1} = \ldots = \frac{du^k}{\varphi^k} \tag{64}$$

and its solution provides the differential invariants.

Definition 0.10. *[[Ol], Def. 2.51]* Let G be a local group of transformations acting on $M \subset X \times U$, where X is the space of independent variables and U is the space of dependent variables. An n-th order *differential invariant* of G is a smooth function $\eta : M^{(n)} \to \mathbb{R}$, depending on x, u and derivatives of u, such that η is an invariant under the prolonged group action

$$\eta(\mathrm{pr}^{(n)} g \cdot (x, u^{(n)})) = \eta(x, u^{(n)}), \qquad (x, u^{(n)}) \in M^{(n)} \tag{65}$$

for all $g \in G$ such that $\mathrm{pr}^{(n)} g \cdot (x, u^{(n)})$ is defined.

Here $M^{(n)}$ is an n-jet space of M that is the extension of the $X \times U$ and consists of the coordinates that represent the independent variables, dependent variables, and the derivatives of dependent variables up to order n.

Lemma 1. *The differential equation with 1 dependent and 1 independent variables that admits the symmetry group with an infinitesimal generator* $\boldsymbol{v} = \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial u}$*, the characteristic equation is*

$$\frac{dx}{\xi} = \frac{du}{\varphi} = \frac{du_x}{0} \tag{66}$$

and we have the invariants $\eta(x, u) = c_1$ *and* $\zeta(x, u) = c_2$ *where* c_1, c_2 *are the constants of integration from the characteristic (66).*

Proof of Lemma 1. By directly using the characteristic equation (64). □

Proposition 1. *[[Ol], Prop. 2.56] Let* $\eta^1(x, u^{(n)}), \ldots, \eta^k(x, u^{(n)})$ *be a complete set of functionally independent n-th order differential invariants. An n-th order differential equation* $\Delta(x, u^{(n)}) = 0$ *admits G as a symmetry group if and only if there is an equivalent equation*

$$\tilde{\Delta}(\eta^1(x, u^{(n)}), \ldots, \eta^k(x, u^{(n)})) = 0 \tag{67}$$

involving only the differential invariants of G. In particular, if G is a one-parameter group of transformations, any n-th order differential equation having G as a symmetry group is equivalent to an $(n-1)$*-st order equation*

$$\tilde{\Delta}(y, w, dw/dy, \ldots, d^{n-1}w/dy^{n-1}) = 0 \tag{68}$$

involving the invariants $y = \eta(x, u)$ *and* $w = \zeta(x, u, u_x)$ *of* $\mathrm{pr}^{(1)}G$ *and their derivatives.*

The advantage of this method over the reduction method 1 is that it does not require the replacement $z = w_y$. This allows us to use also the second infinitesimal generator of the symmetry group of both the Euler-Bernoulli and the modified Euler-Bernoulli beam equations in order to obtain the reductions. More details will be demonstrated in the following chapters.

VI.1 Reduction Method 2 For Euler-Bernoulli Beam Equation

Suppose we apply the reduction method 2 to the Euler-Bernoulli beam equation that admits its symmetry groups with $\mathbf{v}_1 = \frac{\partial}{\partial x}$ and $\mathbf{v}_2 = x\partial_x + 4\partial_u$ (as presented in chapter IV.1). Then the following theorem is obtained.

Theorem 6. *The Euler-Bernoulli beam equation (4) that admits the symmetry groups with* $\boldsymbol{v}_1 = \frac{\partial}{\partial x}$ *and* $\boldsymbol{v}_2 = x\partial_x + 4\partial_u$ *reduces to the following third order differential equation (69) and third order integro-differential equation (70)*

$$w^3 w_{yyy} + w_y w_{yy}(3w^{-5} + w^2) + w_y^3(3w^{-8} - 2w) = e^{-y}, \tag{69}$$

where $w = u_x$, $y = u$, *and*

$$w_{yyy}B^4 + 2(u_{xxx} - w_y A)A + (u_{xxx}A + u_{xx}C)B = e^{-u}, \tag{70}$$

where $A = -\frac{1}{x^2} - \frac{1}{4}u_{xx}$, $B = \frac{1}{x} - \frac{1}{4}w$, $C == \frac{2}{x^3} - \frac{1}{4}u_{xxx}$, $u_{xx} = w_y B$ *and* $u_{xxx} = w_{yy}B^2 + w_y A$, $u = \int w dx$, *and* $x = e^{y + \frac{1}{4}\int w dx}$.

Proof of Theorem 6. We will use the reduction method 2 as explained in the beginning of this chapter.

For this system, the characteristic equation from the infinitesimal generator $\mathbf{v}_1 = \frac{\partial}{\partial x}$ from Lemma 1 is

$$\frac{dx}{1} = \frac{du}{0} = \frac{du_x}{0} \tag{71}$$

and its solution gives us the following change of variables

$$y = u \qquad w = u_x \tag{72}$$

with $w = f(y)$, that is $u_x = f(u)$. The derivative expressions in terms of these new coordinates are

$$\begin{aligned} w_y &= \frac{dw}{dy} = \frac{dw/dx}{dy/dx} = \frac{u_{xx}}{u_x} \\ w_{yy} &= \frac{dw_y/dx}{dy/dx} = \frac{u_{xxx}w - (ww_y)^2}{w^3} \\ w_{yyy} &= \frac{1}{w^3}\left[u_{xxxx} - w_y^3(3w^{-8} - 2w) - w_y w_{yy}(3w^{-5} + w^2)\right]. \end{aligned} \tag{73}$$

So, in terms of (y, w) coordinates, the Euler-Bernoulli beam equation reduces to the third order differential equation

$$w^3 w_{yyy} + w_y w_{yy}(3w^{-5} + w^2) + w_y^3(3w^{-8} - 2w) = e^{-y}. \tag{74}$$

Similarly, the characteristic equation from the infinitesimal generator $\mathbf{v}_2 = x\partial_x + 4\partial_u$ from

Lemma 1 is

$$\frac{dx}{x} = \frac{du}{4} = \frac{du_x}{0} \tag{75}$$

and its solution gives us the following change of variables

$$y = ln(x) - \frac{1}{4}u \qquad w = u_x \tag{76}$$

with $w = f(y)$, that is $u_x = f(ln(x) - \frac{1}{4}u)$. Since the derivative expressions in terms of these new coordinates are complicated (see APPENDIX C), we show the final result, that is the integro-differential equation of order 3

$$w_{yyy}B^4 + 2(u_{xxx} - w_y A)A + (u_{xxx}A + u_{xx}C)B = e^{-u}, \tag{77}$$

where $A = -\frac{1}{x^2} - \frac{1}{4}u_{xx}$, $B = \frac{1}{x} - \frac{1}{4}w$, $C = \frac{2}{x^3} - \frac{1}{4}u_{xxx}$, $u_{xx} = w_y B$ and $u_{xxx} = w_{yy}B^2 + w_y A$, $u = \int w dx$, and $x = e^{y + \frac{1}{4}\int w dx}$. □

As can be seen, we were able to obtain the reduction (third order integro-differential equation) by using the infinitesimal generator $\mathbf{v}_2$. Now we will try this method for our modified equation in the following section.

VI.2 Reduction Method 2 For Modified Euler-Bernoulli Beam Equation

Suppose we apply the reduction method 2 to the modified Euler-Bernoulli beam equation that admits its symmetry groups with $\mathbf{v}_1 = \frac{\partial}{\partial x}$ and $\mathbf{v}_2 = \frac{1}{4}x\partial_x + \partial_u$ (as presented in chapter IV.2). Then the following theorem is obtained.

Theorem 7. *The modified Euler-Bernoulli beam equation (7) that admits the symmetry groups with* $\boldsymbol{v}_1 = \frac{\partial}{\partial x}$ *and* $\boldsymbol{v}_2 = \frac{1}{4}x\partial_x + \partial_u$ *reduces to the following second order differential equation (78) and third order integro-differential equation (79)*

$$-\frac{1}{2}(ww_y)^2 - w_{yy}w^3 = e^{-y} + C, \tag{78}$$

where $w = u_x$, $y = u$, *and*

$$\frac{1}{2}w_y^2(\frac{4}{x} - w)^2 - w(w_{yy}(4/x - w)^2 + w_y(-\frac{4}{x^2} - w_y(\frac{4}{x} - w))) = e^{-u} + C, \tag{79}$$

where $x = e^{(y+\int wdx)/4}$ *and* $u = \int wdx$.

Proof of Theorem 7. We will use the reduction method 2 as explained in the beginning of this chapter.

The characteristic equation from the infinitesimal generator $\mathbf{v}_1 = \frac{\partial}{\partial x}$ from Lemma 1 is

$$\frac{dx}{1} = \frac{du}{0} = \frac{du_x}{0} \tag{80}$$

and its solution gives us the following change of variables

$$y = u \qquad w = u_x \tag{81}$$

with $w = f(y)$, that is $u_x = f(u)$. The derivative expressions in terms of these new coordinates are

$$
\begin{aligned}
w_y = \frac{dw}{dy} = \frac{dw/dx}{dy/dx} = \frac{u_{xx}}{u_x} \\
w_{yy} = \frac{dw_y/dx}{dy/dx} = \frac{u_{xxx}u_x - u_{xx}^2}{u_x^3}.
\end{aligned} \tag{82}
$$

So our equation in terms of (y, w) coordinates becomes the second order differential equation

$$
-\frac{1}{2}(ww_y)^2 - w_{yy}w^3 = e^{-y} + C. \tag{83}
$$

Similarly, the characteristic equation from the infinitesimal generator $\mathbf{v}_2 = \frac{1}{4}x\partial_x + \partial_u$ from Lemma 1 is

$$
\frac{dx}{1/4x} = \frac{du}{1} = \frac{du_x}{0} \tag{84}
$$

and its solution gives us the following change of variables

$$
y = 4ln(x) - u \qquad w = u_x \tag{85}
$$

with $w = f(y)$, that is $u_x = f(4ln(x) - u)$. The derivative expressions in terms of these new coordinates are

$$
\begin{aligned}
w_y = \frac{dw}{dy} = \frac{dw/dx}{dy/dx} = \frac{u_{xx}}{4/x - w} \\
w_{yy} = \frac{dw_y/dx}{dy/dx} = \frac{u_{xxx} - w_y(-4/x^2 - u_{xx})}{(4/x - w)^2}.
\end{aligned} \tag{86}
$$

So, our equation in terms of (y, w) coordinates becomes the second order integro-differential equation

$$\frac{1}{2}w_y^2(\frac{4}{x}-w)^2 - w(w_{yy}(4/x-w)^2 + w_y(-\frac{4}{x^2} - w_y(\frac{4}{x}-w))) = e^{-u} + C, \tag{87}$$

where $x = e^{(y+\int wdx)/4}$ and $u = \int wdx$. □

Again, we were able to derive the reduced form (second order integro-differential equation) by using the reduction method 2 since it works with $\mathbf{v}_2$.

CHAPTER VII: ANALYTICAL AND NUMERICAL SOLUTIONS

In this section, we bring some examples of using the reductions of Euler-Bernoulli beam equation for our purpose of finding potential analytical and numerical solutions. We will compare our results with the existing numerical solution found with finite element method by Shakipov [Sh] which is shown below in Figure 3. Finite element method is widely

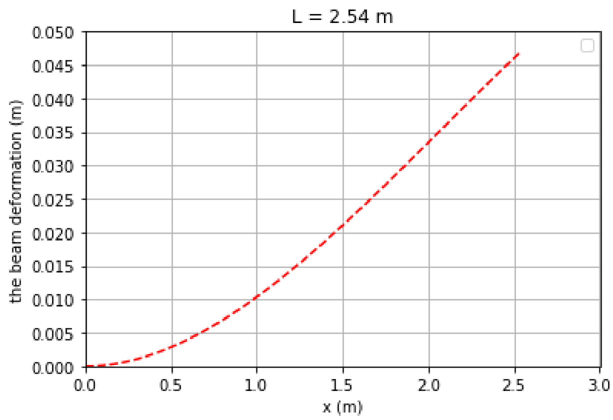


Figure 3: Finite element solution of $u_{xxxx} = Ke^{-Bu}$.

used numerical method that is used to solve differential equations with many application in engineering and other fields [Hu]. The main point of this method is to divide the domain of the whole system into smaller finite parts (so-called ”finite elements”) and transform them into a set of linear equations that is solved by using any of existing numerical methods [Am]. Finite element is the method of choice in many scientific fields as it is able to solve highly complicated problems with small error.

VII.1 One Analytical Solution Of The Euler-Bernoulli Beam Equation Using Its Reduced Form

We try to find the explicit solution of our equation of interest (4) by using the following reduced form as shown in Theorem 4, but with K and B being unknown which yields to

$$10z^{-6}z_y z_{yy} - z^{-5}z_{yyy} - 15z^{-7}z_y^3 = Ke^{-By}, \tag{88}$$

where $z = u_x^{-1}$ and $y = u$. Our guess function is $z = e^{ay+b}$. We substitute this expression into the reduced equation (88) and we obtain

$$-6a^3e^{3ay} = Ke^{4b}e^{(7a-B)y}. \tag{89}$$

Then we get the following equations

$$3a = 7a - B \qquad -6a^3 = Ke^{4b}. \tag{90}$$

Solving for a and b, the solution of the equation will take the form

$$u = \frac{1}{a}ln(ax + e^b) - \frac{b}{a}, \tag{91}$$

where $a = \frac{B}{4}$ and $b = \frac{1}{4}ln(\frac{6a}{K}) + i\frac{\pi(2n+1)}{4}$ for $n = 1, 2, 3, \ldots$. Fitting this solution to the solution from finite element approximation gives us the good nonlinear least-squares fit shown in Figure 4. The main problem of this analytical solution is that the third and fourth boundary conditions in (5) are not met. However, we still find the solution useful since the

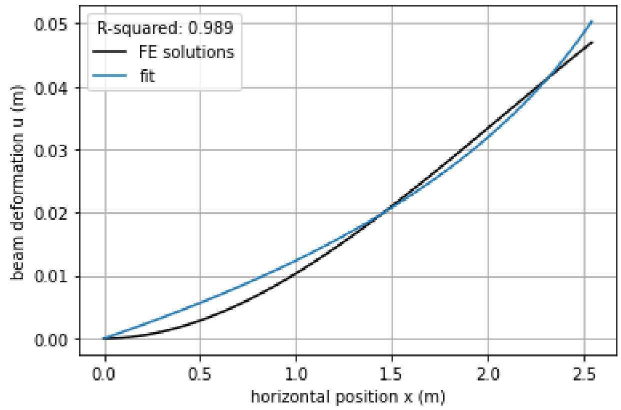


Figure 4: Fitting the finite element solution of Euler-Bernoulli beam equation with $u = \frac{1}{a}ln(ax + e^b) - \frac{b}{a}$ with R-squared = 0.988.

Theorem 2 tells us that the followings are also the solutions of the Euler-Bernoulli beam equation.

$$u_1 = \frac{1}{a}ln(a(x-\varepsilon) + e^b) - \frac{b}{a} \tag{92}$$

$$u_2 = \frac{1}{a}ln(e^{-\varepsilon}ax + e^b) - \frac{b}{a} - 4\varepsilon \tag{93}$$

for any real number ε, $a = \frac{B}{4}$ and $b = \frac{1}{4}\left[ln(\frac{6a}{K}) + i\pi + 2i\pi n\right]$ for $n = 1, 2, 3, \ldots$.

VII.2 Numerical Solution Of The Reduced Form Of The Euler-Bernoulli Beam Equation

Now we try to demonstrate the approach to numerically solve the reduced form of the Euler-Bernoulli beam equation from Theorem 5 but with K and B being unknown which yields

$$zz_{yy} - \frac{5}{2}z_y^2 = \frac{K}{B}z^6e^{-By} \tag{94}$$

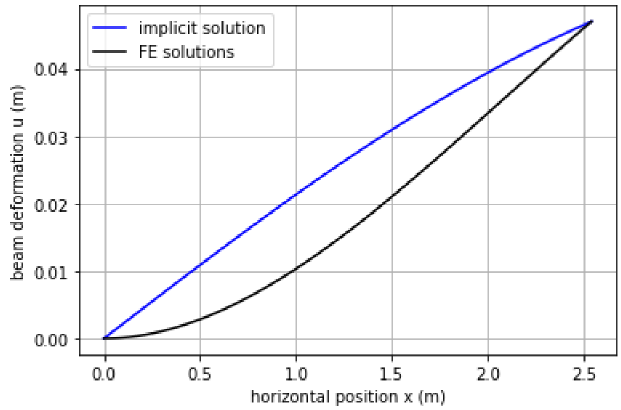


Figure 5: Numerical solution of $zz_{yy} - \frac{5}{2}z_y^2 = z^6 e^{-y}$ vs. finite element solutions of Euler-Bernoulli beam equation.

where $z = u_x^{-1}$ and $y = u$. Slight modification can be done to this equation by letting $f = z$ and $g = z_y$ which gives us the following pair of differential equations

$$\begin{aligned} f_y &= g \\ g_y &= \frac{1}{f}\left(\frac{5}{2}g^2 + \frac{K}{B}e^{-By}f^6\right), \end{aligned} \tag{95}$$

where $f_y = \frac{df}{dy}$ and $g_y = \frac{dg}{dy}$. This pair resembles the dynamic system that can be solved numerically with given initial values. We use the Python package called GEKKO [BeHMH], which is the modern tool that uses various numerical methods for solving complicated systems, to obtain the solution of (95). Figure 5 shows the numerical result and comparison with the finite element approximation of Euler-Bernoulli beam equation (Python code is included in APPENDIX D). The interesting idea behind the solution of this reduced form is that it reverses the way of our original problem is solved. In particular, if we wanted to solve the beam deflection u for some given length x, then by (95) we solve for the length x of the beam for given deflection values u. As we can see from Figure 5, the maximum length given by finite element result coincides with the maximum length found by solving the equation

(95). This is indeed the useful approach for the retaining wall deformation problems, since the maximum deflection can be enough information for engineering solutions.

CHAPTER VIII: RESULTS AND DISCUSSION

At this point, other than classifying the Lie symmetry groups for both original and modified Euler-Bernoulli beam equations, our work resulted in 6 different reductions of the Euler-Bernoulli beam equation. In particular, below is the summary of results.

Original Euler-Bernoulli beam equation: $\boxed{u_{xxxx} = e^{-u}}$.

The corresponding reduced forms are

- Third order differential equation

$$\boxed{10z^{-6}z_y z_{yy} - z^{-5}z_{yyy} - 15z^{-7}z_y^3 = e^{-y}}$$

where $z = u_x^{-1}$ and $y = u$.

- Third order differential equation

$$\boxed{w^3 w_{yyy} + w_y w_{yy}(3w^{-5} + w^2) + w_y^3(3w^{-8} - 2w) = e^{-y}}$$

where $w = u_x$ and $y = u$.

- Third order integro-differential equation

$$\boxed{w_{yyy}B^4 + 2(u_{xxx} - w_y A)A + (u_{xxx}A + u_{xx}C)B = e^{-u}}$$

where $A = -\frac{1}{x^2} - \frac{1}{4}u_{xx}$, $B = \frac{1}{x} - \frac{1}{4}w$, $C == \frac{2}{x^3} - \frac{1}{4}u_{xxx}$, $u_{xx} = w_y B$ and $u_{xxx} = w_{yy}B^2 + w_y A$, $u = \int w dx$, $x = e^{y + \frac{1}{4}\int w dx}$, $w = u_x$ and $y = ln(x) - \frac{1}{4}u$.

Modified form of Euler-Bernoulli beam equation: $\boxed{\frac{1}{2}u_{xx}^2 - u_x u_{xxx} = e^{-u} + C}$. The corresponding reduced forms are

- The second order differential equation

$$\boxed{zz_{yy} - \frac{5}{2}z_y^2 = z^6 e^{-y}}$$

where $z = u_x^{-1}$ and $y = u$.

- The second order differential equation

$$\boxed{-\frac{1}{2}(ww_y)^2 - w_{yy}w^3 = e^{-y} + C}$$

where $w = u_x$ and $y = u$

- The second order integro-differential equation

$$\boxed{\frac{1}{2}w_y^2(\frac{4}{x} - w)^2 - w(w_{yy}(4/x - w)^2 + w_y(-\frac{4}{x^2} - w_y(\frac{4}{x} - w))) = e^{-u} + C}$$

where $x = e^{(y+\int wdx)/4}$, $u = \int wdx$, $w = u_x$, and $y = 4ln(x) - u$.

Most of these reductions are highly nonlinear and seem hard to be solved analytically, however, we now know some classification of equations that admit the same Lie symmetries as Euler-Bernoulli beam equation does. Furthermore, our attempt to solve it both analytically and numerically by using some of its reduced forms shows that there are possibilities of understanding the properties of the solution by analyzing the rest of its transformed equations.

Also, we created the new perspective of working with the Euler-Bernoulli beam equation. In particular, by numerically solving one of its reduced forms, we understood that instead of solving the beam deformation values for some given beam length, we can also find the beam length for some given deformation values. This observation may be unusual for the civil engineers, but we think this question is worth considering.

Finally, several future directions of our work can be listed with the following research questions, such as

- Are there any Lie symmetry groups of the reduced equations?
- Can we obtain further reductions of the reduced equations by applying the Lie groups methods?
- What are the possible applications of the reduced equations? For instance, what are the applications of the transformation for when the wall deflection becomes an independent variable, and the wall length becomes a dependent variable.
- What are the reductions for other types of fourth-order differential equations using the methods described in this work?

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APPENDIX A: COMPUTATIONS AND MATHEMATICA CODE FOR APPLYING THE FOURTH PROLONGATION ON $u_{xxxx} = e^{-u}$

$$\begin{aligned}
\mathrm{pr}^{(4)}\mathbf{v}(u_{xxxx}) &= \mathrm{pr}^{(4)}\mathbf{v}(e^{-u}) \\
\xi u_{xxxx} + \varphi^{xxxx} &= -\xi u_x e^{-u} - \varphi e^{-u} \\
\varphi^{xxxx} + \varphi e^{-u} &= -\xi(u_{xxxxx} + u_x e^{-u})
\end{aligned}$$

since $\frac{d}{dx}(u_{xxxx}) = \frac{d}{dx}(e^{-u})$, we have

$$\varphi^{xxxx} + \varphi e^{-u} = 0$$

Below is the code and its corresponding output of computing φ^{xxxx}.

Input:

$$\boxed{\left[\frac{\partial^4}{\partial x\,\partial x\,\partial x\,\partial x}\left(\varphi(x,u(x)) - \xi(x,u(x))\frac{\partial u(x)}{\partial x}\right) + \xi(x,u(x))\frac{\partial^5 u(x)}{\partial x\,\partial x\,\partial x\,\partial x\,\partial x}\right]}$$

Output:

$$-10u''(x)u^{(3)}(x)\xi^{(0,1)}(x,u(x)) - 5u'(x)u^{(4)}(x)\xi^{(0,1)}(x,u(x)) + u^{(4)}(x)\varphi^{(0,1)}(x,u(x))$$

$$-15u'(x)u''(x)^2\xi^{(0,2)}(x,u(x)) - 10u'(x)^2u^{(3)}(x)\xi^{(0,2)}(x,u(x)) + 3u''(x)^2\varphi^{(0,2)}(x,u(x))$$

$$+4u'(x)u^{(3)}(x)\varphi^{(0,2)}(x,u(x)) - 10u'(x)^3u''(x)\xi^{(0,3)}(x,u(x)) + 6u'(x)^2u''(x)\varphi^{(0,3)}$$

$$(x,u(x)) - u'(x)^5\xi^{(0,4)}(x,u(x)) + u'(x)^4\varphi^{(0,4)}(x,u(x)) - 4u^{(4)}(x)\xi^{(1,0)}(x,u(x)) - 12u''(x)^2$$

$$\xi^{(1,1)}(x,u(x)) - 16u'(x)u^{(3)}(x)\xi^{(1,1)}(x,u(x)) + 4u^{(3)}(x)\varphi^{(1,1)}(x,u(x)) - 24u'(x)^2u''(x)$$

$$\xi^{(1,2)}(x,u(x)) + 12u'(x)u''(x)\varphi^{(1,2)}(x,u(x)) - 4u'(x)^4\xi^{(1,3)}(x,u(x)) + 4u'(x)^3\varphi^{(1,3)}$$

$$(x,u(x)) - 6u^{(3)}(x)\xi^{(2,0)}(x,u(x)) - 18u'(x)u''(x)\xi^{(2,1)}(x,u(x)) + 6u''(x)\varphi^{(2,1)}(x,u(x))$$

$$-6u'(x)^3\xi^{(2,2)}(x,u(x)) + 6u'(x)^2\varphi^{(2,2)}(x,u(x)) - 4u''(x)\xi^{(3,0)}(x,u(x)) - 4u'(x)^2$$

$$\xi^{(3,1)}(x,u(x)) + 4u'(x)\varphi^{(3,1)}(x,u(x)) - u'(x)\xi^{(4,0)}(x,u(x)) + \varphi^{(4,0)}(x,u(x))$$

APPENDIX B: COMPUTATIONS AND MATHEMATICA CODE FOR APPLYING THE THIRD PROLONGATION ON $\frac{1}{2}u_{xx}^2 - u_x u_{xxx} = e^{-u} + C$

$$\begin{aligned}
\mathrm{pr}^{(3)}\mathbf{v}(\tfrac{1}{2}u_{xx}^2 - u_x u_{xxx}) &= \mathrm{pr}^{(3)}\mathbf{v}(e^{-u} + C) \\
-\xi u_x u_{xxxx} - \varphi^x u_{xxx} + \varphi^{xx} u_{xx} - \varphi^{xxx} u_x &= -\xi u_x e^{-u} - \varphi e^{-u} \\
-\varphi^x u_{xxx} + \varphi^{xx} u_{xx} - \varphi^{xxx} u_x &= \xi(u_x u_{xxxx} - u_x e^{-u}) - \varphi e^{-u}
\end{aligned}$$

since $u_{xxxx} = e^{-u}$ and $u_x u_{xxxx} - u_x e^{-u} = 0$ assuming $u_x \neq 0$, we have

$$\varphi^{xx} u_{xx} - \varphi^x u_{xxx} - \varphi^{xxx} u_x = -\varphi e^{-u}$$

The code and its corresponding output of computing φ^x.

Input:

$$\boxed{\frac{\partial}{\partial x}\left(\varphi(x,u(x)) - \xi(x,u(x))\frac{\partial u(x)}{\partial x}\right) + \xi(x,u(x))\frac{\partial^2 u(x)}{\partial x\,\partial x}}$$

Output:

$$-u'(x)\left(u'(x)\xi^{(0,1)}(x,u(x)) - \varphi^{(0,1)}(x,u(x)) + \xi^{(1,0)}(x,u(x))\right)$$
$$+\varphi^{(1,0)}(x,u(x))$$

The code and its corresponding output of computing φ^{xx}.

Input:

$$\boxed{\frac{\partial^2}{\partial x\,\partial x}\left(\varphi(x,u(x))-\xi(x,u(x))\frac{\partial u(x)}{\partial x}\right)+\xi(x,u(x))\frac{\partial^3 u(x)}{\partial x\,\partial x\,\partial x}}$$

Output:

$$-3u'(x)u''(x)\xi^{(0,1)}(x,u(x))+u''(x)\varphi^{(0,1)}(x,u(x))-u'(x)^3\xi^{(0,2)}(x,u(x))$$

$$+u'(x)^2\varphi^{(0,2)}(x,u(x))-2u''(x)\xi^{(1,0)}(x,u(x))-2u'(x)^2\xi^{(1,1)}(x,u(x))$$

$$+2u'(x)\varphi^{(1,1)}(x,u(x))-u'(x)\xi^{(2,0)}(x,u(x))+\varphi^{(2,0)}(x,u(x))$$

The code and corresponding output of computing φ^{xxx}.

Input:

$$\boxed{\frac{\partial^3}{\partial x\,\partial x\,\partial x}\left(\varphi(x,u(x))-\xi(x,u(x))\frac{\partial u(x)}{\partial x}\right)+\xi(x,u(x))\frac{\partial^4 u(x)}{\partial x\,\partial x\,\partial x\,\partial x}}$$

Output:

$$u^{(3)}(x)\varphi^{(0,1)}(x,u(x)) - 3u^{(3)}(x)\left(u'(x)\xi^{(0,1)}(x,u(x)) + \xi^{(1,0)}(x,u(x))\right)$$

$$+u''(x)\varphi^{(1,1)}(x,u(x)) + 2u''(x)\left(u'(x)\varphi^{(0,2)}(x,u(x)) + \varphi^{(1,1)}(x,u(x))\right)$$

$$-3u''(x)\left(u''(x)\xi^{(0,1)}(x,u(x)) + u'(x)\xi^{(1,1)}(x,u(x)) + u'(x)\left(u'(x)\xi^{(0,2)}(x,u(x))\right.\right.$$

$$+\xi^{(1,1)}(x,u(x)) + \xi^{(2,0)}(x,u(x)) + u'(x)\varphi^{(2,1)}(x,u(x)) + u'(x)\,(u'(x)$$

$$\varphi^{(1,2)}(x,u(x)) + \varphi^{(2,1)}(x,u(x)) + u'(x)\left(u''(x)\varphi^{(0,2)}(x,u(x)) + u'(x)\right.$$

$$\varphi^{(1,2)}(x,u(x)) + u'(x)\left(u'(x)\varphi^{(0,3)}(x,u(x)) + \varphi^{(1,2)}(x,u(x))\right)$$

$$+\varphi^{(2,1)}(x,u(x)) - u'(x)\left(u^{(3)}(x)\xi^{(0,1)}(x,u(x)) + u''(x)\xi^{(1,1)}(x,u(x))\right.$$

$$+2u''(x)\left(u'(x)\xi^{(0,2)}(x,u(x)) + \xi^{(1,1)}(x,u(x))\right) + u'(x)\xi^{(2,1)}(x,u(x))$$

$$+u'(x)\left(u'(x)\xi^{(1,2)}(x,u(x)) + \xi^{(2,1)}(x,u(x))\right) + u'(x)\left(u''(x)\xi^{(0,2)}(x,u(x))\right.$$
$$+u'(x)\xi^{(1,2)}(x,u(x)) + u'(x)\left(u'(x)\xi^{(0,3)}(x,u(x)) + \xi^{(1,2)}(x,u(x))\right)$$
$$+\xi^{(2,1)}(x,u(x)) + \xi^{(3,0)}(x,u(x)) + \varphi^{(3,0)}(x,u(x))$$

APPENDIX C: COMPUTATIONAL PART OF THE PROOF OF THEOREM 6

$$w_y = \frac{u_{xx}}{\frac{1}{x} - \frac{1}{4}u_x} = \frac{u_{xx}}{\frac{1}{x} - \frac{1}{4}w}$$

$$w_{yy} = \frac{u_{xxx} - w_y\left(-\frac{1}{x^2} - \frac{1}{4}u_{xx}\right)}{\left(\frac{1}{x} - \frac{1}{4}w\right)^2}$$

$$\begin{aligned} w_{yyy} &= ((u_{xxxx} - (u_{xxx}(-\tfrac{1}{x^2} - \tfrac{1}{4}u_{xx}) + u_{xx}(-\tfrac{2}{x^3} - \tfrac{1}{4}u_{xxx})))(\tfrac{1}{x} - \tfrac{1}{4}w)^2 \\ &= -2(u_{xxx} - w_y(-\tfrac{1}{x^2} - \tfrac{1}{4}u_{xx}))(\tfrac{1}{x} - \tfrac{1}{4}w)(-\tfrac{1}{x^2} - \tfrac{1}{4}u_{xx}))/(\tfrac{1}{x} - \tfrac{1}{4}w)^5 \end{aligned}$$

APPENDIX D: PYTHON CODE WITH GEKKO SOLVER

```
!pip install gekko
from gekko import GEKKO
import numpy as np
import matplotlib.pyplot as plt
from shapely.geometry import LineString
import math

m = GEKKO()
m.time = np.linspace(0,0.0469779687,num=101)

s = 76.2
sigma = 1000000
E = 20.76
I = 0.00249739
c = 0.03
d = 4.572
K = s*sigma/(E*I)
B = np.log(10)/(c*d)

t = m.Var(0)
z1 = m.Var(0.04565)
z2 = m.Var(0)

m.Equation(t.dt()==1)
m.Equation(z1.dt()==z2)
m.Equation(z2.dt()==1/z1*(5/2*z2**2+K/B*m.exp(-B*t)*z1**6))

m.options.IMODE = 4
m.options.NODES = 3
m.solve()

plt.plot(t,z1,'b-',label=r'$\frac{dz_1}{dy} = z_2$')
```

```
plt.plot(t,z2,'r--',label=r'$\frac{dz_2}{dy}= 5/2z_2^2z_1^{-1}+K/B\exp(-By)z_1^5$')
plt.ylabel('response')
plt.xlabel('y')
plt.legend(loc='best')
plt.show()

plt.plot(t,z1)
plt.xlabel('u')
plt.ylabel('dx/du')
plt.show()
y=t

x = [0]
for i in range(len(z1)-1):
  x.append((y[i+1]-y[i])*z1[i]+x[i])

plt.plot(x,y)
plt.xlabel('x')
plt.ylabel('u')
plt.show()

x = [i*1000 for i in x]

yData1 = np.array([0, 7.90602967e-06, 3.14944931e-05, 7.05709520e-05,
1.24940968e-04, 1.94410103e-04, 2.78783918e-04, 3.77867975e-04,
4.91467835e-04, 6.19389061e-04, 7.61437214e-04, 9.17417856e-04,
1.08713655e-03, 1.27039885e-03, 1.46701033e-03, 1.67677654e-03,
1.89950305e-03, 2.13499542e-03, 2.38305921e-03, 2.64349998e-03,
2.91612329e-03, 3.20073471e-03, 3.49713979e-03, 3.80514411e-03,
4.12455321e-03, 4.45517267e-03, 4.79680803e-03, 5.14926488e-03,
5.51234876e-03, 5.88586524e-03, 6.26961987e-03, 6.66341823e-03,
7.06706588e-03, 7.48036837e-03, 7.90313126e-03, 8.33516012e-03,
8.77626052e-03, 9.22623800e-03, 9.68489814e-03, 1.01520465e-02,
1.06274886e-02, 1.11110301e-02, 1.16024764e-02, 1.21016333e-02,
```

```
1.26083061e-02, 1.31223006e-02, 1.36434221e-02, 1.41714764e-02,
1.47062690e-02, 1.52476054e-02, 1.57952912e-02, 1.63491319e-02,
1.69089332e-02, 1.74745005e-02, 1.80456395e-02, 1.86221556e-02,
1.92038546e-02, 1.97905418e-02, 2.03820230e-02, 2.09781036e-02,
2.15785892e-02, 2.21832853e-02, 2.27919977e-02, 2.34045317e-02,
2.40206929e-02, 2.46402870e-02, 2.52631194e-02, 2.58889958e-02,
2.65177217e-02, 2.71491027e-02, 2.77829443e-02, 2.84190521e-02,
2.90572316e-02, 2.96972884e-02, 3.03390281e-02, 3.09822563e-02,
3.16267784e-02, 3.22724001e-02, 3.29189269e-02, 3.35661644e-02,
3.42139181e-02, 3.48619937e-02, 3.55101966e-02, 3.61583324e-02,
3.68062067e-02, 3.74536251e-02, 3.81003931e-02, 3.87463163e-02,
3.93912001e-02, 4.00348503e-02, 4.06770724e-02, 4.13176719e-02,
4.19564543e-02, 4.25932253e-02, 4.32277904e-02, 4.38599552e-02,
4.44895252e-02, 4.51163060e-02, 4.57401031e-02, 4.63607222e-02,
4.69779687e-02])

xData = np.array([0, 0.0254, 0.0508, 0.0762, 0.1016, 0.127,  0.1524, 0.1778, 0.2032, 0.2286,
 0.254,  0.2794, 0.3048, 0.3302, 0.3556, 0.381,  0.4064, 0.4318, 0.4572, 0.4826,
 0.508,  0.5334, 0.5588, 0.5842, 0.6096, 0.635,  0.6604, 0.6858, 0.7112, 0.7366,
 0.762,  0.7874, 0.8128, 0.8382, 0.8636, 0.889,  0.9144, 0.9398, 0.9652, 0.9906,
 1.016,  1.0414, 1.0668, 1.0922, 1.1176, 1.143,  1.1684, 1.1938, 1.2192, 1.2446,
 1.27,  1.2954, 1.3208, 1.3462, 1.3716, 1.397,  1.4224, 1.4478, 1.4732, 1.4986,
 1.524,  1.5494, 1.5748, 1.6002, 1.6256, 1.651,  1.6764, 1.7018, 1.7272, 1.7526,
 1.778,  1.8034, 1.8288, 1.8542, 1.8796, 1.905,  1.9304, 1.9558, 1.9812, 2.0066,
 2.032,  2.0574, 2.0828, 2.1082, 2.1336, 2.159,  2.1844, 2.2098, 2.2352, 2.2606,
 2.286,  2.3114, 2.3368, 2.3622, 2.3876, 2.413,  2.4384, 2.4638, 2.4892, 2.5146,
 2.54])

plt.plot(x,y, color = 'blue', label = 'implicit solution')
plt.plot(xData,yData1, color = 'black', label = 'FE solutions')
plt.xlabel('horizontal position x (m)')
plt.ylabel('beam deformation u (m)')
plt.legend()
plt.grid()
```

```
plt.show()
```