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Embedding Factorizations

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EMBEDDING FACTORIZATIONS

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72 Pages

Let V be a set of n vertices for some $n \in \mathbb{N}$ and let E be a collection of h-subsets of V. Then $\mathcal{G} = (V, E)$ is an h-unifrom hypergraph and we refer to V as its vertex set and to E as its edge set. We say that $\mathcal{G}$ is complete and denote it by K_n^h if every h-subset of V is contained in E. If every edge in E is repeated λ times, we say G is λ-fold. Specifically, λK_n^h is the complete λ-fold n-vertex h-uniform hypergraph with an edge set containing λ copies of every h-subset of V. In this case, we denote the edge set by $E(\lambda K_n^h)$.

Let $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ for some $r_1, r_2, \ldots, r_k \in \mathbb{N}$. An $\mathbf{r}$-*factorization* of λK_n^h is a partition of $E(\lambda K_n^h)$ into subsets $F_1, \ldots, F_k$ such that all elements of V are included at least once in F_i and are included exactly r_i times in F_i for all $i \in \{1, \ldots, k\}$. Each such subset F_i is called an r_i-factor. A *partial* $\mathbf{r}$-*factorization* of λK_m^h is a partition of $E(\lambda K_m^h)$ into $F_1, \ldots, F_k$ such that each vertex in $V(\lambda K_m^h)$ is included at most r_i times in each color class F_i for $i \in \{1, \ldots, k\}$. Two vertices are adjacent in a hypergraph if some edge in the hypergraph contains both vertices. An r_i-factor F_i is connected if for any arbitrary pair of vertices $x, y \in V$, there is some sequence of vertices $x, w_1, w_2, \ldots, y$ with each consecutive pair adjacent in F_i. In this case, we say that F_i consists of only one component. If we assign some color i to every h-subset in $E(\lambda K_n^h)$ for $i \in \{1, \ldots, k\}$, we call this a k-coloring of λK_n^h. An $\mathbf{r}$-factorization of λK_n^h is a k-coloring of $E(\lambda K_n^h)$ such that edges of each color $i \in \{1, \ldots, k\}$ induce an r_i-factor.

Let $\mathbf{r} = (r_1, r_2, \ldots, r_q)$ and let $\mathbf{s} = (s_1, s_2, \ldots, s_k)$ where $r_i, s_j \in \mathbb{N}$ for all $i \in \{1, \ldots, q\}, j \in \{1, \ldots, k\}$. Motivated by an embedding problem of Peter Cameron and the work of many others, we show that for $n \geq hm$, the obvious necessary conditions that ensure that an $\mathbf{r}$-factorization of λK_m^h can be extended to an $\mathbf{s}$-factorization of λK_n^h are

also sufficient. For $n \geq hm$, we also establish the necessary and sufficient conditions under which an $\mathbf{r}$-factorization of λK_m^h can be extended to a connected $\mathbf{s}$-factorization of λK_n^h.

For $n \geq (h-1)(2m-1)$, we find necessary and sufficient conditions under which a partial $\mathbf{r}$-factorization of λK_m^h can be extended to an $\mathbf{r}$-factorization of λK_n^h in which each r_i-factor is connected. We also prove a similar result extending a given partition of *any* sub-hypergraph $\mathcal{G}$ of λK_m^h to a connected $\mathbf{r}$-factorization of λK_n^h.

KEYWORDS: r-factorizations, embedding, edge-coloring, hypergraphs, amalgamation, detachment

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CONTENTS

CHAPTER I: INTRODUCTION

I.1 Definitions and Notation

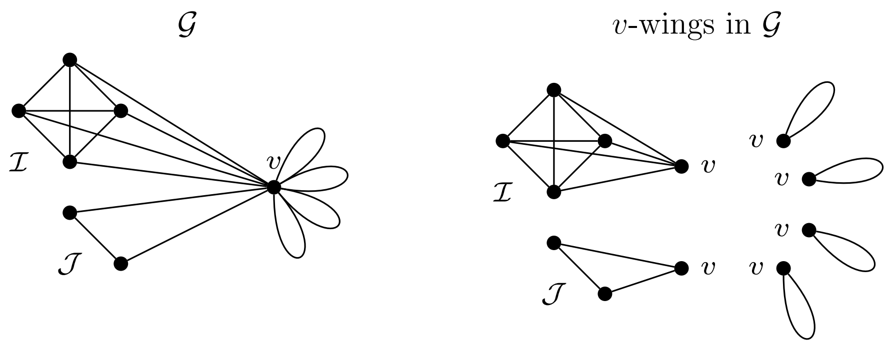
A hypergraph $\mathcal{G}$ is a vertex set $V(\mathcal{G})$ paired with an edge set $E(\mathcal{G})$ containing subsets of the vertex set. All hypergraphs in this paper are h-uniform; that is, $|e| = h$ for all $e \in E(\mathcal{G})$. We write $\mathcal{G} := K_n^h$ if $\mathcal{G}$ is the complete n-vertex h-uniform hypergraph whose edge set $E(\mathcal{G})$ is the collection of all h-subsets of the vertex set. The degree of a vertex $v \in V(\mathcal{G})$, denoted by $\deg(v)$, is the number of edges $e \in E(\mathcal{G})$ such that $v \in e$. In this thesis, we allow multiple edges. The *multiplicity* of an edge e, denoted by $\textsf{mult}(e)$, is the number of times e occurs in $E(\mathcal{G})$. In particular, in λK_m^h, each h-subset of $V(\lambda K_m^h)$ occurs λ times, so $\textsf{mult}(e) = \lambda$ for all edges $e \in E(\lambda K_m^h)$. The degree of a vertex $v \in V(\mathcal{G})$, denoted by $\textsf{deg}(v)$, is the number of occurrences of the vertex v in edges in $E(\mathcal{G})$. If the degree of every vertex in $V(\mathcal{G})$ is exactly r, then $\mathcal{G}$ is *r-regular*.

A *k-coloring* of $\mathcal{G}$ is a mapping $f : E(\mathcal{G}) \to [k]$ where $[k] := \{1, \dots, k\}$. We may also consider such a coloring as a partition of $\mathcal{G}$ into color classes $\mathcal{G}(j)$ each induced by edges with color j for $j \in [k]$. An *r-factor* is a spanning r-regular subhypergraph of $\mathcal{G}$; that is, a subhypergraph of $\mathcal{G}$ which spans all vertices in $V(\mathcal{G})$ and in which the degree of every vertex is r. For $\mathbf{r} = (r_1, \dots, r_k)$, an **r**-*factorization* of $\mathcal{G}$ is a k-coloring of $\mathcal{G}$ where for $j \in [k]$, $\mathcal{G}(j)$ induces an r_j-factor of $\mathcal{G}$ and a *partial* **r**-*factorization* of $\mathcal{G}$ is a k-coloring of $\mathcal{G}$ where for $j \in [k]$, $\mathcal{G}(j)$ induces a spanning subhypergraph of $\mathcal{G}$ in which the degree of each vertex is at most r_j. An (partial) r-factorization is an (partial) **r**-factorization with $\mathbf{r} = (r, \dots, r)$. A partial 1-factorization is often called a *proper coloring.*

Given a 1-factorization of K_n^h, if you think of the set V of vertices as the set of points, the set E of edges as the set of lines, and 1-factors as parallel classes, then for every point $v \in V$ and for each line ℓ in E, there is exactly another line ℓ' which is parallel to ℓ (that is, contained in the same parallel class as ℓ) and contains v. Hence, a 1-factorization is sometimes called a *parallelism.*

A vertex v in a connected hypergraph $\mathcal{G}$ is a *cut vertex* if there exist two non-trivial sub-hypergraphs I, J of $\mathcal{G}$ such that $I \cup J = \mathcal{G}$, $V(I \cap J) = \{v\}$, and $E(I \cap J) = \varnothing$. A sub-hypergraph W of a hypergraph $\mathcal{G}$ is an *v-wing* of $\mathcal{G}$ if (i) W is non-trivial and connected, (ii) v is not a cut vertex of W, and (iii) no edge in $E(\mathcal{G})\backslash E(W)$ is incident with a vertex in $V(W)\backslash\{v\}$. A v-wing W is *large* if $V(W) \neq \{v\}$, and is *small* if $V(W) = \{v\}$. Let $\omega_v(\mathcal{G})$, and $\omega_v^L(\mathcal{G})$ be the number of v-wings, and the number of large v-wings in $\mathcal{G}$, respectively. Let $c(\mathcal{G})$ denote the number of components of $\mathcal{G}$.

Example: Hypergraph $\mathcal{G}$ with cut vertex v, $\omega_v(\mathcal{G}) = 6$, and $\omega_v^L(\mathcal{G}) = 2$.



We say that a hypergraph $\mathcal{G}$ is *connected* if it consists of only one component. Equivalently, $\mathcal{G}$ is connected if for any arbitrary pair of vertices $x, y \in V(\mathcal{G})$, there is some sequence of vertices $x, w_1, w_2, \dots, y \in V(\mathcal{G})$ such that each consecutive pair of vertices is adjacent in $\mathcal{G}$. Note that a 1-factor in h-uniform $\mathcal{G}$ with more than h vertices is not connected. Thus, if an r_j-factor in $\mathcal{G}$ is connected (and $|V(\mathcal{G})| > h$), then we must have that $r_j \geq 2$. Moreover, if a component of a color class of λK_m^h is r_j-regular, then there is no way to extend it to a connected r_j-factor in λK_n^h.

I.2 Problem Statement and Motivation

In this thesis we find:

1. conditions that ensure an r-factorization of K_m^6 can be extended to an s-factorization of K_n^6

2. conditions that ensure an $\mathbf{r} := (r_1, \dots, r_q)$-factorization of λK_m^h can be extended to

an $\mathbf{s} := (s_1, \dots, s_k)$-factorization of λK_n^h,

3. conditions that ensure a partial $\mathbf{r} := (r_1, \dots, r_k)$-factorization of λK_m^h can be extended to an $\mathbf{r}$-factorization of λK_n^h, and

4. conditions that ensure a partial $\mathbf{r} := (r_1, \dots, r_k)$-factorization of any subhypergraph of λK_m^h can be extended to an $\mathbf{r}$-factorization of λK_n^h.

We are particularly interested to complete our extensions in such a way that each color class is connected and we identify the conditions under which this is possible.
The main source of interest in these problems dates back to an 18th century problem of Sylvester [12] which asked for a 1-factorization of K_n^h. Another source of interest is non-associative algebra and design theory. In 1945, Hall proved the following result.

Theorem I.2.1 (Hall, 1945, [18])**.** *Given a rectangle of $n-r$ rows and n columns such that each of the numbers $1, 2, \dots, n$ occurs once in every row and no number occurs twice in any column, there exist r rows which may be added to the given rectangle to form a Latin square.*

Nearly seventy years ago, Ryser generalized this result and found the necessary and sufficient conditions that ensure an $r \times s$ Latin rectangle can be embedded into an $n \times n$ Latin square.

Theorem I.2.2 (Ryser, 1951, [29] Theorem 2)**.** *Let T be an $r \times s$ Latin rectangle based upon the integers $1, 2, \dots, n$. Let $N(i)$ denote the number of times that the integer i occurs in T. A necessary and sufficient condition for T to be extended to an $n \times n$ Latin square is that for each $i \in \{1, 2, \dots, n\}$,*

$$N(i) \geq r + s - n.$$

In graph theoretic terms, this is equivalent to finding the necessary and sufficient conditions under which a proper edge-coloring of the complete bipartite graph $K_{r,s}$ can be extended to a a one-factorization of $K_{n,n}$.

In 1960, Evans proved the following further result.

Theorem I.2.3 (Evans, 1960, Theorem 2 [15]). *For any n, an incomplete $n \times n$ Latin square can be embedded in a $t \times t$ Latin square for any $t \geq 2n$.*

In graph theoretic terms, this is equivalent to extending a partial 1-factorization of $F \subseteq K_{n,n}$ using n colors to a 1-factorization of $K_{t,t}$, given $t \geq 2n$.
Cruse provided a symmetric analogue of Ryser's theorem by finding the necessary and sufficient conditions under which an $r \times r$ symmetric Latin rectangle can be embedded into an $n \times n$ symmetric Latin square.

Theorem I.2.4 (Cruse, 1972, [14] Theorem 1). *Let T be an $r \times r$ symmetric Latin rectangle based on the symbols $1, 2, \ldots, n$, where $n > r$. Denote by $N(i)$ the number of occurrences of the symbol i in T. In order for T to be extendible to an $n \times n$ symmetric Latin square based on the symbols $1, 2, \ldots, n$, it is both necessary and sufficient that*

1. *$N(i) \geq 2r - n$ for every symbol $i \in \{1, 2, \ldots, n\}$, and*

2. *$N(i) \equiv n \pmod 2$ for at least r of the symbols $i \in \{1, 2, \ldots, n\}$.*

In graph theoretic terms, this result is equivalent to finding conditions under which a proper edge-coloring of the complete graph $\mathbb{K}_r$ (this is the complete graph K_r with a loop on each vertex) can be extended to a one-factorization of $\mathbb{K}_n$.
In the 1970s, Baranyai resolved the problem posed by Sylvester in the 18th century, establishing two necessary and sufficient conditions under which K_n^h is r-factorable.

Theorem I.2.5 (Baranyai, 1973, [12] Corollary 2). *K_n^h is r-factorable if and only if $h \mid rn$ and $r \mid \binom{n-1}{h-1}$.*

Following the resolution of Sylvester's problem by Baranyai, Cameron asked the following question.

Problem 1 (Cameron, 1976, [13] Question 1.2). *Under what conditions can partial parallelisms be extended to parallelisms?*

Embedding structured factorizations has been studied by various authors. In the 1980's, Hilton introduced a technique called *amalgamation*, which has been very effective in solving a wide range of problems [1, 2, 3, 16, 20, 23, 27]. One example of this is an early result of Hilton in which he found conditions that ensure a partial 2-factorization of K_m can be extended to a connected 2-factorization of K_n [21].

Theorem I.2.6 (Hilton, 1982, [21] Theorem 2)**.** *Let $1 \leq r \leq 2n+1$. An edge-coloring of K_r with n colors $c_1, \ldots, c_n$ can be extended to a Hamiltonian decomposition of K_{2n+1} in which each color class of the edge-coloring of K_r is incorporated into a Hamiltonian circuit of K_{2n+1} if and only if each color class of the edge-coloring of K_r consists of at most $2n+1-r$ disjoint paths (counting a vertex of K_r with no edges of color c_i on it as a path (of length 0) of c_i).*

Hilton also used this technique in results pertaining to Latin squares in 1987 [22].
A third source of interest is Marcotte and Seymour's theorem (1990) [26] that establishes necessary and sufficient conditions for a proper k-coloring of a subgraph of a multiforest G to be extended to a proper k-coloring of G.
A proper coloring can be viewed as a partial 1-factorization or as an *almost 1-factorization* (in an *almost r-factor*, we allow the vertex degrees to be r or $r-1$).
A breakthrough in the direction of settling Cameron's question was a result of Häagkvist and Hellgren [17] in 1993 that settled the problem of embedding a 1-factorization of K_m^h into a 1-factorization of K_n^h, proving that a 1-factorization of K_m^h can be embedded into a 1-factorization of K_n^h if and only if $h \mid m$, $h \mid n$, and $n \geq 2m$.
Higher edge-connected analogues of Hilton's results have also been proven. For example, Rodger and Wantland proved an analogue of Hilton's result giving necessary and sufficient conditions for embedding a proper edge coloring of K_m into a 2-edge-connected k-factorization of K_n.

Theorem I.2.7 (Rodger and Wantland, 1995, [28] Theorem 3.2)**.** *Let $v < kn+1$. An*

n-edge-coloring of K_v can be embedded into a 2-edge-connected k-factorization of K_{kn+1} if and only if

(i) $\deg_{K_v(i)}(v) \leq k$ *for every* $v \in V(K_v(i))$,

(ii) *for every component C of K_v,* $\sum_{v\in V(C)} \deg_{K_v(i)}(v) \leq k|V(C)| - 2$,

(iii) *if C is a component of $K_v(i)$ and e is a cut edge of C with C_1 and C_2 being the components of $C - e$, then there exist $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$ with* $\deg_{K_v(i)}(v_1) < k$ *and* $\deg_{K_v(i)}(v_2) < k$,

(iv) *n is odd if k is odd, and*

(v) $kn + 1 \geq 2v - 2\epsilon/k$,

where $\epsilon = \min_{1\leq i\leq n} |E(H_i)|$ and H_i is the subgraph induced by the edges colored c_i.

Johnson proved a more general analogue of Hilton's result giving necessary and sufficient conditions for embedding a factorization of K_m into a $\mathbf{s} := (s_1, \ldots, s_k)$-factorization of K_n where each s_i factor is ℓ_i-connected given $\boldsymbol{\ell} := (\ell_1, \ldots, \ell_k)$.

Theorem I.2.8 (Johnson, 2003, [25] Theorem 8)**.** *Let n, t, K, and L satisfy the following conditions*

(A1) $\sum_{i=1}^{t} k_i = n - 1$,

(A2) *if n is odd, then each k_i is even,*

(A3) *for $1 \leq i \leq t$, $\ell_i \leq k_i$, and*

(A4) *if $n \geq 3$, then $\ell_i = 0$ if k_i=1.*

and let $\alpha = n - m$. A factorization $G_1, \ldots, G_t$ of K_m can be embedded into a factorization $F_1, \ldots, F_t$ of K_n in which, for $1 \leq i \leq t$, F_i is a k_i-regular ℓ_i-edge-connected graph if and only if the following conditions hold:

(I) *for* $1 \leq i \leq t$, $\deg_{G(i)}(v) \leq k_i$ *for each* $v \in V(K_m)$,

(II) *for* $1 \leq i \leq t$, $\alpha k_i \geq \epsilon_i$,

(III) *for* $1 \leq i \leq t$, *if* $\ell_i = 1$, *then* $\alpha \geq \dfrac{2\omega_i - \epsilon_i - 2}{k_i - 2}$,

(IV) *for* $1 \leq i \leq t$, *if* $\ell_i = k_i$, ℓ_i *is odd and* $\omega_i \geq 2$, *then* $\alpha \neq 2$, *and*

(V) *for* $1 \leq i \leq t$, H_i *is* ℓ_i*-edge-connected.*

In 2003, Hilton et al. [24] found necessary and sufficient conditions for a given coloring of K_t to be embedded into a connected k-factorization of K_m.

Theorem I.2.9 (Hilton et al., Theorem 3.1, [24])**.** *For* $1 \leq i \leq n$ *and* $1 \leq j \leq \omega_i$, *let* $G(i)$ *be the subgraph of* K_t *that is induced by the edges colored* i, *and let* $C_{i,1}, \dots, C_{i,\omega_i}$ *be the components of* $G(i)$. *If* $G(i)$ *has maximum degree at most* k, *let* $\epsilon_{i,j} = \sum_{v \in V(C_{i,j})}(k - \deg_{G(i)}(v))$ *and let* $\epsilon_i = \sum_{1 \leq j \leq \omega_i} \epsilon_{i,j}$. *Let* $k \geq 3$ *and* $\alpha = kn + 1 - t$. *An* n*-edge-colored* K_t, *with color classes* $G(1), G(2), \dots, G(n)$ *(where possibly some color classes contain no edges), can be embedded in an* n*-edge-colored* K_{kn+1} *in which each color class is a connected* k*-factor if and only if*

1. $\deg_{G(i)}(v) \leq k$ *for each* $v \in V(K_t)$ *and for* $1 \leq i \leq n$,

2. $\epsilon_{i,j} \geq 1$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq \omega_i$,

3. $\alpha \geq max\{\epsilon_i/k : 1 \leq i \leq n\}$,

4. $\alpha \geq max\{\frac{2\omega_i - \epsilon_i - 2}{k-2} : 1 \leq i \leq n\}$, *and*

5. *if* k *is odd, then* n *is odd.*

Hilton's amalgamation technique has recently been applied to hypergraphs by Bahmanian [8, 5, 11] and inspires the proofs of the results in this thesis. In fact, the following theorem concerning the reverse of amalgamation, called detachment, is foundational to the results in this thesis.

Theorem I.2.10 (Bahmanian, 2012 [5] Theorem 4.1)**.** *Let $\mathcal{F}$ be a k-edge-colored hypergraph and let $g : V(\mathcal{F}) \to \mathbb{N}$ be a simple function. Then there exists a simple g-detachment $\mathcal{G}$ (possibly with multiple edges) of $\mathcal{F}$ with amalgamation function $\Psi : V(\mathcal{G}) \to V(\mathcal{F})$, g being the number function associated with Ψ, such that:*

(A1) $\textbf{deg}_{\mathcal{G}}(v) \approx \textbf{deg}_{\mathcal{F}}(u)/g(u)$ *for each* $u \in V(\mathcal{F})$ *and each* $v \in \Psi^{-1}(u)$,

(A2) $\textbf{deg}_{\mathcal{G}(j)}(v) \approx \textbf{deg}_{\mathcal{F}(j)}(u)/g(u)$ *for each* $u \in V(\mathcal{F})$, *each* $v \in \Psi^{-1}(u)$ *and* $1 \leq j \leq k$,

(A3) $\textbf{mult}_{\mathcal{G}}(U_1, \dots, U_r) \approx \textbf{mult}_{\mathcal{F}}(u_1^{m_1}, \dots, u_r^{m_r})/\Pi_{i=1}^r(\binom{g(u_i)}{m_i})$ *for distinct* $u_1, \dots, u_r \in V(\mathcal{F})$ *and* $U_i \subset \Psi^{-1}(u_i)$ *with* $|U_i| = m_i \leq g(u_i)$ *for* $1 \leq i \leq r$, *and*

(A4) $\textbf{mult}_{\mathcal{G}(j)}(U_1, \dots, U_r) \approx \textbf{mult}_{\mathcal{F}(j)}(u_1^{m_1}, \dots, u_r^{m_r})/\Pi_{i=1}^r(\binom{g(u_i)}{m_i})$ *for distinct* $u_1, \dots, u_r \in V(\mathcal{F})$ *and* $U_i \subset \Psi^{-1}(u_i)$ *with* $|U_i| = m_i \leq g(u_i)$ *for* $1 \leq i \leq r$ *and* $1 \leq j \leq k$.

In 2016, Bahmanian and Newman completely settled the question 'Under what conditions can an r-factorization of K_m^3 be extended to an s-factorization of K_n^3?' with the following theorem.

Theorem I.2.11 (Bahmanian and Newman, 2016, [4] Theorem 16)**.** *An r-factorization of K_m^3 can be embedded into an s-factorization of K_n^3 if and only if*

(C1) $3 \mid rm$, $3 \mid sn$;

(C2) $r \mid \binom{m-1}{2}$, $s \mid \binom{n-1}{2}$;

(C3) $1 \leq s/r \leq \binom{n-1}{2} \Big/ \binom{m-1}{2}$;

(C4) $n \geq 3m/2$ *if* $1 < s/r < \binom{n-1}{2} \Big/ \binom{m-1}{2}$;

(C5) $n \geq 2m$ *if* $s = r$;

(C6) $sm\binom{n-m}{2} \geq \binom{n-1}{2}$ *if* $m(s-r)$ *is odd and* $s/r = \binom{n-1}{2} \Big/ \binom{m-1}{2}$.

Bahmanian and Newman also generalized Häagkvist and Hellgren's result by proving the following theorems.

Theorem I.2.12 (Bahmanian and Newman, 2017, [10] Theorem 1.7)**.** *Let n, m, h, and r be positive integers satisfying the following necessary conditions.*

$$n \geq 2m, \qquad h \mid rm, \qquad h \mid rn, \qquad r \mid \binom{m-1}{h-1}, \qquad r \mid \binom{n-1}{h-1}.$$

Additionally, assume that

$$\gcd(m, n, h) = \gcd(n, h).$$

Then there exists an r-factorization of K_n^h containing an embedded r-factorization of K_m^h.

Theorem I.2.13 (Bahmanian and Newman, 2017, [10] Theorem 1.8)**.** *Let n, m, h, r, and s be positive integers satisfying the following necessary conditions.*

$$h \mid rm, \qquad h \mid sn, \qquad r \mid \binom{m-1}{h-1}, \qquad s \mid \binom{n-1}{h-1}.$$

Additionally, assume that the following conditions hold, where $k = \gcd(m, n, h)$.

$$\gcd(m, n, h) = \gcd(n, h), \qquad n \geq 2m \qquad 1 \leq \frac{s}{r} \leq \frac{m}{k}\left[1 - \binom{m-k}{h} \Big/ \binom{m}{h}\right].$$

Then there exists an s-factorization of K_n^h containing an embedded r-factorization of K_m^h.

The results in chapter IV are generalizations of the following recent results with some added conditions pertaining to connectivity of factorizations.

Theorem I.2.14 (Bahmanian, 2018, [7] Theorem 4.1)**.** *For $n \geq 4.847323m$, any partial r-factorization of K_m^4 can be extended to an r-factorization of K_n^4 if and only if $4 \mid rn$ and $r \mid \binom{n-1}{3}$.*

Theorem I.2.15 (Bahmanian, 2018, [7] Theorem 5.1)**.** *For $n \geq 6.285214m$, any partial r-factorization of K_m^5 can be extended to an r-factorization of K_n^5 if and only if* $5 \mid rn$ *and* $r \mid \binom{n-1}{4}$.

A recent result along the same lines as Marcotte and Seymour's theorem from 1990 is that of Harrelson, McDonald, and Puelo [19] in 2018, which makes progress towards similar results for planar graphs that are not necessarily multiforests. Among other things, they establish the necessary and sufficient conditions for a proper k-coloring of a subgraph H of a planar graph G to be extended to a proper k-coloring of G when $k \geq \Delta(H) + \Delta(G) + 4$. Another recent result of Bahmanian is the following generalization of Baranyai's theorem, establishing necessary and sufficient conditions under which λK_n^h is $\mathbf{r}$-factorable into connected r_i factors.

Theorem I.2.16 (Bahmanian, 2019, [6] Theorem 1.2)**.** *Let $\mathbf{r} = (r_1, \ldots, r_k)$. Then λK_n^h is $\mathbf{r}$-factorable if and only if $h \mid r_i n$ for $1 \leq i \leq k$ and $\sum_{i=1}^k r_i = \lambda\binom{n-1}{h-1}$. Moreover, for $1 \leq i \leq k$, if $r_i \geq 2$, then we can guarantee that the r_i-factor is connected.*

Baranyai's theorem and this generalization identify the 'obvious' necessary conditions for the results in this thesis; that is, the necessary divisibility conditions for λK_n^h to be r-factorable or $\mathbf{r}$-factorable, respectively.

The result in chapter II mirrors the following results found in [9].

Theorem I.2.17 (Bahmanian and Haghshenas, 2019, [9] Theorem 1.2)**.** *For $n \geq 4m$, an r-factorization of K_m^4 can be extended to an s-factorization of K_n^4 if and only if the following conditions hold.*

$$4 \mid rm, \quad 4 \mid sn, \quad r \mid \binom{m-1}{3}, \quad s \mid \binom{n-1}{3}, \quad 1 \leq s/r \leq \binom{n-1}{3} \Big/ \binom{m-1}{3}.$$

Theorem I.2.18 (Bahmanian and Haghshenas, 2019, [9] Theorem 1.3)**.** *For $n \geq 5m$, an r-factorization of K_m^5 can be extended to an s-factorization of K_n^5 if and only if the* *following conditions hold.*

$$5 \mid rm, \quad 5 \mid sn, \quad r \mid \binom{m-1}{4}, \quad s \mid \binom{n-1}{4}, \quad 1 \leq s/r \leq \binom{n-1}{4} \Big/ \binom{m-1}{4}.$$

The following theorem is a generalization of these results with an added condition to guarantee connectivity.

Theorem I.2.19 (Bahmanian, Johnsen, Napirata, 2021+)**.** *For $n \geq hm$, an r-factorization of λK_m^h can be extended to a connected s-factorization of λK_n^h if and only if $s \geq r+1$ and*

$$h \mid rm, \quad h \mid sn, \quad r \mid \lambda\binom{m-1}{h-1}, \quad s \mid \lambda\binom{n-1}{h-1}, \quad 1 \leq \frac{s}{r} \leq \binom{n-1}{h-1} \Big/ \binom{m-1}{h-1}.$$

In chapter III, we further generalize this theorem and also identify the necessary conditions to guarantee connectivity of the resulting factorization.

In both chapters, the theorems pertaining to connectivity of r-factors found in [8] are also directly applied. In particular, we use the following result.

Theorem I.2.20 (Bahmanian, 2020, [8] Corollary 7.4)**.** *Let $\mathcal{G}$ be a partial r-factorization of λK_m^h and let $\mathcal{H}$ be the hypergraph obtained by adding a new vertex α and new edges to $\mathcal{G}$ so that*

$$\textbf{mult}(\alpha^i, X) = \lambda\binom{n-m}{i} \textit{for each } 1 \leq i \leq h, \textit{ and } X \subseteq V(\lambda K_m^h) \textit{ with } |X| = h-i.$$

A partial r-factorization of λK_m^h can be extended to a connected r-factorization of λK_n^h if and only if the new edges of $\mathcal{H}$ can be colored such that

$$\textbf{deg}_{H(i)}(v) = \begin{cases} r & \textit{if } v \neq \alpha, \\ r(n-m) & \textit{if } v = \alpha, \end{cases} \forall i \in [k], \tag{I.1}$$

and

$$\omega_\alpha(H(j)) \leq (r-1)(n-m)+1 \quad \forall i \in [k], \tag{I.2}$$

where $k := \lambda\binom{n-1}{h-1}\Big/ r \in \mathbb{N}$.

Here, the first condition guarantees that we can extend the partial r-factorization of λK_m^h to an r-factorization of λK_n^h and the second condition guarantees that each individual color class $\lambda K_n^h(j)$ will be connected. This condition is also useful when extending partial $\mathbf{r}$-factorizations to connected $\mathbf{r}$-factorizations, where $\mathbf{r} = (r_1, \dots, r_k)$.

I.3 Identities

The following combinatorial identities are used often in this thesis.

Lemma I.3.1. *For* $n \geq m \geq h$

$$\sum\nolimits_{i=0}^{h} \binom{m}{i}\binom{n-m}{h-i} = \binom{n}{h} \tag{I.3}$$

$$\sum\nolimits_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i} = m\left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right]. \tag{I.4}$$

Proof. First, we will show that

$$\sum\nolimits_{i=0}^{h} \binom{m}{i}\binom{n-m}{h-i} = \binom{n}{h}.$$

Suppose we have a set X of n vertices. We could consider two subsets of these vertices: one subset U including m vertices and one subset V including the remaining $n-m$ vertices. We count the number of ways to choose h-subsets from the set X. There are $\binom{n}{h}$ ways to choose an arbitrary h-subset of X.

These subsets may all be defined based upon the number of vertices from U contained in them. Each subset contains some i vertices from U, where $0 \leq i \leq h$. The remaining $h-i$

vertices in any subset are contained in V. Since there are $\binom{m}{i}\binom{n-m}{h-i}$ edges containing exactly i vertices from the subset U, it follows that there are $\sum_{i=0}^{h}\binom{m}{i}\binom{n-m}{h-i}$ total h-subsets of X.

Hence we conclude that

$$\sum_{i=0}^{h}\binom{m}{i}\binom{n-m}{h-i}=\binom{n}{h},$$

as desired.

Next, we will show that

$$\sum_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i}=m\left[\binom{n-1}{h-1}-\binom{m-1}{h-1}\right].$$

A proof of this second identity can be found in [9]. We include the proof here for the sake of completeness.

Let $\mathcal{F}$ be the hypergraph with $V(\mathcal{F})=\{u,v\}$ and edge set $E(\mathcal{F})=\{u^iv^{h-i}, 0\le i\le h-1\}$ such that $\mathsf{mult}(u^iv^{h-i})=\binom{m}{i}\binom{n-m}{h-i}$ for $0\le i\le h-1$. Each edge of the form u^iv^{h-i} adds i to the degree of u in $\mathcal{F}$. Thus, the degree of u in $\mathcal{F}$ is equal to $\sum_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i}$.

Let $u_1,\dots,u_m$ be the m vertices in K_m^h and let K_n^h have vertex set $u_1,\dots,u_m,v_1,\dots,v_{n-m}$. Then we may obtain the hypergraph $\mathcal{F}$ by identifying m vertices $u_1,\dots,u_m$ in $K_n^h\backslash K_m^h$ by the vertex u and the remaining $n-m$ vertices by the vertex v. The degree of any u_i in $K_n^h\backslash K_m^h$ is equal to $\binom{n-1}{h-1}-\binom{m-1}{h-1}$. Thus, the degree of u in $\mathcal{F}$ is equal to $m\left[\binom{n-1}{h-1}-\binom{m-1}{h-1}\right]$.

Therefore,

$$\sum_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i}=\mathsf{deg}_{\mathcal{F}}\, u=m\left[\binom{n-1}{h-1}-\binom{m-1}{h-1}\right],$$

as desired. □

CHAPTER II: EMBEDDING FACTORIZATIONS WHEN $h = 6$

Suppose that an r-factorization of K_m^6 is given. Each color class is an r-factor and within each r-factor, the number of edges (which is an integer) is equal to $mr/6$ (the number of vertices multiplied by the degree of each vertex divided by the number of vertices per edge). Thus, the existence of an r-factorization of K_m^6 implies that $6 \mid rm$. Moreover, since the degree of each vertex in K_m^6 is $\binom{m-1}{5}$ and the degree of each vertex in color class $K_m^6(j)$ is r, we must have that $r \mid \binom{m-1}{5}$. Thus, in order to extend the given r-factorization of K_m^6 to an s-factorization of K_n^6, the following conditions are necessary.

$$6 \mid rm, \qquad 6 \mid sn, \qquad r \mid \binom{m-1}{5}, \qquad s \mid \binom{n-1}{5}. \tag{II.1}$$

Let the number of colors in the r-factorization of K_m^6 and in the s-factorization of K_n^6 be q and k, respectively. Then we have the following additional necessary conditions.

$$r \leq s, \qquad q \leq k \tag{II.2}$$

In this chapter, a quintuple $(n, m, s, r, 6)$ is *admissible* if it satisfies conditions (II.1) and (II.2). We shall show that

Theorem II.0.1. *For $n \geq 6m$, an r-factorization of K_m^6 can be extended to an s-factorization of K_n^h if and only if $(n, m, s, r, 6)$ is admissible.*

Throughout the rest of this chapter, we shall assume that $(n, m, s, r, 6)$ is admissible, the number of colors in the given r-factorization of K_m^6 is q, the number of colors in the s-factorization of K_n^6 is k, and

$$\kappa_1 := \{1, \ldots, q\}, \qquad \kappa_2 := \{q+1, \ldots, k\}, \qquad \kappa := \kappa_1 \cup \kappa_2.$$

Let $\mathcal{G}$ be the hypergraph $K_n^6 \backslash K_m^6$. We refer to the m vertices in $V(\mathcal{G}) \cap K_m^6$ as the old vertices in $\mathcal{G}$ and we refer to the remaining $n-m$ vertices in $V(\mathcal{G}) \backslash K_m^6$ as the new vertices in $\mathcal{G}$. We shall reduce the problem of extending an r-factorization of K_m^6 to an s-factorization of K_n^6 to the problem of coloring a 2-vertex hypergraph $\mathcal{F}$ with $V(\mathcal{F}) = \{u, v\}$. Before describing $E(\mathcal{F})$, we need to introduce some more notation. An *edge of the form* $u^i v^j$ (or a $u^i v^j$*-edge*) is an edge in which vertex u occurs i times and vertex v occurs j times. When we color the edges of $\mathcal{F}$, we use $\mathbf{deg}_{\mathcal{F}(j)}(v)$ and $\mathbf{mult}_{\mathcal{F}(j)}(e)$ to denote the degree of v and the multiplicity of e, respectively, in color class $\mathcal{F}(j)$. The following describes the edge set of $\mathcal{F}$.

$$\mathbf{mult}(u^i v^{6-i}) = \binom{m}{i}\binom{n-m}{6-i} \quad \text{for } 0 \leq i \leq 5. \tag{II.3}$$

In fact, $\mathcal{F}$ is obtained by identifying all the m old vertices of $\mathcal{G}$ with a vertex u, and identifying all the remaining $n-m$ new vertices with a vertex v. We say that $\mathcal{G}$ is a *detachment* of $\mathcal{F}$, and $\mathcal{F}$ is an *amalgamation* of $\mathcal{G}$.

In order to extend the r-factorization of K_m^6 to an s-factorization of K_n^6, we need to color $\mathcal{G}$ with k colors such that each color class of K_n^6 with edge set $E(\mathcal{G}) \cup E(K_m^6)$ induces an s-factor for color j. In each r factor in the given r-factorization of K_m^6, we have that the degree of each vertex is r. Thus, coloring in such a way would require the degree of the m old vertices in $V(\mathcal{G})$ to be $s-r$ in $\mathcal{G}(j)$ for all $j \in \kappa_1$, the degree of the m old vertices to be s in $\mathcal{G}(j)$ for all $j \in \kappa_2$, and the degree of the $n-m$ new vertices in $V(\mathcal{G})$ to be s in $\mathcal{G}(j)$ for all $j \in \kappa$. If we can obtain such a coloring, then in the amalgamation $\mathcal{F}$ of $\mathcal{G}$, $\mathbf{deg}_{\mathcal{F}(j)}(u) = m(s-r)$ for $j \in \kappa_1$, $\mathbf{deg}_{\mathcal{F}(j)}(u) = sm$ for $j \in \kappa_2$, and $\mathbf{deg}_{\mathcal{F}(j)}(v) = s(n-m)$ for

$j \in \kappa$. More importantly, by the following lemma, which is an immediate consequence of a result of Bahmanian (see [5, Theorem 4.1]), the converse of the previous statement is also true.

Lemma II.0.2. *An r-factorization of K_m^6 can be extended to an s-factorization of K_n^6 provided the hypergraph $\mathcal{F}$ described above can be colored so that*

$$\deg_{\mathcal{F}(j)}(x) = \begin{cases} m(s-r) & \text{if } x = u, j \in \kappa_1, \\ sm & \text{if } x = u, j \in \kappa_2, \\ s(n-m) & \text{if } x = v, j \in \kappa. \end{cases} \tag{II.4}$$

II.1 Proof of Theorem II.0.1

Suppose $(n, m, s, r, 6)$ is admissible. We proceed by coloring the six types of edges in $\mathcal{F}$ to meet the degree conditions in Lemma II.0.2.

Lemma II.1.1. *We can color the edges of the form u^5v such that*

$$5\,\mathrm{mult}_{\mathcal{F}(j)}(u^5v) \leq \begin{cases} (s-r)m & \text{for } j \in \kappa_1; \\ sm & \text{for } j \in \kappa_2. \end{cases} \tag{II.5}$$

Proof. To show that this is possible, we need to show that

$$\mathrm{mult}(u^5v) \leq q \left\lfloor \frac{(s-r)m}{5} \right\rfloor + (k-q) \left\lfloor \frac{sm}{5} \right\rfloor.$$

We have

$$
\begin{aligned}
5q\left\lfloor\frac{(s-r)m}{5}\right\rfloor+5(k-q)\left\lfloor\frac{sm}{5}\right\rfloor &\geq q((s-r)m-4)+(k-q)(sm-4) \\
&= ksm-qrm-4k \\
&\geq ksm-qrm-4ks \\
&= m\left[\binom{n-1}{5}-\binom{m-1}{5}\right]-4\binom{n-1}{5}.
\end{aligned}
$$

Thus, it suffices to show that

$$
m\left[\binom{n-1}{5}-\binom{m-1}{5}\right]-4\binom{n-1}{5}-5\binom{m}{5}(n-m)\geq 0.
$$

We have

$$
\begin{aligned}
&m\left[\binom{n-1}{5}-\binom{m-1}{5}\right]-4\binom{n-1}{5}-5\binom{m}{5}(n-m)\\
&\quad=m\binom{n-m}{5}+2\binom{m}{2}\binom{n-m}{4}+3\binom{m}{3}\binom{n-m}{3}+4\binom{m}{4}\binom{n-m}{2}\\
&\qquad+5\binom{m}{5}(n-m)-5\binom{m}{5}(n-m)-4\binom{n-1}{5}\\
&\quad=m\binom{n-m}{5}+2\binom{m}{2}\binom{n-m}{4}+3\binom{m}{3}\binom{n-m}{3}+4\binom{m}{4}\binom{n-m}{2}\\
&\qquad-4\left[\binom{m-1}{5}+\binom{m-1}{4}(n-m)+\binom{m-1}{3}\binom{n-m}{2}\right]\\
&\qquad-4\left[\binom{m-1}{2}\binom{n-m}{3}+(m-1)\binom{n-m}{4}+\binom{n-m}{5}\right]\\
&\quad=(m-4)\binom{m-1}{3}\binom{n-m}{2}-4\binom{m-1}{5}-4\binom{m-1}{4}(n-m)\\
&\qquad+(m-4)\left[\binom{n-m}{5}+(m-1)\binom{n-m}{4}+\binom{m-1}{2}\binom{n-m}{3}\right]\\
&\quad\geq(m-4)\binom{m-1}{3}\left[\binom{n-m}{2}-\frac{m-5}{5}-(n-m)\right]\\
&\quad\geq\binom{n-m}{2}-\frac{m-5}{5}-(n-m)\\
&\quad\geq\binom{n-m}{2}-\frac{1}{5}(m-5)(n-m)-(n-m)\\
&\quad=(n-m)\left[\frac{1}{2}(n-m-1)-\frac{1}{5}(m-5)-1\right]\\
&\quad\geq\frac{1}{2}n-\frac{7}{10}m-\frac{1}{2}\geq\frac{23}{10}m-\frac{1}{2}\geq 0.
\end{aligned}
$$

□

Lemma II.1.2. *We can color the edges of the form* u^4v^2 *such that*

$$
4\,\mathbf{mult}_{\mathcal{F}(j)}(u^4v^2)\leq\begin{cases}m(s-r)-5\,\mathbf{mult}_{\mathcal{F}(j)}(u^5v) & \text{for } j\in\kappa_1;\\ sm-5\,\mathbf{mult}_{\mathcal{F}(j)}(u^5v) & \text{for } j\in\kappa_2.\end{cases}\tag{II.6}
$$

Proof. Based upon our coloring of the first type of edges (see II.5), the right-hand sides of the inequalities in II.6 are nonnegative. Thus, to show that this is possible, we need to show that

$$\mathrm{mult}(u^4v^2) \le \sum_{j\in\kappa_1} \left\lfloor \frac{m(s-r) - 5\,\mathrm{mult}_{\mathcal{F}(j)}(u^5v)}{4} \right\rfloor + \sum_{j\in\kappa_2} \left\lfloor \frac{sm - 5\,\mathrm{mult}_{\mathcal{F}(j)}(u^5v)}{4} \right\rfloor.$$

We have

$$\begin{aligned}
&4\sum_{j\in\kappa_1} \left\lfloor \frac{m(s-r) - 5\,\mathrm{mult}_{\mathcal{F}(j)}(u^5v)}{4} \right\rfloor + 4\sum_{j\in\kappa_2} \left\lfloor \frac{sm - 5\,\mathrm{mult}_{\mathcal{F}(j)}(u^5v)}{4} \right\rfloor \\
&\quad \ge \sum_{j\in\kappa_1} (m(s-r) - 5\,\mathrm{mult}_{\mathcal{F}(j)}(u^5v) - 3) + \sum_{j\in\kappa_2} (sm - 5\,\mathrm{mult}_{\mathcal{F}(j)}(u^5v) - 3) \\
&\quad = q(m(s-r)-3) + (k-q)(sm-3) - 5\,\mathrm{mult}(u^5v) \\
&\quad = ksm - qrm - 5\binom{m}{5}(n-m) - 3k \\
&\quad \ge ksm - qrm - 5\binom{m}{5}(n-m) - 3ks \\
&\quad = m\left[\binom{n-1}{5} - \binom{m-1}{5}\right] - 5\binom{m}{5}(n-m) - 3\binom{n-1}{5}.
\end{aligned}$$

Thus, it suffices to show that

$$m\left[\binom{n-1}{5} - \binom{m-1}{5}\right] - 5\binom{m}{5}(n-m) - 3\binom{n-1}{5} - 4\binom{m}{4}\binom{n-m}{2} \ge 0.$$

We have

$$
\begin{aligned}
&m\left[\binom{n-1}{5}-\binom{m-1}{5}\right]-5\binom{m}{5}(n-m)-3\binom{n-1}{5}-4\binom{m}{4}\binom{n-m}{2} \\
&=m\binom{n-m}{5}+2\binom{m}{2}\binom{n-m}{4}+3\binom{m}{3}\binom{n-m}{3}-3\binom{n-1}{5} \\
&=m\binom{n-m}{5}+m(m-1)\binom{n-m}{4}+m\binom{m-1}{2}\binom{n-m}{3} \\
&\quad-3\left[\binom{m-1}{5}+\binom{m-1}{4}(n-m)+\binom{m-1}{3}\binom{n-m}{2}\right] \\
&\quad-3\left[\binom{m-1}{2}\binom{n-m}{3}+(m-1)\binom{n-m}{4}+\binom{n-m}{5}\right] \\
&=(m-3)\left[\binom{m-1}{2}\binom{n-m}{3}+(m-1)\binom{n-m}{4}+\binom{n-m}{5}\right] \\
&\quad-(m-3)\left[\frac{(m-4)(m-5)}{20}\binom{m-1}{2}+\frac{m-4}{4}\binom{m-1}{2}(n-m)+\binom{m-1}{2}\binom{n-m}{2}\right] \\
&\geq\binom{m-1}{2}\left[\binom{n-m}{3}-\frac{(m-4)(m-5)}{20}-\frac{m-4}{4}(n-m)-\binom{n-m}{2}\right] \\
&\geq\binom{n-m}{3}-\frac{(m-4)(m-5)}{20}-\frac{m-4}{4}(n-m)-\binom{n-m}{2} \\
&\geq(n-m)\left[\frac{1}{3}\binom{n-m-1}{2}-\frac{(m-4)(m-5)}{20}-\frac{m-4}{4}-\frac{1}{2}(n-m-1)\right] \\
&\geq\frac{1}{6}(n-m-1)(n-m-2)-\frac{(m-4)(m-5)}{20}-\frac{m-4}{4}-\frac{1}{2}(n-m-1) \\
&=\frac{1}{6}(n-m-1)(n-m-5)-\frac{(m-4)m}{20} \\
&\geq\frac{1}{6}(5m-1)(5m-5)-\frac{1}{20}(m-4)m \\
&=\frac{1}{60}\left(50(5m-1)(m-1)-3m^2+12m\right) \\
&=\frac{1}{60}\left(247m^2-288m+50\right) \\
&\geq 0 \text{ when } m=6.
\end{aligned}
$$

The following completes the proof.

$$
\frac{d}{dm}\left(247m^2-288m+50\right)=494m-288>0 \text{ when } m>6.
$$

Thus $247m^2 - 288m + 50$ is increasing and positive for $m \geq 6$, so we conclude $247m^2 - 288m + 50 \geq 0$. $\square$

Lemma II.1.3. *We can color the edges of the form u^3v^3 such that*

$$3\,\mathbf{mult}_{\mathcal{F}(j)}(u^3v^3) \leq \begin{cases} m(s-r) - 4\,\mathbf{mult}_{\mathcal{F}(j)}(u^4v^2) - 5\,\mathbf{mult}_{\mathcal{F}(j)}(u^5v) & \text{for } j \in \kappa_1; \\ sm - 4\,\mathbf{mult}_{\mathcal{F}(j)}(u^4v^2) - 5\,\mathbf{mult}_{\mathcal{F}(j)}(u^5v) & \text{for } j \in \kappa_2. \end{cases} \tag{II.7}$$

Proof. Based upon our coloring of the first two types of edges (see II.6), the right-hand sides of the inequalities in II.7 are nonnegative. Thus, to show that this is possible, we need to show that

$$\begin{aligned} \mathbf{mult}(u^3v^3) \leq & \sum\nolimits_{j\in\kappa_1} \left\lfloor \frac{m(s-r) - 5\,\mathbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\,\mathbf{mult}_{\mathcal{F}(j)}(u^4v^2)}{3} \right\rfloor \\ & + \sum\nolimits_{j\in\kappa_2} \left\lfloor \frac{sm - 5\,\mathbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\,\mathbf{mult}_{\mathcal{F}(j)}(u^4v^2)}{3} \right\rfloor. \end{aligned}$$

We have

$$
\begin{aligned}
3\sum\nolimits_{j\in\kappa_1} & \left\lfloor \frac{m(s-r) - 5\,\textbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\,\textbf{mult}_{\mathcal{F}(j)}(u^4v^2)}{3} \right\rfloor \\
&\quad + 3\sum\nolimits_{j\in\kappa_2} \left\lfloor \frac{sm - 5\,\textbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\,\textbf{mult}_{\mathcal{F}(j)}(u^4v^2)}{3} \right\rfloor \\
&\geq \sum\nolimits_{j\in\kappa_1} (m(s-r) - 5\,\textbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\,\textbf{mult}_{\mathcal{F}(j)}(u^4v^2) - 2) \\
&\quad + \sum\nolimits_{j\in\kappa_2} (sm - 5\,\textbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\,\textbf{mult}_{\mathcal{F}(j)}(u^4v^2) - 2) \\
&= q(m(s-r)-2) + (k-q)(sm-2) - 5\,\textbf{mult}(u^5v) - 4\,\textbf{mult}(u^4v^2) \\
&= ksm - qrm - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} - 2k \\
&\geq ksm - qrm - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} - 2ks \\
&= m\left[\binom{n-1}{5} - \binom{m-1}{5}\right] - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} - 2\binom{n-1}{5}.
\end{aligned}
$$

Thus, it suffices to show that

$$
\begin{aligned}
m\left[\binom{n-1}{5} - \binom{m-1}{5}\right] - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} \\
- 2\binom{n-1}{5} - 3\binom{m}{3}\binom{n-m}{3} \geq 0.
\end{aligned}
$$

We have

$$
\begin{aligned}
&m\left[\binom{n-1}{5}-\binom{m-1}{5}\right]-5(n-m)\binom{m}{5}-4\binom{n-m}{2}\binom{m}{4}\\
&\quad-2\binom{n-1}{5}-3\binom{m}{3}\binom{n-m}{3}\\
&=m\binom{n-m}{5}+2\binom{m}{2}\binom{n-m}{4}-2\binom{n-1}{5}\\
&=m\binom{n-m}{5}+2\binom{m}{2}\binom{n-m}{4}-2\left[\binom{m-1}{5}+\binom{m-1}{4}(n-m)\right]\\
&\quad-2\left[\binom{m-1}{3}\binom{n-m}{2}+\binom{m-1}{2}\binom{n-m}{3}+(m-1)\binom{n-m}{4}+\binom{n-m}{5}\right]\\
&\geq(m-2)(m-1)(n-m)\frac{1}{4}\binom{n-m-1}{3}-2\binom{m-1}{2}\left[\frac{1}{60}(m-3)(m-4)(m-5)\right]\\
&\quad-2\binom{m-1}{2}(n-m)\left[\frac{1}{12}(m-3)(m-4)+\frac{1}{6}(m-3)(n-m-1)+\frac{1}{3}\binom{n-m-1}{2}\right]\\
&\geq\frac{1}{4}\binom{n-m-1}{3}-\frac{1}{60}(m-3)(m-4)(m-5)-\frac{1}{12}(m-3)(m-4)\\
&\quad-\frac{1}{6}(m-3)(n-m-1)-\frac{1}{3}\binom{n-m-1}{2}\\
&=\frac{1}{12}(n-m-1)\left(\frac{1}{2}(n-m-2)(n-m-3)-2(m-3)-2(n-m-2)\right)\\
&\quad-\frac{1}{60}(m-3)(m-4)m\\
&=\frac{1}{12}(n-m-1)\left(\frac{1}{2}(n-m-2)(n-m-7)-2(m-3)\right)-\frac{1}{60}(m-3)(m-4)m\\
&\geq\frac{1}{12}(5m-1)\left(\frac{1}{2}(5m-2)(5m-7)-2(m-3)\right)-\frac{1}{60}(m-3)(m-4)m\\
&=\frac{5m-7}{12}\binom{5m-1}{2}-\frac{1}{60}(m-3)\left(10(5m-1)+m(m-4)\right)\\
&=\frac{5m-7}{12}\binom{5m-1}{2}-\frac{1}{60}(m-3)(m^2+46m-10)\\
&=\frac{1}{120}(623m^3-1336m^2+871m-130)\\
&=\frac{1}{120}\left(m^2(623m-1336)+871m-130\right)\\
&\quad\geq 0.
\end{aligned}
$$

$\square$

Lemma II.1.4. *We can color the edges of the form u^2v^4 such that*

$$2\,\mathbf{mult}_{\mathcal{F}(j)}(u^2v^4) \leq \begin{cases} 2\rho_{4,j} & \text{for } j \in \kappa_1; \\ 2\sigma_{4,j} & \text{for } j \in \kappa_2, \end{cases} \tag{II.8}$$

where

$$\rho_{4,j} = \frac{(s-r)m}{2} - \sum_{\ell=1}^{3} \frac{6-\ell}{2}\,\mathbf{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell})$$
$$\sigma_{4,j} = \frac{sm}{2} - \sum_{\ell=1}^{3} \frac{6-\ell}{2}\,\mathbf{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell})$$

Proof. Based upon our coloring of the first three types of edges (see II.7), the right-hand sides of the inequalities in II.8 are nonnegative. Thus, to show that this is possible, we need to show that

$$\mathbf{mult}(u^2v^4) \leq \sum\nolimits_{j\in\kappa_1} \lfloor \rho_{4,j} \rfloor + \sum\nolimits_{j\in\kappa_2} \lfloor \sigma_{4,j} \rfloor .$$

We have

$$
\begin{aligned}
&2\sum\nolimits_{j\in\kappa_1}\lfloor\rho_{4,j}\rfloor + 3\sum\nolimits_{j\in\kappa_2}\lfloor\sigma_{4,j}\rfloor \\
&\quad\geq \sum\nolimits_{j\in\kappa_1}(m(s-r) - 5\,\textbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\,\textbf{mult}_{\mathcal{F}(j)}(u^4v^2) - 3\,\textbf{mult}_{\mathcal{F}(j)}(u^3v^3) - 1) \\
&\qquad + \sum\nolimits_{j\in\kappa_2}(sm - 5\,\textbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\,\textbf{mult}_{\mathcal{F}(j)}(u^4v^2) - 3\,\textbf{mult}_{\mathcal{F}(j)}(u^3v^3) - 1) \\
&\quad= q(m(s-r)-1) + (k-q)(sm-1) - 5\,\textbf{mult}(u^5v) - 4\,\textbf{mult}(u^4v^2) - 3\,\textbf{mult}_{\mathcal{F}(j)}(u^3v^3) \\
&\quad= ksm - qrm - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} - 3\,\textbf{mult}_{\mathcal{F}(j)}(u^3v^3) - k \\
&\quad\geq ksm - qrm - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} - 3\,\textbf{mult}_{\mathcal{F}(j)}(u^3v^3) - ks \\
&\quad= m\left[\binom{n-1}{5} - \binom{m-1}{5}\right] - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} - 3\binom{m}{3}\binom{n-m}{3} \\
&\qquad - \binom{n-1}{5}.
\end{aligned}
$$

Thus, it suffices to show that

$$
\begin{aligned}
&m\left[\binom{n-1}{5} - \binom{m-1}{5}\right] - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} - 3\binom{m}{3}\binom{n-m}{3} \\
&\qquad - \binom{n-1}{5} - 2\binom{m}{2}\binom{n-m}{4} \geq 0
\end{aligned}
$$

We have

$$
\begin{aligned}
&m\left[\binom{n-1}{5}-\binom{m-1}{5}\right]-5\binom{m}{5}(n-m)-4\binom{m}{4}\binom{n-m}{2}-3\binom{m}{3}\binom{n-m}{3}\\
&\quad-\binom{n-1}{5}-2\binom{m}{2}\binom{n-m}{4}\\
&=m\binom{n-m}{5}-\binom{n-1}{5}\\
&=m\binom{n-m}{5}-\binom{n-m}{5}-(m-1)\binom{n-m}{4}-\binom{m-1}{2}\binom{n-m}{3}\\
&\quad-\binom{m-1}{3}\binom{n-m}{2}-\binom{m-1}{4}(n-m)-\binom{m-1}{5}\\
&\geq(m-1)(n-m)\left[\frac{1}{5}\binom{n-m-1}{4}-\frac{1}{4}\binom{n-m-1}{3}-\frac{1}{6}\binom{n-m-1}{2}(m-2)\right]\\
&\quad+(m-1)(n-m)\left[-\frac{1}{6}(n-m-1)\binom{m-2}{2}-\frac{1}{4}\binom{m-2}{3}-\frac{1}{5}\binom{m-2}{4}\right]\\
&\geq\frac{1}{5}\binom{n-m-1}{4}-\frac{1}{4}\binom{n-m-1}{3}-\frac{1}{6}\binom{n-m-1}{2}(m-2)\\
&\quad-\frac{1}{6}(n-m-1)\binom{m-2}{2}-\frac{1}{4}\binom{m-2}{3}-\frac{1}{5}\binom{m-2}{4}\\
&\quad=\frac{1}{120}\left(n^3(n-4m-15)+5n(17n-45+6m^2n+35mn-4m^3-35m^2-90m)\right)\\
&\qquad+\frac{1}{120}\left(m^3(5m-46)+m(344m-591)+754\right)\\
&\quad\geq\frac{1}{120}\left((6m)^3(2m-15)+30m(102m-45+36m^3+210m^2-4m^3-35m^2-90m)\right)\\
&\qquad+\frac{1}{120}\left(m^3(5m-46)+m(344m-591)+754\right)\\
&\quad=\frac{1}{120}(1397m^4+1964m^3+704m^2-1941m+754)\\
&\quad\geq 0
\end{aligned}
$$

□

Lemma II.1.5. *We can color the edges of the form* uv^5 *such that*

$$
\mathbf{mult}_{\mathcal{F}(j)}(uv^5)=\begin{cases}\rho_{5,j} & \textit{for } j\in\kappa_1,\\ \sigma_{5,j} & \textit{for } j\in\kappa_2,\end{cases}\tag{II.9}
$$

where

$$\rho_{5,j} = (s-r)m - \sum_{\ell=1}^{4}(6-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell})$$
$$\sigma_{5,j} = sm - \sum_{\ell=1}^{4}(6-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell})$$

Proof. Based upon our coloring of the first four types of edges (see II.8), the right-hand sides of the inequalities in II.9 are nonnegative. Thus, the following proves that this coloring is possible.

$$\begin{aligned}
\sum\nolimits_{j\in\kappa} \mathbf{mult}_{\mathcal{F}(j)}(uv^5) &= \sum\nolimits_{j\in\kappa_1} \mathbf{mult}_{\mathcal{F}(j)}(uv^5) + \sum\nolimits_{j\in\kappa_2} \mathbf{mult}_{\mathcal{F}(j)}(uv^5) \\
&= qm(s-r) + (k-q)sm - 5\sum\nolimits_{j\in\kappa} \mathbf{mult}_{\mathcal{F}(j)}(u^5v) - 4\sum\nolimits_{j\in\kappa} \mathbf{mult}_{\mathcal{F}(j)}(u^4v^2) \\
&\quad - 3\sum\nolimits_{j\in\kappa} \mathbf{mult}_{\mathcal{F}(j)}(u^3v^3) - 2\sum\nolimits_{j\in\kappa} \mathbf{mult}_{\mathcal{F}(j)}(u^2v^4) \\
&= ksm - qrm - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} - 3\binom{m}{3}\binom{n-m}{3} \\
&\quad - 2\binom{m}{2}\binom{n-m}{4} \\
&= m\left[\binom{n-1}{5} - \binom{m-1}{5}\right] - 5\binom{m}{5}(n-m) - 4\binom{m}{4}\binom{n-m}{2} \\
&\quad - 3\binom{m}{3}\binom{n-m}{3} - 2\binom{m}{2}\binom{n-m}{4} \\
&= m\binom{n-m}{5} = \mathbf{mult}(uv^5).
\end{aligned}$$

□

Lemma II.1.6. *We can color the edges of the form v^6 such that*

$$\mathbf{mult}_{\mathcal{F}(j)}(v^6) = \begin{cases} s\left(\dfrac{n}{6} - m\right) + \sum_{\ell\in[4]}(5-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^\ell) + rm\left(\dfrac{5}{6}\right) & \textit{for } j \in \kappa_1, \\ s\left(\dfrac{n}{6} - m\right) + \sum_{\ell\in[4]}(5-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^\ell) & \textit{for } j \in \kappa_2. \end{cases} \tag{II.10}$$

Proof. Since $(n, m, s, r, 6)$ is admissible, $\mathbf{mult}_{\mathcal{F}(j)}(v^6)$ is an integer and since $n \geq 6m$, $\mathbf{mult}_{\mathcal{F}(j)}(v^6) \geq 0$ for all $j \in \kappa$. Thus, the following completes the proof.

$$\begin{aligned} \sum_{j\in\kappa} \mathbf{mult}_{\mathcal{F}(j)}(v^6) &= \sum_{j\in\kappa_1} \mathbf{mult}_{\mathcal{F}(j)}(v^6) + \sum_{j\in\kappa_2} \mathbf{mult}_{\mathcal{F}(j)}(v^6) \\ &= q(s(n/6-m) + rm(5/6)) + (k-q)(s(n/6-m)) \\ &\quad + \sum_{\ell\in[4]}(5-\ell)\,\mathbf{mult}(u^{6-\ell}v^\ell) \\ &= ksn/6 - ksm + 5qrm/6 + \sum_{\ell\in[4]}(5-\ell)\,\mathbf{mult}(u^{6-\ell}v^\ell) \\ &= \binom{n}{6} - m\binom{n-1}{5} + 5\binom{m}{6} + 4\binom{m}{5}(n-m) + 3\binom{m}{4}\binom{n-m}{2} \\ &\quad + 2\binom{m}{3}\binom{n-m}{3} + \binom{m}{2}\binom{n-m}{4} \\ &= \binom{n}{6} - m\binom{n-1}{5} + 5\binom{m}{6} + m\left[\binom{n-1}{5} - \binom{m-1}{5}\right] - \binom{n}{6} \\ &\quad + \binom{n-m}{6} + \binom{m}{6} \\ &= \binom{n-m}{6} = \mathbf{mult}(v^6). \end{aligned}$$

$\square$

Lemma II.1.7. *Coloring according to Lemmas II.1.1–II.1.6 satisfies the degree conditions stated in lemma II.0.2.*

Proof. For $j \in \kappa$, we have

$$
\begin{aligned}
\deg_{\mathcal{F}(j)}(u) &= \sum\nolimits_{\ell\in[5]}(6-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell})\\
&= \sum\nolimits_{\ell\in[4]}(6-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell}) + \mathrm{mult}_{\mathcal{F}(j)}(uv^5)\\
&= \sum\nolimits_{\ell\in[4]}(6-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell}) + \begin{cases}\rho_{5,j} & \text{for } j\in\kappa_1\\ \sigma_{5,j} & \text{for } j\in\kappa_2\end{cases}\\
&= \begin{cases} m(s-r) & \text{for } j\in\kappa_1\\ sm & \text{for } j\in\kappa_2\end{cases},
\end{aligned}
$$

and

$$
\begin{aligned}
\deg_{\mathcal{F}(j)}(v) &= \sum\nolimits_{\ell\in[6]}\ell\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell})\\
&= \sum\nolimits_{\ell\in[4]}\ell\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell}) + 5\,\mathrm{mult}_{\mathcal{F}(j)}(uv^5) + 6\,\mathrm{mult}_{\mathcal{F}(j)}(v^6)\\
&= \sum\nolimits_{\ell\in[4]}\ell\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell}) + 6\sum\nolimits_{\ell\in[4]}(5-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell})\\
&\quad + 6s\left(\frac{n}{6}-m\right) + \begin{cases}5\rho_{5,j} + 6rm\left(\dfrac{5}{6}\right) & \text{for } j\in\kappa_1,\\ 5\sigma_{5,j} & \text{for } j\in\kappa_2.\end{cases}\\
&= \sum\nolimits_{\ell\in[4]}\ell\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell}) + 6\sum\nolimits_{\ell\in[4]}(5-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell})\\
&\quad - 5\sum\nolimits_{\ell\in[4]}(6-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{6-\ell}v^{\ell}) + sn - 6sm\\
&\quad + \begin{cases}5(s-r)m + 5rm & \text{for } j\in\kappa_1,\\ 5sm & \text{for } j\in\kappa_2.\end{cases}\\
&= sn - 6sm + 5sm\\
&= s(n-m).
\end{aligned}
$$

$\square$

This completes the proof of Theorem II.0.1.

In the next chapter, we will address a generalization of Theorem II.0.1 where the value of h is unknown. In fact, Theorem II.0.1 is a corollary of Theorem III.0.1 in the next section.

CHAPTER III: EMBEDDING CONNECTED FACTORIZATIONS

Let $q, k \in \mathbb{N}$ and let $\mathbf{s} = (s_1, \dots, s_k)$ and $\mathbf{r} = (r_1, \dots, r_q)$. Suppose that an $\mathbf{r}$-factorization of λK_m^h is given using q colors. Then the edges of color j in each color class $\lambda K_m^h(j)$ induce an r_j factor. Within each r_j-factor, the number of edges (which is an integer) is equal to mr_j/h. Thus, the existence of a $\mathbf{r}$-factorization of λK_m^h implies that $h \mid r_j m$ for each color j. Moreover, since the degree of each vertex in λK_m^h is $\lambda\binom{m-1}{h-1}$ and the degree of each vertex in color class $\lambda K_m^h(j)$ is r_j, we must have that $\sum_{j=1}^{q} r_j = \lambda\binom{m-1}{h-1}$. If it is possible to extend this $\mathbf{r}$-factorization of λK_m^h to an $\mathbf{s}$-factorization of λK_n^h, then the existence of an $\mathbf{s}$-factorization of λK_n^h similarly implies that $h \mid s_j n$ for all $j \in [k]$ and $\sum_{j=1}^{k} s_j = \lambda\binom{n-1}{h-1}$. Thus, in order to extend the given $\mathbf{r}$-factorization of λK_m^h to an $\mathbf{s}$-factorization of λK_n^h with k color classes, the following conditions are necessary.

$$h \mid r_j m, \qquad h \mid s_j n, \qquad \sum_{j=1}^{q} r_j = \lambda\binom{m-1}{h-1}, \qquad \sum_{j=1}^{k} s_j = \lambda\binom{n-1}{h-1}. \tag{III.1}$$

Since the number of colors in the $\mathbf{r}$-factorization of λK_m^h and in the $\mathbf{s}$-factorization of λK_n^h are q and k, respectively, we have the following additional necessary conditions.

$$r_j \leq s_j, \qquad\qquad q \leq k \tag{III.2}$$

These conditions arise from the fact that we cannot embed an edge-coloring of a hypergraph using q colors into an edge-coloring of a larger hypergraph using fewer than q colors and from the fact that we cannot embed an r-factor into an s-factor if $s < r$.

In this chapter, a sextuple $(n, m, \mathbf{s}, \mathbf{r}, h, \lambda)$ is *admissible* if it satisfies conditions (III.1) and (III.2). We shall show that

Theorem III.0.1. *For $n \geq hm$, an $\mathbf{r}$-factorization of λK_m^h can be extended to an*

$\mathbf{s}$*-factorization of* λK_n^h *if and only if* $(n, m, \mathbf{s}, \mathbf{r}, h, \lambda)$ *is admissible.*

In fact, we prove something much stronger. By a connected factorization, we mean a factorization in which each color class is connected.

Theorem III.0.2. *For* $n \geq hm$*, an* $\mathbf{r}$*-factorization of* λK_m^h *can be extended to a connected* $\mathbf{s}$*-factorization of* λK_n^h *if and only if* $s_j \geq r_j + 1$ *and* $(n, m, \mathbf{s}, \mathbf{r}, h, \lambda)$ *is admissible.*

III.1 Proof: Overview

Throughout the rest of this chapter, we shall assume that $(n, m, \mathbf{s}, \mathbf{r}, h, \lambda)$ is admissible, the number of colors in the given $\mathbf{r}$-factorization of λK_m^h is q, the number of colors in the $\mathbf{s}$-factorization of λK_n^h is k, and

$$\kappa_1 := \{1, \dots, q\}, \qquad \kappa_2 := \{q+1, \dots, k\}, \qquad \kappa := \kappa_1 \cup \kappa_2.$$

Let $\mathcal{G}$ be the hypergraph $K_n^h \backslash K_m^h$. We refer to the m vertices in $V(\mathcal{G}) \cap K_m^h$ as the old vertices in $\mathcal{G}$ and we refer to the remaining $n - m$ vertices in $V(\mathcal{G}) \backslash K_m^h$ as the new vertices in $\mathcal{G}$. We shall reduce the problem of extending an $\mathbf{r}$-factorization of λK_m^h to an $\mathbf{s}$-factorization of λK_n^h to the problem of coloring a 2-vertex hypergraph $\mathcal{F}$ with $V(\mathcal{F}) = \{u, v\}$. Before describing $E(\mathcal{F})$, we need to introduce some more notation. An *edge of the form* $u^i v^j$ (or a $u^i v^j$*-edge*) is an edge containing i copies of vertex u and j copies of vertex v. When we color the edges of $\mathcal{F}$, we use $\mathbf{deg}_{\mathcal{F}(j)}(v)$ and $\mathbf{mult}_{\mathcal{F}(j)}(e)$ to denote the degree of v and the multiplicity of e, respectively, in color class $\mathcal{F}(j)$. The following describes the edge set of $\mathcal{F}$.

$$\mathbf{mult}(u^i v^{h-i}) = \lambda \binom{m}{i} \binom{n-m}{h-i} \quad \text{for } 0 \leq i \leq h-1. \tag{III.3}$$

In fact, $\mathcal{F}$ is obtained by identifying all the m old vertices of $\mathcal{G} := \lambda K_n^h \backslash \lambda K_m^h$ with a vertex u, and identifying all the remaining $n-m$ new vertices with v. We say that $\mathcal{G}$ is a *detachment* of $\mathcal{F}$, and $\mathcal{F}$ is an *amalgamation* of $\mathcal{G}$.

In order to extend the $\mathbf{r}$-factorization of λK_m^h to an $\mathbf{s}$-factorization of λK_n^h, we need to color $\mathcal{G}$ with k colors such that each color class of the hypergraph λK_n^h with edge set $E(\mathcal{G}) \cup E(\lambda K_m^h)$ induces an s_j-factor for color j. In each r_j factor in λK_m^h, we have that the degree of each vertex is r_j. Thus, coloring in such a way would require the degree of the m old vertices in $V(\mathcal{G})$ to be $s_j - r_j$ in $\mathcal{G}(j)$ for all $j \in \kappa_1$, the degree of the m old vertices to be s_j in $\mathcal{G}(j)$ for all $j \in \kappa_2$ and the degree of the $n-m$ vertices in $V(\mathcal{G})$ to be s_j in $\mathcal{G}(j)$ for all $j \in \kappa$. If we can obtain such a coloring, then in the amalgamation $\mathcal{F}$ of $\mathcal{G}$, $\mathbf{deg}_{\mathcal{F}(j)}(u) = m(s_j - r_j)$ for $j \in \kappa_1$, $\mathbf{deg}_{\mathcal{F}(j)}(u) = s_j m$ for $j \in \kappa_2$, and $\mathbf{deg}_{\mathcal{F}(j)}(v) = s_j(n-m)$ for $j \in \kappa$. More importantly, by the following lemma, which is a consequence of a result of Bahmanian (see [5, Theorem 4.1]), the converse of the previous statement is also true.

Lemma III.1.1. *An $\mathbf{r}$-factorization of λK_m^h can be extended to an $\mathbf{s}$-factorization of λK_n^h provided the hypergraph $\mathcal{F}$ described in* (III.3) *can be colored so that*

$$\mathbf{deg}_{\mathcal{F}(j)}(x) = \begin{cases} m(s_j - r_j) & \text{if } x = u, j \in \kappa_1, \\ s_j m & \text{if } x = u, j \in \kappa_2, \\ s_j(n-m) & \text{if } x = v, j \in \kappa. \end{cases} \tag{III.4}$$

Proof. Suppose we are given an $\mathbf{r}$-factorization of λK_m^h. Let $\mathcal{F}$ be the hypergraph defined above and suppose we have colored the edges of $\mathcal{F}$ such that Equation III.4 holds.

Then by [5, Theorem 4.1], there exists an n-vertex hypergraph $\tilde{\mathcal{H}}$, obtained by replacing the vertex u of $\mathcal{F}$ by m vertices $u_1, \dots, u_m$, replacing the vertex v of $\mathcal{F}$ by $n-m$ vertices $v_1, \dots, v_{n-m}$, and replacing each $u^{h-i}v^i$-edge in $\mathcal{F}$ by an edge of the form $U_1 \cup U_2$ where $U_1 \subseteq \{u_1, \dots, u_m\}, |U_1| = h-i$, $U_2 \subseteq \{v_1, \dots, v_{n-m}\}, |U_2| = i$ such that the edges incident with u (in each color class of $\mathcal{F}$) are shared as evenly as possible among $u_1, \dots, u_m$ (in each color class of $\tilde{\mathcal{H}}$) and the edges incident with v (in each color class of $\mathcal{F}$) are shared as evenly as possible among $v_1, \dots, v_{n-m}$ (in each color class of $\tilde{\mathcal{H}}$) in the following way.

(a) For $i \in [m], j \in \kappa_1$,

$$\mathbf{deg}_{\tilde{\mathcal{H}}(j)}(u_i) = \frac{\mathbf{deg}_{\mathcal{F}(j)}(u)}{m} = \frac{m(s_j - r_j)}{m} = s_j - r_j;$$

(b) For $i \in [m], j \in \kappa_2$,

$$\mathbf{deg}_{\tilde{\mathcal{H}}(j)}(u_i) = \frac{\mathbf{deg}_{\mathcal{F}(j)}(u)}{m} = \frac{m(s_j)}{m} = s_j;$$

(c) For $i \in [n-m], j \in \kappa$,

$$\mathbf{deg}_{\tilde{\mathcal{H}}(j)}(v_i) = \frac{\mathbf{deg}_{\mathcal{F}(j)}(v)}{n-m} = \frac{s_j(n-m)}{n-m} = s_j;$$

(d) For $U_1 \subseteq \{u_1, \dots, u_m\}, |U_1| = h - i, U_2 \subseteq \{v_1, \dots, v_{n-m}\}, |U_2| = i, i \in [h]$,

$$\mathbf{mult}_{\tilde{\mathcal{H}}}(U_1 \cup U_2) = \frac{\mathbf{mult}_{\mathcal{F}}(u^{h-i}v^i)}{\binom{m}{h-i}\binom{n-m}{i}} = \frac{\lambda\binom{m}{h-i}\binom{n-m}{i}}{\binom{m}{h-i}\binom{n-m}{i}} = \lambda.$$

Let $\mathcal{H}$ be λK_n^h. First color the edges in $E(\lambda K_m^h) \subseteq E(\mathcal{H})$ according to the given $\mathbf{r}$-factorization of λK_m^h and then color the remaining edges of $\mathcal{H}$ according to the coloring of the edges in $E(\tilde{\mathcal{H}})$. Then the degree of each vertex u_i in $V(\mathcal{H}) \cap V(\lambda K_m^h)$ will be $s_j - r_j + r_j = s_j$ in color class $\mathcal{H}(j)$ for $j \in \kappa_1$, the degree of each vertex u_i in

$V(\mathcal{H}) \cap V(\lambda K_m^h)$ will be s_j in each color class $\mathcal{H}(j)$ for $j \in \kappa_2$, and the degree of each vertex $v_i \in V(\mathcal{H}) \backslash V(\lambda K_m^h)$ will be s_j in each color class $\mathcal{H}(j)$, $j \in \kappa$.

Thus $\mathcal{H}$ is an $\mathbf{s}$-factorization of λK_n^h with the given $\mathbf{r}$-factorization of λK_m^h embedded in it, as desired. $\square$

When needed, we will use the following combinatorial identities without further explanation. For $n \geq m \geq h$

$$\sum_{i=0}^{h} \binom{m}{i}\binom{n-m}{h-i} = \binom{n}{h}, \quad \sum_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i} = m\left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right].$$

These identities and their proofs can be found in Lemma I.3.1 in the Introduction.

III.2 Proof of Theorem III.0.1: Details

In this section, we shall assume that $n \geq hm$. The following inequality will be crucial in our argument.

Lemma III.2.1. *We have*

$$\sum_{\ell \in [h-i-1]} m\binom{m-1}{\ell-1}\binom{n-m}{h-\ell} \geq (h-i-1)\binom{n-1}{h-1}, \quad \forall i \in [h-2]. \tag{III.5}$$

Proof. We need to show that $f(i) \geq 0$ for $i \in [h-2]$ where

$$\begin{aligned}
f(i) &:= \sum_{\ell \in [h-i-1]} m\binom{m-1}{\ell-1}\binom{n-m}{h-\ell} - (h-i-1)\binom{n-1}{h-1}, \quad i \in [h-2],\\
g(i) &:= f(i+1) - f(i) = \sum_{\ell \in [h-i-2]} m\binom{m-1}{\ell-1}\binom{n-m}{h-\ell} - (h-i-2)\binom{n-1}{h-1}\\
&\qquad - \sum_{\ell \in [h-i-1]} m\binom{m-1}{\ell-1}\binom{n-m}{h-\ell} + (h-i-1)\binom{n-1}{h-1}\\
&= \binom{n-1}{h-1} - m\binom{m-1}{h-i-2}\binom{n-m}{i+1}, \quad i \in [h-3].
\end{aligned}$$

We will show that g is decreasing. If g is decreasing, one of three cases holds:

Case 1: g is decreasing and $g(i) \geq 0$ for $i \in [h-3]$. Then $f(i+1) \geq f(i)$ for $i \in [h-3]$, so if $f(1) \geq 0$, then we have that $f(i) \geq 0$ for $i \in [h-2]$.

Case 2: g is decreasing and $g(i) \leq 0$ for $i \in [h-3]$. Then $f(i+1) \leq f(i)$ for $i \in [h-3]$, so if $f(h-2) \geq 0$, then we have that $f(i) \geq 0$ for $i \in [h-2]$.

Case 3: g is decreasing with $g(1) > 0$ and $g(h-3) < 0$, so

$g(1) > ... > g(i) \geq 0 \geq g(i+1) > ... > g(h-3)$ for some $i \in [h-3]$. Then

$f(i+1) \geq f(i) \geq ... \geq f(1)$ and $f(i+2) \geq f(i+3) \geq ... \geq f(h-2)$, so if $f(1) \geq 0$ and $f(h-2) \geq 0$, then we have that $f(i) \geq 0$ for $i \in [h-2]$.

Hence, it suffices to show that g is decreasing, $f(1) \geq 0$, and $f(h-2) \geq 0$.

Note that

$$\begin{aligned} g(i) - g(i+1) &= \binom{n-1}{h-1} - m\binom{m-1}{h-i-2}\binom{n-m}{i+1} \\ &\quad - \binom{n-1}{h-1} + m\binom{m-1}{h-i-3}\binom{n-m}{i+2} \\ &= m\binom{m-1}{h-i-3}\binom{n-m}{i+2} - m\binom{m-1}{h-i-2}\binom{n-m}{i+1}. \end{aligned}$$

Thus for $i \in [h-3]$, we have the following which proves that g is decreasing.

$$\begin{aligned} \frac{g(i)-g(i+1)}{m} &= \binom{m-1}{h-i-3}\binom{n-m}{i+2} - \binom{m-1}{h-i-2}\binom{n-m}{i+1} \\ &= \binom{m-1}{h-i-3}\binom{n-m}{i+1}\left(\frac{n-m-i-1}{i+2} - \frac{m-h+i+2}{h-i-2}\right) \\ &= \binom{m-1}{h-i-3}\binom{n-m}{i+1}\left(\frac{(n-i-1)(h-i-2)+(h-i-2)(i+2)}{(i+2)(h-i-2)}\right) \\ &\quad + \binom{m-1}{h-i-3}\binom{n-m}{i+1}\left(\frac{-m(h-i-2+i+2)}{(i+2)(h-i-2)}\right) \\ &= \binom{m-1}{h-i-3}\binom{n-m}{i+1}\frac{(n+1)(h-i-2)-hm}{(i+2)(h-i-2)} \\ &\geq \binom{m-1}{h-i-3}\binom{n-m}{i+1}\frac{(hm+1)(h-i-2)-hm}{(i+2)(h-i-2)} \geq 0, \end{aligned}$$

since $n \geq hm$ and $i \leq h-3$.

Since

$$
\begin{aligned}
f(1) &= m \sum\nolimits_{\ell \in [h-2]} \binom{m-1}{\ell-1}\binom{n-m}{h-\ell} - (h-2)\binom{n-1}{h-1} \\
&= m\left[\sum\nolimits_{\ell \in [h]} \binom{m-1}{\ell-1}\binom{n-m}{h-\ell} - \binom{m-1}{h-2}(n-m) - \binom{m-1}{h-1}\right] - (h-2)\binom{n-1}{h-1} \\
&= m\left[\binom{n-1}{h-1} - \binom{m-1}{h-2}(n-m) - \binom{m-1}{h-1}\right] - (h-2)\binom{n-1}{h-1} \\
&= (m-h+2)\binom{n-1}{h-1} \ - m(n-m)\binom{m-1}{h-2} - m\binom{m-1}{h-1} \\
&= (m-h+2)\binom{n-1}{h-1} \ - m\left(n-m+\frac{m-h+1}{h-1}\right)\binom{m-1}{h-2} \\
&\geq (m-h+2)\binom{n-1}{h-1} \ - m(n-1)\binom{m-1}{h-2},
\end{aligned}
$$

the following proves that $f(1) \geq 0$.

$$
\begin{aligned}
(m-h+2)&\binom{n-1}{h-1}\Big/\left[m(n-1)\binom{m-1}{h-2}\right] \\
&= \frac{(m-h+2)(n-2)!(m-h+1)!}{m(h-1)(n-h)!(m-1)!} \\
&= \frac{1}{h-1}\prod\nolimits_{i\in[h-2]} \frac{n-1-i}{m+1-i} \\
&= \frac{1}{h-1}\prod\nolimits_{i\in[h-2]} \left(1+\frac{n-m-2}{m+1-i}\right) \\
&\geq \frac{1}{h-1}\left(1+\frac{n-m-2}{m}\right)^{h-2} \\
&= \frac{(n-2)^{h-2}}{(h-1)m^{h-2}} \\
&\geq \frac{(hm-2)^{h-2}}{(h-1)m^{h-2}} \geq 1.
\end{aligned}
$$

Using Bernoulli's inequality ($\forall\, r \geq 1, \forall\, x \geq -1 : (1+x)^r \geq 1+rx$) we have the following which proves that $f(h-2) \geq 0$.

$$
\begin{aligned}
m\binom{n-m}{h-1}\Big/\binom{n-1}{h-1} &= \frac{m(n-m)!(n-h)!}{(n-1)!(n-m-h+1)!} \\
&= m\prod\nolimits_{i\in[m-1]} \frac{n-m-h+i+1}{n-m+i} \\
&= m\prod\nolimits_{i\in[m-1]} \left(1-\frac{h-1}{n-m+i}\right) \\
&\geq m\prod\nolimits_{i\in[m-1]} \left(1-\frac{h-1}{n-m}\right) \\
&\geq m\left(1-\frac{1}{m}\right)^{m-1} \\
&\geq m\left(1-\frac{m-1}{m}\right) = 1.
\end{aligned}
$$

□

Using Lemma III.2.1, we show that we can color all the $u^{h-i}v^i$-edges for $i \in [h-2]$.

Lemma III.2.2. *We can color the edges of the form $u^{h-1}v, u^{h-2}v^2, \ldots, u^2v^{h-2}$ in $\mathcal{F}$ in that particular order such that*

$$
\mathbf{mult}_{\mathcal{F}(j)}(u^{h-i}v^i) \leq \begin{cases} \rho_{ij} & \textit{for } j \in \kappa_1, \\ \sigma_{ij} & \textit{for } j \in \kappa_2, \end{cases} \qquad \textit{for } i \in [h-2], \tag{III.6}
$$

where for $i \in [h-1]$,

$$
\begin{cases} \rho_{ij} = \dfrac{1}{h-i}\left(m(s_j-r_j) - \sum_{\ell\in[i-1]}(h-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^\ell)\right) & \textit{for } j \in \kappa_1, \\ \sigma_{ij} = \dfrac{1}{h-i}\left(ms_j - \sum_{\ell\in[i-1]}(h-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^\ell)\right) & \textit{for } j \in \kappa_2. \end{cases}
$$

Proof. By letting $\rho_{0j} = m(s_j - r_j)/h, \sigma_{0j} = ms_j/h$, we have

$$\text{for } i \in [h-1], \qquad \begin{cases} \rho_{ij} = \dfrac{h-i+1}{h-i} \left(\rho_{i-1,j} - \text{mult}_{\mathcal{F}(j)}(u^{h-i+1}v^{i-1})\right) & \text{for } j \in \kappa_1, \\ \sigma_{ij} = \dfrac{h-i+1}{h-i} \left(\sigma_{i-1,j} - \text{mult}_{\mathcal{F}(j)}(u^{h-i+1}v^{i-1})\right) & \text{for } j \in \kappa_2. \end{cases}$$

Therefore, for $i \in [h-1]$, $\rho_{ij} \geq 0$ (if $j \in \kappa_1$) and $\sigma_{ij} \geq 0$ (if $j \in \kappa_2$). Since $f(i) \geq 0$ for $i \in [h-2]$, we have

$$
\begin{aligned}
0 &\le \lambda \sum\nolimits_{\ell\in[h-i-1]} \ell\binom{m}{\ell}\binom{n-m}{h-\ell} - \lambda(h-i-1)\binom{n-1}{h-1} \\
&= \lambda \sum\nolimits_{\ell=i+1}^{h-1}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} - \lambda(h-i-1)\binom{n-1}{h-1} \\
&= \lambda \sum\nolimits_{\ell\in[h-1]}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} - \lambda \sum\nolimits_{\ell\in[i]}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} \\
&\qquad - \lambda(h-i-1)\binom{n-1}{h-1} \\
&= \lambda m\left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right] - \lambda \sum\nolimits_{\ell\in[i]}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} \\
&\qquad - \lambda(h-i-1)\binom{n-1}{h-1} \\
&= \lambda m\left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right] - \lambda \sum\nolimits_{\ell\in[i-1]}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} \\
&\qquad - \lambda(h-i)\binom{m}{h-i}\binom{n-m}{i} - (h-i-1)\sum\nolimits_{j\in\kappa} s_j \\
&\le \lambda m\left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right] - \lambda \sum\nolimits_{\ell\in[i-1]}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} \\
&\qquad - \lambda(h-i)\binom{m}{h-i}\binom{n-m}{i} - (h-i-1)(k) \\
&= m\left[\sum\nolimits_{j\in\kappa} s_j - \sum\nolimits_{j\in\kappa_1} r_j\right] - \sum\nolimits_{\ell\in[i-1]}(h-\ell)\,\mathrm{mult}(u^{h-\ell}v^{\ell}) \\
&\qquad - \lambda(h-i)\binom{m}{h-i}\binom{n-m}{i} - k(h-i-1) \\
&= \sum\nolimits_{j\in\kappa_1}(s_j-r_j)m + \sum\nolimits_{j\in\kappa_2} s_j m - \sum\nolimits_{j\in\kappa}\sum\nolimits_{\ell\in[i-1]}(h-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&\qquad - \lambda(h-i)\binom{m}{h-i}\binom{n-m}{i} - \sum\nolimits_{j\in\kappa}(h-i-1) \\
&= \sum\nolimits_{j\in\kappa_1}\left[(s_j-r_j)m - \sum\nolimits_{\ell\in[i-1]}(h-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) - (h-i-1)\right] \\
&\qquad + \sum\nolimits_{j\in\kappa_2}\left[s_j m - \sum\nolimits_{\ell\in[i-1]}(h-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) - (h-i-1)\right] - (h-i)\,\mathrm{mult}(u^{h-i}v^{i}) \\
&\le (h-i)\sum\nolimits_{j\in\kappa_1}\lfloor\rho_{ij}\rfloor + (h-i)\sum\nolimits_{j\in\kappa_2}\lfloor\sigma_{ij}\rfloor - (h-i)\,\mathrm{mult}(u^{h-i}v^{i}).
\end{aligned}
$$

Therefore, for $i \in [h-2]$,

$$\mathrm{mult}(u^{h-i}v^i) \le \sum\nolimits_{j\in\kappa_1} \lfloor \rho_{ij} \rfloor + \sum\nolimits_{j\in\kappa_2} \lfloor \sigma_{ij} \rfloor.$$

□

Coloring the remaining edges of $\mathcal{F}$ is straightforward.

Lemma III.2.3. *We can color the uv^{h-1}-edges and the v^h-edges of $\mathcal{F}$ such that*

$$\mathrm{mult}_{\mathcal{F}(j)}(uv^{h-1}) = \begin{cases} \rho_{h-1,j} & \text{for } j \in \kappa_1, \\ \sigma_{h-1,j} & \text{for } j \in \kappa_2, \end{cases}$$

$$\mathrm{mult}_{\mathcal{F}(j)}(v^h) = \begin{cases} s_j\left(\frac{n}{h} - m\right) + \sum\nolimits_{\ell\in[h-2]}(h-\ell-1)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^\ell) + r_j m\left(\frac{h-1}{h}\right) \\ \quad \text{for } j \in \kappa_1, \\ s_j\left(\frac{n}{h} - m\right) + \sum\nolimits_{\ell\in[h-2]}(h-\ell-1)\,\mathrm{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^\ell) \\ \quad \text{for } j \in \kappa_2. \end{cases}$$

Proof. By Lemma III.2.2, both $\rho_{h-1,j}$ and $\sigma_{h-1,j}$ are nonnegative integers. We have

$$
\begin{aligned}
\sum\nolimits_{j\in\kappa} \mathbf{mult}_{\mathcal{F}(j)}(uv^{h-1}) &= \sum\nolimits_{j\in\kappa_1} \rho_{h-1,j} + \sum\nolimits_{j\in\kappa_2} \sigma_{h-1,j} \\
&= m\sum\nolimits_{j\in\kappa_1}(s_j - r_j) - \sum\nolimits_{j\in\kappa_1}\sum\nolimits_{\ell\in[h-2]}(h-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&\qquad + \sum\nolimits_{j\in\kappa_2} s_j m - \sum\nolimits_{j\in\kappa_2}\sum\nolimits_{\ell\in[h-2]}(h-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&= m\sum\nolimits_{j\in\kappa} s_j - m\sum\nolimits_{j\in\kappa_1} r_j - \sum\nolimits_{\ell\in[h-2]}(h-\ell)\sum\nolimits_{j\in\kappa}\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&= \lambda m\left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right] - \lambda\sum\nolimits_{\ell\in[h-2]}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} \\
&= \lambda\sum\nolimits_{\ell\in[h-1]}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} - \lambda\sum\nolimits_{\ell\in[h-2]}(h-\ell)\binom{m}{h-\ell}\binom{n-m}{\ell} \\
&= \lambda m\binom{n-m}{h-1}.
\end{aligned}
$$

Since $(n, m, \mathbf{s}, \mathbf{r}, h, \lambda)$ is admissible and $n \geq hm$, we have that $\mathbf{mult}_{\mathcal{F}(j)}(v^h)$ is a nonnegative integer for all $j \in \kappa$. Thus, the following completes the proof.

$$
\begin{aligned}
\sum_{j\in\kappa} \mathrm{mult}_{\mathcal{F}(j)}(v^h) &= \sum_{j\in\kappa_1} \mathrm{mult}_{\mathcal{F}(j)}(v^h) + \sum_{j\in\kappa_2} \mathrm{mult}_{\mathcal{F}(j)}(v^h) \\
&= \sum_{j\in\kappa} s_j \left(\frac{n}{h} - m\right) + m \sum_{j\in\kappa_1} r_j \left(\frac{h-1}{h}\right) \\
&\quad + \sum_{j\in\kappa} \sum_{\ell\in[h-2]} (h-\ell-1)\, \mathrm{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&= \left(\frac{n}{h} - m\right) \lambda \binom{n-1}{h-1} + m \left(\frac{h-1}{h}\right) \lambda \binom{m-1}{h-1} \\
&\quad + \sum_{\ell\in[h-2]} (h-\ell-1) \lambda \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \frac{\lambda n}{h} \binom{n-1}{h-1} - \frac{\lambda m}{h} \binom{m-1}{h-1} - \lambda m \left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right] \\
&\quad + \sum_{\ell\in[h-2]} \lambda (h-\ell) \binom{m}{h-\ell} \binom{n-m}{\ell} - \sum_{\ell\in[h-2]} \lambda \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \binom{n}{h} - \lambda \binom{m}{h} - \sum_{\ell\in[h-1]} \lambda (h-\ell) \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&\quad + \sum_{\ell\in[h-2]} \lambda (h-\ell) \binom{m}{h-\ell} \binom{n-m}{\ell} - \sum_{\ell\in[h-2]} \lambda \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \binom{n}{h} - \lambda \binom{m}{h} - \lambda m \binom{n-m}{h-1} - \sum_{\ell\in[h-2]} \lambda \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \binom{n}{h} - \lambda \binom{m}{h} - \sum_{\ell\in[h-1]} \lambda \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \binom{n}{h} - \lambda \binom{m}{h} - \lambda \left[\binom{n}{h} - \binom{m}{h} - \binom{n-m}{h}\right] \\
&= \lambda \binom{n-m}{h}.
\end{aligned}
$$

□

Lemma III.2.4. *Coloring of $\mathcal{F}$ satisfies* (III.4).

Proof. For $j \in \kappa$, we have

$$
\begin{aligned}
\deg_{\mathcal{F}(j)}(u) &= \sum\nolimits_{\ell\in[h-1]}(h-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&= \sum\nolimits_{\ell\in[h-2]}(h-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) + \mathbf{mult}_{\mathcal{F}(j)}(uv^{h-1}) \\
&= \sum\nolimits_{\ell\in[h-2]}(h-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) + \begin{cases} \rho_{h-1,j} & \text{for } j\in\kappa_1 \\ \sigma_{h-1,j} & \text{for } j\in\kappa_2 \end{cases} \\
&= \begin{cases} m(s_j-r_j) & \text{for } j\in\kappa_1 \\ s_j m & \text{for } j\in\kappa_2 \end{cases},
\end{aligned}
$$

and

$$
\begin{aligned}
\deg_{\mathcal{F}(j)}(v) &= \sum\nolimits_{\ell\in[h]}\ell\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&= \sum\nolimits_{\ell\in[h-2]}\ell\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) + (h-1)\,\mathbf{mult}_{\mathcal{F}(j)}(uv^{h-1}) + h\,\mathbf{mult}_{\mathcal{F}(j)}(v^{h}) \\
&= \sum\nolimits_{\ell\in[h-2]}\ell\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) + h\sum\nolimits_{\ell\in[h-2]}(h-\ell-1)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&\qquad + hs_j\left(\frac{n}{h}-m\right) + \begin{cases} (h-1)\rho_{h-1,j} + hr_jm\left(\dfrac{h-1}{h}\right) & \text{for } j\in\kappa_1, \\ (h-1)\sigma_{h-1,j} & \text{for } j\in\kappa_2. \end{cases} \\
&= \sum\nolimits_{\ell\in[h-2]}\ell\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) + h\sum\nolimits_{\ell\in[h-2]}(h-\ell-1)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) \\
&\qquad - (h-1)\sum\nolimits_{\ell\in[h-2]}(h-\ell)\,\mathbf{mult}_{\mathcal{F}(j)}(u^{h-\ell}v^{\ell}) + s_jn - hs_jm \\
&\qquad + \begin{cases} (h-1)m(s_j-r_j) + (h-1)r_jm & \text{for } j\in\kappa_1, \\ (h-1)ms_j & \text{for } j\in\kappa_2. \end{cases} \\
&= s_jn - hs_jm + (h-1)s_jm \\
&= s_j(n-m).
\end{aligned}
$$

$\square$

This completes the proof of Theorem III.0.1.

III.3 Connected Factorizations

Let $\mathcal{F}$ be the hypergraph defined in section III.1 and colored in section III.2. Let $\mathcal{H}$ be the $\mathbf{s}$-factorization of λK_n^h obtained by first detaching the colored hypergraph $\mathcal{F}$ by detaching u into m vertices and detaching v into $n-m$ vertices and then coloring the remaining edges according to the given $\mathbf{r}$-factorization of λK_m^h, as described in the proof of Lemma III.1.1. Now amalgamate $\mathcal{H}$ by mapping the $n-m$ vertices in $V(\mathcal{H})\backslash V(\lambda K_m^h)$ to one vertex v. Let $\tilde{\mathcal{F}}$ be the resulting $(m+1)$-vertex hypergraph with $V(\tilde{\mathcal{F}}) = V(\lambda K_m^h) \cup \{v\}$ and with an edge set containing $E(\lambda K_m^h) \subseteq E(\tilde{\mathcal{F}})$ and containing remaining edges incident to v such that

$$\textbf{mult}_{\tilde{\mathcal{F}}}(Xv^{h-i}) = \lambda\binom{n-m}{h-i} \qquad \forall X \subseteq V(\lambda K_m^h), |X| = i, 0 \leq i \leq h-1. \tag{III.7}$$

Assume that the edges in $E(\lambda K_m^h) \subseteq E(\tilde{\mathcal{F}})$ are colored according to the given $\mathbf{r}$-factorization of λK_m^h. Let U_{h-i} be the set containing all sets $U \subseteq V(\mathcal{H})\backslash V(\lambda K_m^h)$ with $|U| = h-i$ and assume that the remaining edges in $E(\tilde{\mathcal{F}})$ are colored such that for each $X \subseteq V(\lambda K_m^h), |X| = i$, $0 \leq i \leq h-1$,

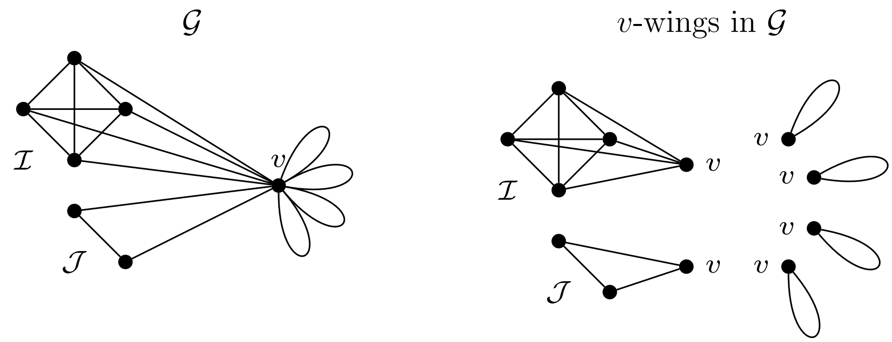
$$\textbf{mult}_{\tilde{\mathcal{F}}(j)}(Xv^{h-i}) = \sum\nolimits_{U \subseteq U_{h-i}} \textbf{mult}_{\mathcal{H}(j)}(X \cup U). \tag{III.8}$$

By an Xv^j-edge or $*v^j$-edge for short, we mean an edge of the form $X \cup \{v^j\}$ (so it contains j copies of v). Observe that the edges of λK_m^h are the $*v^0$-edges.

Recall that a vertex v in some hypergraph $\mathcal{G}$ is a *cut vertex* if there exist two non-trivial sub-hypergraphs I, J of $\mathcal{G}$ such that (i) $I \cup J = \mathcal{G}$, (ii) $V(I \cap J) = \{v\}$, and (iii)

$E(I \cap J) = \varnothing$. A sub-hypergraph W of a hypergraph $\mathcal{G}$ is an *v-wing* of $\mathcal{G}$ if (i) W is non-trivial and connected, (ii) v is not a cut vertex of W, and (iii) no edge in $E(\mathcal{G}) \backslash E(W)$ is incident with a vertex in $V(W) \backslash \{v\}$. A v-wing W is *large* if $V(W) \neq \{v\}$. Let $\omega_v(\mathcal{G})$, and $\omega_v^L(\mathcal{G})$ be the number of v-wings, and the number of large v-wings in $\mathcal{G}$, respectively. Let $c(\mathcal{G})$ denote the number of components of $\mathcal{G}$.

Example: Hypergraph $\mathcal{G}$ with cut vertex v, $\omega_v(\mathcal{G}) = 6$, and $\omega_v^L(\mathcal{G}) = 2$.



For $j \in \kappa$, we have $\textbf{mult}_{\tilde{\mathcal{F}}(j)}(v^h)$ loops containing h copies of vertex v, each of which is a v-wing in $\tilde{\mathcal{F}}(j)$. The remaining edges in $\tilde{\mathcal{F}}(j)$ contain at least one vertex $u \neq v$, so any remaining v-wings must be large v wings. Let $\tilde{\mathcal{F}} \backslash \{v\}$ be the hypergraph with vertex set $V(\tilde{\mathcal{F}}) \backslash \{v\}$ and with edge set $\{e \backslash \{v\} | e \in E(\tilde{\mathcal{F}})\}$. Since $s_j > r_j$ for all colors j, there must be at least one Xv^i edge in $E(\tilde{\mathcal{F}})$ incident to every vertex u in $V(\tilde{\mathcal{F}}) \backslash \{v\}$. The addition of each such edge may decrease, but can not increase, the number of components in $\tilde{\mathcal{F}}(j) \backslash \{v\}$ compared to the number of components in $\lambda K_m^h(j)$. Each component in $\tilde{\mathcal{F}}(j) \backslash \{v\}$ corresponds to a single v-wing in $\tilde{\mathcal{F}}(j)$. It follows that the number of large v-wings in $\tilde{\mathcal{F}}(j)$ cannot exceed the number of components in $\lambda K_m^h(j)$. Thus,

$$\omega_v(\tilde{\mathcal{F}}(j)) = \textbf{mult}_{\tilde{\mathcal{F}}(j)}(v^h) + \omega_v^L(\tilde{\mathcal{F}}(j)) \leq \textbf{mult}_{\tilde{\mathcal{F}}(j)}(v^h) + c(\lambda K_m^h(j)).$$

Moreover, $c(\lambda K_m^h(j)) \leq m/h$ for $j \in \kappa_1$ as there are at most m/h pairwise disjoint edges in a given r_j-factor, and $c(\lambda K_m^h(j)) \leq m$ for $j \in \kappa_2$ since the number of components in a hypergraph cannot exceed the number of vertices in it. Let $z_1 = m(s_j - r_j), z_2 = ms_j$. Using induction, we show that for $2 \le p \le h$ and $j \in \kappa_i, i = 1, 2$, the following holds.

$$\sum\nolimits_{\ell\in[h-2]}(h-\ell-1)\,\mathbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell}) \le \sum\nolimits_{\ell\in[h-p]}\left(\frac{h-\ell}{p-1}-1\right)\mathbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell}) + z_i\left(\frac{p-2}{p-1}\right). \tag{III.9}$$

Clearly (III.9) holds when $p = 2$, so suppose that it holds for some $2 \le p < h$. We have

$$\begin{aligned}
&\sum\nolimits_{\ell\in[h-2]}(h-\ell-1)\,\mathbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell}) \\
&\quad\le \sum\nolimits_{\ell\in[h-p-1]}\left(\frac{h-\ell}{p-1}-1\right)\mathbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell}) + z_i\left(\frac{p-2}{p-1}\right) \\
&\qquad + \left(\frac{p}{p-1}-1\right)\mathbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{p}v^{h-p}) \\
&\quad\le \sum\nolimits_{\ell\in[h-p-1]}\left(\frac{h-\ell}{p-1}-1\right)\mathbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell}) + z_i\left(\frac{p-2}{p-1}\right) \\
&\qquad + \frac{1}{p(p-1)}\left(z_i - \sum\nolimits_{\ell\in[h-p-1]}(h-\ell)\,\mathbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell})\right) \\
&\quad= \sum\nolimits_{\ell\in[h-p-1]}\left(\frac{h-\ell}{p}-1\right)\mathbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell}) + z_i\left(\frac{p-1}{p}\right).
\end{aligned}$$

By [8, Corollary 7.4], in order to complete the proof of Theorem III.0.2, we need to show that the coloring of $\tilde{\mathcal{F}}$ described in section III.2 satisfies the following condition.

$$\omega_v(\tilde{\mathcal{F}}(j)) \le (s_j-1)(n-m)+1 \qquad \forall j \in \kappa. \tag{III.10}$$

Since $n \ge hm$ and $s_j \ge r_j + 1 \ge 2$, for $j \in \kappa_1$, we have

$$
\begin{aligned}
\omega_v(\tilde{\mathcal{F}}(j)) &\leq s_j\left(\frac{n}{h}-m\right)+\sum\nolimits_{\ell\in[h-2]}(h-\ell-1)\,\textbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell})+r_jm\left(\frac{h-1}{h}\right)+\frac{m}{h}\\
&\leq s_j\left(\frac{n}{h}-m\right)+m(s_j-r_j)\left(\frac{h-2}{h-1}\right)+r_jm\left(\frac{h-1}{h}\right)+\frac{m}{h}\\
&\leq \frac{s_jn}{h}-\frac{s_jm}{h-1}+\frac{r_jm}{h(h-1)}+\frac{m}{h}\\
&=\frac{s_jnh-s_jn-s_jmh+r_jm+mh-m}{h(h-1)}\\
&\leq \frac{s_jh(n-m)-(r_j+1)n+r_jm+n-m}{h(h-1)}\\
&=\frac{s_jh(n-m)-r_j(n-m)-m}{h(h-1)}\\
&=\frac{s_j(n-m)}{h-1}-\frac{r_j(n-m)+m}{h(h-1)}\\
&\leq (s_j-1)(n-m),
\end{aligned}
$$

and for $j\in\kappa_2$, we have

$$
\begin{aligned}
\omega_v(\tilde{\mathcal{F}}(j)) &\leq s_j\left(\frac{n}{h}-m\right)+\sum\nolimits_{\ell\in[h-2]}(h-\ell-1)\,\textbf{mult}_{\tilde{\mathcal{F}}(j)}(u^{h-\ell}v^{\ell})+m\\
&\leq s_j\left(\frac{n}{h}-m\right)+s_jm\left(\frac{h-2}{h-1}\right)+m\\
&=\frac{s_jn}{h}-\frac{s_jm}{h-1}+m\\
&=\frac{s_jnh-s_jn-s_jmh+mh^2-mh}{h(h-1)}\\
&=\frac{s_j(n-m)+mh}{(h-1)}-\frac{s_jn+mh}{h(h-1)}\\
&\leq (s_j-1)(n-m).
\end{aligned}
$$

This completes the proof of Theorem III.0.2.

CHAPTER IV: EMBEDDING IRREGULAR COLORINGS INTO CONNECTED FACTORIZATIONS

Suppose that a partial $\mathbf{r}$-factorization P of λK_m^h is extended to an $\mathbf{r}$-factorization of λK_n^h. For $i \in [k]$, the existence of an r_i-factor in λK_n^h implies that $h \mid r_i n$ since the number of edges in an r_i-factor (which is an integer) is equal to mr_j/h. Since the degree of each vertex in λK_n^h is $\lambda\binom{n-1}{h-1}$ and the degree of each vertex in color class $\lambda K_n^h(j)$ is r_j, we must have that $\sum_{j=1}^{k} r_j = \lambda\binom{n-1}{h-1}$. Thus, in order to extend P to an $\mathbf{r}$-factorization of λK_n^h the following conditions are necessary.

$$h \mid r_i n \qquad \forall i \in [k], \qquad\qquad \sum_{i=1}^{k} r_i = \lambda\binom{n-1}{h-1}. \tag{IV.1}$$

For further explanation of the necessity of these conditions, see the paragraph prior to Lemma III.1.1 in section III.1.

We shall show that as long as n is big enough, these obvious necessary (divisibility) conditions are sufficient. In this chapter, a quadruple $(n, h, \lambda, \mathbf{r})$ is *admissible* if it satisfies (IV.1). Here are our first two results.

Theorem IV.0.1. *For $n \geq (h-1)(2m-1)$, a partial $\mathbf{r}$-factorization of λK_m^h can be extended to an $\mathbf{r}$-factorization of λK_n^h if and only if $(n, h, \lambda, \mathbf{r})$ is admissible.*

Theorem IV.0.2. *Let $n \geq (h-1)(2m-1)$, $\mathbf{s} = (s_1, \dots, s_q), \mathbf{r} = (r_1, \dots, r_k)$ such that*

$$\sum_{i=1}^{q} \left\lfloor \frac{(r_i - s_i)m}{h} \right\rfloor + \sum_{i=q+1}^{k} \left\lfloor \frac{r_i m}{h} \right\rfloor \geq \lambda\binom{m}{h}.$$

A partial $\mathbf{s}$-factorization of $\mathcal{G} \subseteq \lambda K_m^h$ can be embedded into an $\mathbf{r}$-factorization of λK_n^h if and only if $(n, h, \lambda, \mathbf{r})$ is admissible.

In the following two results, we identify the conditions under which r_j-factors in the extended factorizations in the previous two theorems are connected.

Theorem IV.0.3. *In Theorem IV.0.1, an r_j-factor $\lambda K_n^h(j)$ of λK_n^h is connected if and only if $r_j \geq 2$ and $\lambda K_m^h(j)$ is r_j-irregular for $j \in [k]$.*

Theorem IV.0.4. *In Theorem IV.0.2, let $A \subseteq \{j \in [k] \mid r_j \geq 2\}, B = \{j \in [q] \mid r_j \neq s_j\}$, and define $\overline{r_j} = r_j - 1$ if $j \in A$, and $\overline{r_j} = r_j$ if $j \in [k] \backslash A$. If*

$$\sum_{j \in B} \left\lfloor \frac{(\overline{r_j} - s_j)m}{h} \right\rfloor + \sum_{j \in [k] \backslash [q]} \left\lfloor \frac{\overline{r_j} m}{h} \right\rfloor \geq \lambda \binom{m}{h},$$

then for $j \in A$, the r_j-factor $\lambda K_n^h(j)$ is connected if and only if $\mathcal{G}(j)$ is r_j-irregular.

Before we prove Theorems IV.0.1–IV.0.4, we provide several corollaries in the next section. When needed, we will use Bernoulli's inequality ($\forall\, p \geq 1, \forall\, x \geq -1 : (1+x)^p \geq 1 + px$), and the following combinatorial identities without further explanation. For $n \geq m \geq h$

$$\sum\nolimits_{i=0}^{h} \binom{m}{i}\binom{n-m}{h-i} = \binom{n}{h}, \quad \sum\nolimits_{i=1}^{h-1} i \binom{m}{i}\binom{n-m}{h-i} = m\left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right].$$

These identities and their proofs can be found in Lemma I.3.1 in the Introduction.

IV.1 Corollaries

To exhibit the effectiveness of our main results, here we provide some applications.

Corollary IV.1.1. *Let*

$$n \geq (h-1)(2m-1), \qquad d = \lambda \binom{n-1}{h-1}, \qquad g = \frac{h}{\gcd(n,h)},$$
$$\mathcal{F} = \lambda K_m^h, \qquad \mathcal{H} = \lambda K_n^h.$$

(I) *A proper d-coloring of $\mathcal{F}$ can be extended to a proper d-coloring of $\mathcal{H}$ if and only if $n \equiv 0 \pmod{h}$.*

(II) *A partial g-factorization of $\mathcal{F}$ using $k := d/g$ colors can be extended to a connected g-factorization of $\mathcal{H}$ if and only if $\mathcal{F}(i)$ is g-irregular for $i \in [k]$ and $n \not\equiv 0 \pmod{h}$.*

(III) *A partial 2-factorization of $\mathcal{F}$ using $k := \lfloor d/2 \rfloor$ colors can be extended to a connected 2-factorization of $\mathcal{H}$ if and only if $\mathcal{F}(i)$ is 2-irregular for $i \in [k]$, $2n \equiv 0 \pmod{h}$ and $d \equiv 0 \pmod{2}$.*

(IV) *A partial h-factorization of $\mathcal{F}$ using $k := \lfloor d/h \rfloor$ colors can be extended to a connected h-factorization of $\mathcal{H}$ if and only if $\mathcal{F}(i)$ is h-irregular for $i \in [k]$ and $d \equiv 0 \pmod{h}$.*

Proof. To prove (I), let $k = d, \mathbf{r} = (1, \dots, 1)$. Then $h|n$ and $\sum_{i=1}^{k} 1 = k = d$, so $(n, h, \lambda, 1)$ is admissible. Applying Theorem IV.0.1 completes the proof.

To prove (II), let $\mathbf{r} = (g, \dots, g)$. By definition of greatest common divisor, we have that $\gcd(n, h)|n$. As $h|h$, it follows that $h|\frac{hn}{\gcd(n,h)}$, so $h|gn$. Furthermore, $\sum_{i=1}^{k} g = (d/g)(g) = d = \lambda\binom{n-1}{h-1}$. Thus, $(n, h, \lambda, \mathbf{r})$ is admissible. Moreover, as $h \nmid n$ and $h|gn$, we must have that $h|g$, so as $h \geq 2$, we also have that $g \geq 2$. Applying Theorem IV.0.3 completes the proof.

To prove (III), we apply Theorem IV.0.3 with $\mathbf{r} = (2, \dots, 2)$. As $2n \equiv 0 \pmod{h}$, we have that $h|2n$. As $d \equiv 0 \pmod{2}$, we have that $2|d$, so $\lfloor d/2 \rfloor = (d/2)$. It follows that $\sum_{i=1}^{k} 2 = \lfloor d/2 \rfloor 2 = d = \lambda\binom{n-1}{h-1}$ and we conclude that $(n, h, \lambda, 2)$ is admissible. We assume that $\mathcal{F}(i)$ is 2-irregular. Thus, Theorem IV.0.3 completes the proof.

To prove (IV), we apply Theorem IV.0.3 with $\mathbf{r} = (h, \dots, h)$. As $h|h$, we have $h|hn$. As $d \equiv 0(\pmod{h})$, we have that $h|d$, so $\sum_{i=1}^{k} h = \lfloor d/h \rfloor h = d = \lambda\binom{n-1}{h-1}$ and we conclude that (n, h, λ, h) is admissible. We assume that $\mathcal{F}(i)$ is h-irregular and we have that $h \geq 2$. Thus, Theorem IV.0.3 completes the proof. □

Remark IV.1.2. (I) is a hypergraph analogue of Cruse's theorem [14] which showed that a proper $(n-1)$-coloring of K_m can be extended to a proper $(n-1)$-coloring of K_n

whenever n is even and $n \geq 2m$. (III) is a hypergraph analogue of Hilton's theorem on extending path decompositions of K_m to Hamiltonian decompositions of K_n [21].

IV.2 Proof of Theorem IV.0.1

Let $\mathcal{G} := \lambda K_m^h$. We have established that it is necessary for $(n, h, \lambda, \mathbf{r})$ to be admissible, so suppose that $(n, h, \lambda, \mathbf{r})$ is admissible, $n \geq (h-1)(2m-1)$, and a partial $\mathbf{r}$-factorization of $\mathcal{G}$ is given. Let $\mathcal{F}$ be the hypergraph whose vertex set is $V(\mathcal{G}) \cup \{\alpha\}$, and whose edge set is the (colored) edge set of $\mathcal{G}$ together with further (uncolored) edges (containing the new vertex α) described as follows.

$$\textbf{mult}_{\mathcal{F}}(X\alpha^{h-i}) = \lambda\binom{n-m}{h-i} \qquad \forall X \subseteq V(\mathcal{G}), |X| = i, 0 \leq i \leq h-1. \tag{IV.2}$$

By an $X\alpha^j$-edge or $*\alpha^j$-edge for short, we mean an edge of the form $X \cup \{\alpha^j\}$ (so it contains j copies of α). Observe that the edges of $\mathcal{G}$ are the $*\alpha^0$-edges. In the next three subsections, we color the new (uncolored) edges of $\mathcal{F}$. When we color the edges of $\mathcal{F}$, we use $\textbf{deg}_{\mathcal{F}(j)}(v)$ and $\textbf{mult}_{\mathcal{F}(j)}(e)$ to denote the degree of some vertex v and the multiplicity of some edge e, respectively, in color class j.

IV.2.1 Coloring the $*\boldsymbol{\alpha^i}$-edges, $\boldsymbol{i \in [h-2]}$

We color the $*\alpha$-edges, $*\alpha^2$-edges, $\dots$, $*\alpha^{h-2}$-edges of $\mathcal{F}$, in that particular order, such that

$$\textbf{deg}_{\mathcal{F}(j)}(x) \leq r_j \qquad \forall x \in V(\mathcal{G}), j \in [k]. \tag{IV.3}$$

Suppose to the contrary that for some $i \in [h-2]$, there is an $*\alpha^i$-edge e in $\mathcal{F}$ that cannot be colored. Let $e = X \cup \{\alpha^i\}$ where X is an $(h-i)$-subset of $V(\mathcal{G})$. Then for each $j \in [k]$, there is some $x \in X$ such that $\textbf{deg}_{\mathcal{F}(j)}(x) = r_j$, and consequently, for all $j \in [k]$,

$\sum_{x \in X} \deg_{\mathcal{F}(j)}(x) \geq r_j$ (otherwise, the edge could be colored).

Since $|X| = h - i$, the multiplicity of every edge is λ, and for each $x \in X$, we have

- $\deg_{\mathcal{G}}(x) = \binom{m-1}{h-1}$
- there are $\sum_{\ell=1}^{i} \binom{n-m}{\ell}\binom{m-1}{h-\ell-1}$ new edges which contain x and at most i copies of α, and
- at least one edge containing x cannot be colored.

On the one hand,

$$\sum_{j=1}^{k} \sum_{x \in X} \deg_{\mathcal{F}(j)}(x) \geq \sum_{j=1}^{k} r_j = \lambda \binom{n-1}{h-1},$$

and on the other hand,

$$\sum_{j=1}^{k} \sum_{x \in X} \deg_{\mathcal{F}(j)}(x) \leq \lambda(h-i) \left[\binom{m-1}{h-1} + \sum_{\ell=1}^{i} \binom{n-m}{\ell} \binom{m-1}{h-\ell-1} - 1 \right].$$

Thus, we have

$$\lambda \binom{n-1}{h-1} \leq \lambda(h-i) \left[\binom{m-1}{h-1} + \sum_{\ell=1}^{i} \binom{n-m}{\ell} \binom{m-1}{h-\ell-1} - 1 \right].$$

We shall prove that this is a contradiction by establishing that $f(i) > 0$ for $i \in [h-2]$ where

$$f(i) := \binom{n-1}{h-1} - (h-i) \left[\sum\nolimits_{\ell=0}^{i} \binom{n-m}{\ell} \binom{m-1}{h-\ell-1} - 1 \right], \quad i \in [h-2].$$

Since

$$
\begin{aligned}
f(h-2) = &\binom{n-1}{h-1} - 2\left[\sum\nolimits_{\ell=0}^{h-2} \binom{n-m}{\ell}\binom{m-1}{h-\ell-1} - 1\right] \\
&= \binom{n-1}{h-1} - 2\left[\binom{n-1}{h-1} - \binom{n-m}{h-1}\right] + 2 \\
&= 2\binom{n-m}{h-1} - \binom{n-1}{h-1} + 2,
\end{aligned}
$$

the following proves that $f(h-2) > 0$ for $n \geq (h-1)(2m-1)$.

$$
\begin{aligned}
\binom{n-m}{h-1} \Big/ \binom{n-1}{h-1} &= \frac{(n-m)!(n-h)!}{(n-1)!(n-m-h+1)!} \\
&= \prod_{i=1}^{h-1} \frac{n-m-i+1}{n-i} \\
&= \prod_{i=1}^{h-1} \left(1 - \frac{m-1}{n-i}\right) \\
&\geq \prod_{i=1}^{h-1} \left(1 - \frac{m-1}{n-h+1}\right) \\
&= \left(1 - \frac{m-1}{n-h+1}\right)^{h-1} \\
&\geq 1 - \frac{(h-1)(m-1)}{n-h+1} \\
&\geq 1 - \frac{(h-1)(m-1)}{(h-1)(2m-1)-(h-1)} \\
&= 1 - \frac{(h-1)(m-1)}{2(h-1)(m-1)} = \frac{1}{2}.
\end{aligned}
$$

Now let

$$
\begin{aligned}
g(i) &= f(i+1) - f(i) \\
&= \sum\nolimits_{\ell=0}^{i+1} \binom{n-m}{\ell}\binom{m-1}{h-\ell-1} - (h-i)\binom{m-1}{h-i-2}\binom{n-m}{i+1} - 1, \quad i \in [h-4].
\end{aligned}
$$

We will show that g is strictly decreasing for $i \in [h-4]$. If g is decreasing, then there exists an i with $0 \le i \le h-4$ such that $g(i) \ge 0$ for $1 \le i \le a$ and $g(i) \le 0$ for $a+1 \le i \le h-4$. Therefore, $f(a+1) > f(a) > \cdots > f(1)$ and $f(a+2) > f(a+3) > \cdots > f(h-3)$. Hence, to show $f(i) > 0$ for $i \in [h-3]$, it suffices to show that g is decreasing, $f(1) > 0$, and $f(h-3) > 0$.

Since

$$\begin{aligned} g(i) - g(i+1) &= (h-i-2)\binom{m-1}{h-i-3}\binom{n-m}{i+2} - (h-i)\binom{m-1}{h-i-2}\binom{n-m}{i+1} \\ &= \binom{n-m}{i+1}\binom{m-1}{h-i-3}\left(\frac{(h-i-2)(n-m-i-1)}{i+2} - \frac{(h-i)(m-h+i+2)}{h-i-2}\right), \end{aligned}$$

for $i \in [h-4]$, $g(i) > g(i+1)$ if and only if

$$(h-i-2)^2(n-m-i-1) > (h-i)(i+2)(m-h+i+2). \tag{IV.4}$$

Since $i \le h-4$, we have $\dfrac{1}{h-i-2} \le \dfrac{1}{2}$ and $\dfrac{h-i}{h-i-2} \le 2$, so $\dfrac{h-i}{(h-i-2)^2} \le 1$. Therefore,

$$\frac{(h-i)(i+2)(m-h+i+2)}{(h-i-2)^2} + m + i + 1 \le (h-2)(m-2) + m + h - 3 < n.$$

This proves (IV.4), and consequently, g is strictly decreasing for $i \in [h-4]$.

Since

$$
\begin{aligned}
f(1) &= \binom{n-1}{h-1} - (h-1)\left[\binom{m-1}{h-1} + (n-m)\binom{m-1}{h-2} - 1\right] \\
&> \binom{n-1}{h-1} - (h-1)\binom{m-1}{h-1} - (h-1)(n-m)\binom{m-1}{h-2} \\
&= \binom{n-1}{h-1} - \binom{m-1}{h-2}\Big((h-1)(n-m) + m - h + 1\Big) \\
&> \binom{n-1}{h-1} - (h-1)(n-1)\binom{m-1}{h-2},
\end{aligned}
$$

the following proves that $f(1) > 0$.

$$
\begin{aligned}
\frac{\binom{n-1}{h-1}}{\binom{m-1}{h-2}} &= \frac{(n-1)!(m-h+1)!}{(h-1)(n-h)!(m-1)!} = \frac{n-1}{h-1}\prod_{i=1}^{h-2}\frac{n-i-1}{m-i} \\
&= \frac{n-1}{h-1}\prod_{i=1}^{h-2}\left(1 + \frac{n-m-1}{m-i}\right) \\
&\geq \frac{n-1}{h-1}\prod_{i=1}^{h-2}\left(1 + \frac{n-m-1}{m}\right) \\
&= \frac{n-1}{h-1}\left(1 + \frac{n-m-1}{m}\right)^{h-2} \\
&\geq \frac{n-1}{h-1}\left(1 + \frac{(h-2)(n-m-1)}{m}\right) \\
&\geq \frac{n-1}{h-1}\left(1 + \frac{hm(h-2)}{m}\right) = (n-1)(h-1).
\end{aligned}
$$

Now, we show that $f(h-3) > 0$. Since

$$
\begin{aligned}
f(h-3) &= \binom{n-1}{h-1} - 3\left[\sum_{\ell=0}^{h-3}\binom{n-m}{\ell}\binom{m-1}{h-\ell-1} - 1\right] \\
&= \binom{n-1}{h-1} - 3\left[\binom{n-1}{h-1} - \binom{n-m}{h-1} - (m-1)\binom{n-m}{h-2}\right] + 3 \\
&= 3\binom{n-m}{h-1} + 3(m-1)\binom{n-m}{h-2} - 2\binom{n-1}{h-1} + 3,
\end{aligned}
$$

the following proves that $f(h-3) > 0$.

$$
\begin{aligned}
&\left[\binom{n-m}{h-1} + (m-1)\binom{n-m}{h-2}\right] \Big/ \binom{n-1}{h-1} \\
&\quad = \binom{n-m}{h-2}\left(\frac{n-m-h+2}{h-1} + m - 1\right) \Big/ \binom{n-1}{h-1} \\
&\quad = \left(\frac{n-m-h+2}{h-1} + m - 1\right)\frac{(h-1)(n-m)!(n-h)!}{(n-1)!(n-m-h+2)!} \\
&\quad = \left(\frac{n-m-h+2}{h-1} + m - 1\right)\frac{h-1}{n-m-h+2}\prod_{i=1}^{h-1}\frac{n-m-i+1}{n-i} \\
&\quad \geq \left(1 + \frac{(h-1)(m-1)}{n-h+1}\right)\prod_{i=1}^{h-1}\left(1 - \frac{m-1}{n-i}\right) \\
&\quad \geq \left(1 + \frac{(h-1)(m-1)}{n-h+1}\right)\left(1 - \frac{m-1}{n-h+1}\right)^{h-1} \\
&\quad \geq \left(1 + \frac{(h-1)(m-1)}{n-h+1}\right)\left(1 - \frac{(h-1)(m-1)}{n-h+1}\right) \\
&\quad \geq 1 - \left(\frac{(h-1)(m-1)}{2(h-1)(m-1)}\right)^2 = 1 - \frac{1}{4} > \frac{2}{3}.
\end{aligned}
$$

IV.2.2 **Coloring the $*\boldsymbol{\alpha}^{\boldsymbol{h-1}}$-edges** We color the $*\alpha^{h-1}$-edges such that

$$
\mathbf{mult}_{\mathcal{F}(j)}(x\alpha^{h-1}) = r_j - \mathbf{deg}_{\mathcal{F}(j)}(x) \quad \forall x \in V(\mathcal{G}), j \in [k].
$$

This is possible, because for $x \in V(\mathcal{G})$,

$$
\begin{aligned}
\sum_{j=1}^{k}\Big(r_j - \mathbf{deg}_{\mathcal{F}(j)}(x)\Big) &= \sum_{j=1}^{k} r_j - \sum_{j=1}^{k} \mathbf{deg}_{\mathcal{F}(j)}(x) \\
&= \lambda\binom{n-1}{h-1} - \mathbf{deg}_{\mathcal{F}}(x) \\
&= \lambda\binom{n-1}{h-1} - \sum_{\ell=1}^{h-1}\lambda\binom{m}{\ell}\binom{n-m}{h-\ell-1} \\
&= \lambda\sum_{\ell=0}^{h-1}\binom{m}{\ell}\binom{n-m}{h-\ell-1} - \lambda\sum_{\ell=1}^{h-1}\binom{m}{\ell}\binom{n-m}{h-\ell-1} \\
&= \lambda\binom{n-m}{h-1} = \mathbf{mult}_{\mathcal{F}}(x\alpha^{h-1}).
\end{aligned}
$$

IV.2.3 **Coloring the $\boldsymbol{\alpha^h}$-edges** Recall that $n \geq (h-1)(2m-1)$ and $h \mid r_j n$ for $j \in [k]$. Hence, we color the α^h-edges such that

$$
\mathbf{mult}_{\mathcal{F}(j)}(\alpha^h) = \frac{r_j n}{h} - r_j m + \sum_{\ell=0}^{h-2}(h-\ell-1)\,\mathbf{mult}_{\mathcal{F}(j)}(*\alpha^{\ell}) \quad \forall j \in [k].
$$

Let us verify that this is in fact possible.

$$
\begin{aligned}
\sum_{j=1}^{k} \mathbf{mult}_{\mathcal{F}(j)}(\alpha^h) &= \sum_{j=1}^{k} \left(\frac{r_j n}{h} - r_j m + \sum_{\ell=0}^{h-2} (h-\ell-1)\, \mathbf{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \right) \\
&= \frac{n}{h} \sum_{j=1}^{k} r_j - m \sum_{j=1}^{k} r_j + \sum_{j=1}^{k} \sum_{\ell=0}^{h-2} (h-\ell-1)\, \mathbf{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= \lambda \frac{n}{h} \binom{n-1}{h-1} - \lambda m \binom{n-1}{h-1} + \sum_{\ell=0}^{h-2} (h-\ell-1) \sum_{j=1}^{k} \mathbf{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= \lambda \binom{n}{h} - \lambda m \binom{n-1}{h-1} + \lambda \sum_{\ell=0}^{h-2} (h-\ell-1) \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \sum_{\ell=0}^{h} \binom{m}{h-\ell} \binom{n-m}{\ell} - \lambda m \binom{n-1}{h-1} \\
&\quad + \lambda \sum_{\ell=0}^{h-2} (h-\ell) \binom{m}{h-\ell} \binom{n-m}{\ell} - \lambda \sum_{\ell=0}^{h-2} \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \binom{n-m}{h} + \lambda m \binom{n-m}{h-1} - \lambda m \binom{n-1}{h-1} \\
&\quad + \lambda \sum_{\ell=0}^{h-2} (h-\ell) \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \binom{n-m}{h} - \lambda m \binom{n-1}{h-1} + \lambda \sum_{\ell=0}^{h-1} (h-\ell) \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \binom{n-m}{h} - \lambda m \binom{n-1}{h-1} + \lambda h \binom{m}{h} + \lambda \sum_{\ell=1}^{h-1} (h-\ell) \binom{m}{h-\ell} \binom{n-m}{\ell} \\
&= \lambda \binom{n-m}{h} - \lambda m \binom{n-1}{h-1} + \lambda m \binom{m-1}{h-1} + \lambda m \left[\binom{n-1}{h-1} - \binom{m-1}{h-1} \right] \\
&= \lambda \binom{n-m}{h} = \mathbf{mult}_{\mathcal{F}}(\alpha^h).
\end{aligned}
$$

IV.2.4 **Regularity of the Coloring of $\mathcal{F}$** As a result of the coloring of the $*\alpha^{h-1}$-edges, we have

$$
\mathbf{deg}_{\mathcal{F}(j)}(x) = r_j \quad \forall x \in V(\mathcal{G}), j \in [k], \tag{IV.5}
$$

and so,

$$r_j m = \sum_{x \in V(\mathcal{G})} \mathrm{deg}_{\mathcal{F}(j)}(x) = \sum_{\ell=0}^{h-1} (h-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \quad \forall j \in [k]. \tag{IV.6}$$

Hence,

$$\begin{aligned}
\mathrm{deg}_{\mathcal{F}(j)}(\alpha) &= \sum_{\ell=0}^{h} \ell\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= h\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^h) + \sum_{\ell=0}^{h-1} \ell\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= h\left(\frac{r_j n}{h} - r_j m + \sum_{\ell=0}^{h-2}(h-\ell-1)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell)\right) + \sum_{\ell=0}^{h-1} \ell\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= r_j n - h r_j m + h\sum_{\ell=0}^{h-2}(h-\ell-1)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) + \sum_{\ell=0}^{h-1} \ell\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= r_j n - h\sum_{\ell=0}^{h-1}(h-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) + h\sum_{\ell=0}^{h-2}(h-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&\qquad - h\sum_{\ell=0}^{h-2}\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) + \sum_{\ell=0}^{h-1} \ell\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= r_j n - h\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^{h-1}) - h\sum_{\ell=0}^{h-2}\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) + \sum_{\ell=0}^{h-1} \ell\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= r_j n - h\sum_{\ell=0}^{h-1}\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) + \sum_{\ell=0}^{h-1} \ell\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= r_j n - \sum_{\ell=0}^{h-1}(h-\ell)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^\ell) \\
&= r_j n - r_j m = r_j(n-m) \qquad \forall j \in [k].
\end{aligned}$$

IV.2.5 **A Fair Detachment of $\mathcal{F}$** By [5, Theorem 4.1], there exists an n-vertex hypergraph $\mathcal{H}$, called the *fair* $(\alpha, n-m)$-*detachment* of $\mathcal{F}$, obtained by replacing the vertex α of $\mathcal{F}$ by $n-m$ new vertices $\alpha_1, \dots, \alpha_{n-m}$ in $\mathcal{H}$, and replacing each $X\alpha^i$-edge by an edge of the form $X \cup U$ where $U \subseteq \{\alpha_1, \dots, \alpha_{n-m}\}, |U| = i \in [h]$ (leaving the remaining vertices and edges of $\mathcal{F}$ intact), such that the edges incident with α (in each color class of $\mathcal{F}$) are shared as evenly as possible among $\alpha_1, \dots, \alpha_{n-m}$ in $\mathcal{H}$ in the following way.

(a) For $i \in [n-m], j \in [k]$,

$$\deg_{\mathcal{H}(j)}(\alpha_i) = \frac{\deg_{\mathcal{F}(j)}(\alpha)}{n-m} = \frac{r_j(n-m)}{n-m} = r_j;$$

(b) For $X \subseteq V(\mathcal{G}), U \subseteq \{\alpha_1, \dots, \alpha_{n-m}\}, |X| = h - i, |U| = i \in [h]$,

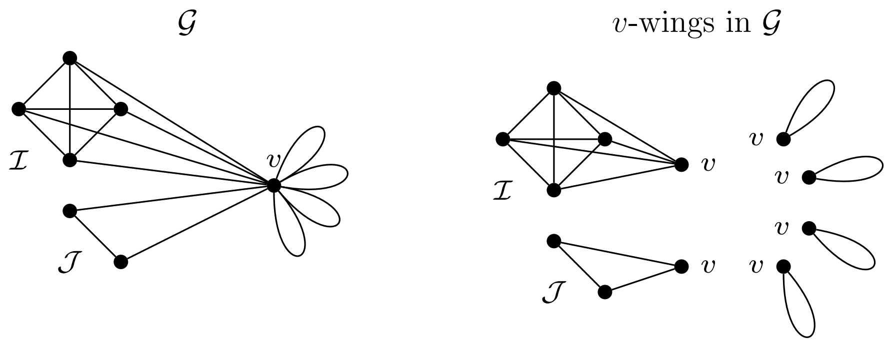
$$\text{mult}_{\mathcal{H}}(XU) = \frac{\text{mult}_{\mathcal{F}}(X\alpha^i)}{\binom{n-m}{i}} = \frac{\lambda\binom{n-m}{i}}{\binom{n-m}{i}} = \lambda. \tag{IV.7}$$

Here, $\text{mult}_{\mathcal{H}}(XU)$ is the number of occurrences of an edge of the form $X \cup U$. Observe that by (b), $\mathcal{H} \cong \lambda K_n^h$, and by (IV.5) and (a), $\mathcal{H}(i)$ is an r_i-factor for $i \in [k]$. This completes the proof of Theorem IV.0.1.

IV.3 Proof of Theorem IV.0.2

A vertex v in a connected hypergraph $\mathcal{G}$ is a *cut vertex* if there exist two non-trivial sub-hypergraphs I, J of $\mathcal{G}$ such that $I \cup J = \mathcal{G}$, $V(I \cap J) = \{v\}$, and $E(I \cap J) = \varnothing$. A sub-hypergraph W of a hypergraph $\mathcal{G}$ is an *v-wing* of $\mathcal{G}$ if (i) W is non-trivial and connected, (ii) v is not a cut vertex of W, and (iii) no edge in $E(\mathcal{G}) \backslash E(W)$ is incident with a vertex in $V(W) \backslash \{v\}$. A v-wing W is *large* if $V(W) \neq \{v\}$, and is *small* if $V(W) = \{v\}$. Let $\omega_v(\mathcal{G})$, and $\omega_v^L(\mathcal{G})$ be the number of v-wings, and the number of large v-wings in $\mathcal{G}$, respectively. Let $c(\mathcal{G})$ denote the number of components of $\mathcal{G}$.

Example: Hypergraph $\mathcal{G}$ with cut vertex v, $\omega_v(\mathcal{G}) = 6$, and $\omega_v^L(\mathcal{G}) = 2$.



An r-factor cannot be connected unless if $r \geq 2$. Moreover, if a component of a color class of λK_m^h is r-regular, then there is no way to extend it to a connected r-factor in λK_n^h. This justifies the necessity of the conditions of Theorem IV.0.3.

Now let $\mathcal{G}_1 := \lambda K_m^h$, $\mathcal{G} \subseteq \mathcal{G}_1$. Suppose that $n \geq (h-1)(2m-1)$ and $(n, h, \lambda, \mathbf{r})$ is admissible where $\mathbf{s} = (s_1, \ldots, s_q), \mathbf{r} = (r_1, \ldots, r_k)$ such that

$$\sum_{i=1}^{q} \left\lfloor \frac{(r_i - s_i)m}{h} \right\rfloor + \sum_{i=q+1}^{k} \left\lfloor \frac{r_i m}{h} \right\rfloor \geq \lambda \binom{m}{h},$$

and suppose a partial $\mathbf{s}$-factorization of $\mathcal{G}$ is given. It suffices to extend the given partial $\mathbf{s}$-factorization of $\mathcal{G}$ to a partial $\mathbf{r}$-factorization of $\mathcal{G}_1$ as by Theorem IV.0.1, we may extend this partial $\mathbf{r}$-factorization of $\mathcal{G}_1$ to an $\mathbf{r}$-factorization of λK_n^h.

Let $\mathcal{F}$ be a hypergraph whose vertex set is $\{\alpha\}$ and has $\lambda\binom{m}{h}$ copies of an edge of the form $\{\alpha^h\}$. In other words,

$$V(\mathcal{F}) = \{\alpha\}, \quad \textbf{mult}_{\mathcal{F}}(\alpha^h) = \lambda \binom{m}{h}.$$

We color the edges of $\mathcal{F}$ such that

$$\mathrm{mult}_{\mathcal{F}(i)}(\alpha^h) \leq \begin{cases} \left\lfloor \dfrac{(r_i - s_i)m}{h} \right\rfloor & \text{for } i \in [q], \\ \left\lfloor \dfrac{r_i m}{h} \right\rfloor & \text{for } i \in [k]\backslash[q]. \end{cases}$$

By [5, Theorem 4.1], there exists an m-vertex hypergraph $\mathcal{H}$ obtained by replacing the vertex α of $\mathcal{F}$ by m new vertices $\alpha_1, \ldots, \alpha_m$ in $\mathcal{H}$, and replacing each α^h-edge by an edge of the form U where $U \subseteq \{\alpha_1, \ldots, \alpha_m\}, |U| = h$ such that the edges incident with α (in each color class of $\mathcal{F}$) are shared as evenly as possible among $\alpha_1, \ldots, \alpha_m$ in $\mathcal{H}$ in the following way.

$$\deg_{\mathcal{H}(j)}(\alpha_i) \approx \frac{\deg_{\mathcal{F}(j)}(\alpha)}{m} \leq \frac{h}{m} \left\lfloor \frac{(r_j - s_j)m}{h} \right\rfloor \leq r_j - s_j \qquad \forall i \in [m], j \in [q]; \tag{IV.8}$$

$$\deg_{\mathcal{H}(j)}(\alpha_i) \approx \frac{\deg_{\mathcal{F}(j)}(\alpha)}{m} \leq \frac{h}{m} \left\lfloor \frac{r_j m}{h} \right\rfloor \leq r_j \qquad \forall i \in [m], j \in [k]\backslash[q]; \tag{IV.9}$$

$$\mathrm{mult}_{\mathcal{H}}(U) = \mathrm{mult}_{\mathcal{F}}(\alpha^h) \Big/ \binom{m}{h} = \lambda \binom{m}{h} \Big/ \binom{m}{h} = \lambda \qquad \forall U \subseteq \{\alpha_1, \ldots, \alpha_n\}, |U| = h. \tag{IV.10}$$

Here, $x \approx y$ means $\lfloor y \rfloor \leq x \leq \lceil y \rceil$. By (IV.10), $\mathcal{H} \cong \lambda K_m^h$, and by (IV.8) and (IV.9), the coloring of $\mathcal{H}$ induces a partial $(r_1 - s_1, \ldots, r_q - s_q, r_{q+1}, \ldots, r_k)$-factorization. We color each edge of $\mathcal{G}_1 \backslash \mathcal{G}$ with color of the corresponding edge in $\mathcal{H}$. This leads to a partial $\mathbf{r}$-factorization of $\mathcal{G}_1$, as desired, and completes the proof of Theorem IV.0.2.

IV.4 Proof of Theorem IV.0.3

Suppose that $(n, h, \lambda, \mathbf{r})$ is admissible, $n \geq (h-1)(2m-1)$, and a partial $\mathbf{r}$-factorization of $\mathcal{G} := \lambda K_m^h$ is given. In the introduction, we established that it is necessary for $\lambda K_m^h(j)$ to be r_j-irregular and for $r_j \geq 2$ for all $j \in [k]$. Let $\mathcal{F}$ be the hypergraph defined in Section IV.2 whose edges are colored according to the coloring described in Subsections IV.2.1–IV.2.4. Let us fix $j \in [k]$ and assume that $r_j \geq 2$ and that no component of $\mathcal{G}(j)$ is r_j-regular. Since $\mathbf{deg}_{\mathcal{F}(j)}(u) = r_j$ for all $u \in V(\mathcal{G})$ and no component of $\mathcal{G}(j)$ is r_j-regular, there must be at least one edge joining α and each component of $\mathcal{G}(j)$. Hence, $\mathcal{F}(j)$ must be connected. We shall prove that $\mathcal{H}(j)$, constructed in Subsection IV.2.5, is connected.

IV.4.1 An Upper Bound for the Number of Wings

We claim that

$$\omega_\alpha(\mathcal{F}(j)) \leq (r_j - 1)(n-m) + 1. \tag{IV.11}$$

Since every $*\alpha^h$-edge is a small α-wing in $\mathcal{F}(j)$, and each component of $\mathcal{G}(j)$ corresponds to at most one large α-wing in $\mathcal{F}(j)$, we have

$$\begin{aligned}
\omega_\alpha(\mathcal{F}(j)) &= \mathrm{mult}_{\mathcal{F}(j)}(\alpha^h) + \omega_\alpha^L(\mathcal{F}(j)) \\
&\leq \mathrm{mult}_{\mathcal{F}(j)}(\alpha^h) + c(\mathcal{G}(j)) \\
&= \frac{r_j n}{h} - r_j m + \sum_{i=1}^{h-1}(h-i)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^{i-1}) + c(\mathcal{G}(j)).
\end{aligned}$$

Thus, to prove (IV.11), it suffices to show that

$$\frac{r_j n}{h} - r_j m + \sum_{i=1}^{h-1}(h-i)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^{i-1}) + c(\mathcal{G}(j)) \leq (r_j - 1)(n-m) + 1,$$

which is equivalent to showing

$$r_j n\left(1-\frac{1}{h}\right)-n+m+1\geq\sum_{i=1}^{h-1}(h-i)\,\mathsf{mult}_{\mathcal{F}(j)}(*\alpha^{i-1})+c(\mathcal{G}(j)). \tag{IV.12}$$

For $i\in[h-1]$, an $*\alpha^{h-i}$-edge in $\mathcal{F}$ contains i vertices of $V(\mathcal{G})$ which are contained in at most i different components of $\mathcal{G}(j)$. Thus, an $*\alpha^{h-i}$-edge connects at most i components of $\mathcal{G}(j)$. It follows that the maximum number of components of $\mathcal{G}(j)$ which are connected by the addition of new edges of the form $*\alpha^{h-i}$ for $i\in[h-1]$ is $\sum_{i=1}^{h-1} i\,\mathsf{mult}_{\mathcal{F}(j)}(*\alpha^{h-i})$. Note that edges of the form α^h do not contain any vertices $v\in V(\mathcal{G})$ and thus do not connect any components of $\mathcal{G}(j)$. Since $\mathcal{F}(j)$ is connected and is obtained by the addition of the edges of the form $*\alpha^{h-i}$ for $0\leq i\leq h-1$ to $\mathcal{G}(j)$, we have

$$c(\mathcal{G}(j))\leq\sum_{i=1}^{h-1} i\,\mathsf{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}).$$

Therefore,

$$\frac{c(\mathcal{G}(j))}{h-1}\leq\sum_{i=1}^{h-1}\frac{i}{h-1}\,\mathsf{mult}_{\mathcal{F}(j)}(*\alpha^{h-i})\leq\sum_{i=1}^{h-1}(h-i)\,\mathsf{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}). \tag{IV.13}$$

Using (IV.6), we have

$$
\begin{aligned}
r_j m \left(1 - \frac{1}{h}\right) - \frac{1}{h} \sum_{i=1}^{h-1} (h-i) \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}) \\
&= \left(1 - \frac{1}{h}\right) \sum_{i=1}^{h} i \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}) - \frac{1}{h} \sum_{i=1}^{h-1} (h-i) \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}) \\
&= \sum_{i=1}^{h-1} \left[i \left(1 - \frac{1}{h}\right) - \left(1 - \frac{i}{h}\right) \right] \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}) + (h-1) \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{0}) \\
&= \sum_{i=1}^{h-1} (i-1) \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}) + (h-1) \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{0}) \\
&= \sum_{i=1}^{h} (i-1) \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}) \\
&= \sum_{i=1}^{h-1} (h-i) \operatorname{mult}_{\mathcal{F}(j)}(*\alpha^{i-1}). \qquad \text{(IV.14)}
\end{aligned}
$$

Moreover, the number of components in $\mathcal{G}$ cannot exceed the number of vertices in $V(\mathcal{G})$, so $c(\mathcal{G}(j)) \leq m$. Now as $n \geq (h-1)(2m-1)$ and $r_j \geq 2$, we have the following which proves (IV.12) for $h > 3$.

$$
\begin{aligned}
r_j n\left(1-\frac{1}{h}\right) - n + m + 1 &\geq hm\left(r_j\left(1-\frac{1}{h}\right)-1\right) + m + 1 \\
&= r_j m\left(1-\frac{1}{h}\right) + (h-1)r_j m\left(1-\frac{1}{h}\right) - hm + m + 1 \\
&= r_j m\left(1-\frac{1}{h}\right) + m\Big((h-1)r_j - h\Big) - m\left(\frac{h-1}{h}r_j - 1\right) + 1 \\
&= r_j m\left(1-\frac{1}{h}\right) + m\Big((h-1)r_j - h\Big)\left(1-\frac{1}{h}\right) + 1 \\
&\geq r_j m\left(1-\frac{1}{h}\right) + m(h-2)\left(1-\frac{1}{h}\right) + 1 \\
&= r_j m\left(1-\frac{1}{h}\right) + m\left(\frac{h^3-4h^2+5h-2}{h(h-1)}\right) \\
&\geq r_j m\left(1-\frac{1}{h}\right) + m\left(\frac{h^2-h-1}{h(h-1)}\right) \\
&= r_j m\left(1-\frac{1}{h}\right) + \left(1-\frac{1}{h(h-1)}\right)m \\
&\geq r_j m\left(1-\frac{1}{h}\right) + \left(1-\frac{1}{h(h-1)}\right)c(\mathcal{G}(j)) \\
&= r_j m\left(1-\frac{1}{h}\right) - \frac{1}{h}\left(\frac{c(\mathcal{G}(j))}{h-1}\right) + c(\mathcal{G}(j)) \\
&\geq r_j m\left(1-\frac{1}{h}\right) - \frac{1}{h}\sum_{i=1}^{h-1}(h-i)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^{h-i}) + c(\mathcal{G}(j)) \\
&= \sum_{i=1}^{h-1}(h-i)\,\mathrm{mult}_{\mathcal{F}(j)}(*\alpha^{i-1}) + c(\mathcal{G}(j)).
\end{aligned}
$$

IV.4.2 **Connected Detachments** By [8, Theorem 1.1], in the fair $(\alpha, n-m)$-detachment $\mathcal{H}$ of $\mathcal{F}$, $\mathcal{H}(j)$ is connected if and only if

$$\mathrm{deg}_{\mathcal{F}(j)}(\alpha) - \omega_\alpha(\mathcal{F}(j)) \geq n - m - 1. \tag{IV.15}$$

In (IV.11), we showed that

$$\omega_\alpha(\mathcal{F}(j)) \leq (r_j - 1)(n-m) + 1.$$

Moreover, recall that $\mathbf{deg}_{\mathcal{F}(j)}(\alpha) = r_j(n-m)$. Hence

$$\mathbf{deg}_{\mathcal{F}(j)}(\alpha) - \omega_\alpha(\mathcal{F}(j)) \geq r_j(n-m) - (r_j-1)(n-m) - 1 = n-m-1.$$

This completes the proof of Theorem IV.0.3.

IV.5 Proof of Theorem IV.0.4

Suppose that $n \geq (h-1)(2m-1)$ and $(n, h, \lambda, \mathbf{r})$ is admissible where $\mathbf{s} = (s_1, \ldots, s_q), \mathbf{r} = (r_1, \ldots, r_k)$ such that

$$\sum_{i \in B} \left\lfloor \frac{(\overline{r_i} - s_i)m}{h} \right\rfloor + \sum_{i \in [k] \backslash [q]} \left\lfloor \frac{\overline{r_i} m}{h} \right\rfloor \geq \lambda \binom{m}{h},$$

where $A \subseteq \{i \in [k] \mid r_i \geq 2\}, B = \{i \in [q] \mid r_i \neq s_i\}$, $\overline{r_i} := r_i - 1$ if $i \in A$, and $\overline{r_i} := r_i$ if $i \in [k] \backslash A$. Assume that a partial $\mathbf{s}$-factorization of $\mathcal{G} \subseteq \mathcal{G}_1 := \lambda K_m^h$ is given such that $\mathcal{G}(i)$ is r_i-irregular for $i \in A$. By Theorem IV.0.3, it suffices to extend the given partial $\mathbf{s}$-factorization of $\mathcal{G}$ to a partial $\mathbf{r}$-factorization of $\mathcal{G}_1$ in such a way that no component of color class i of the partial $\mathbf{r}$-factorization of $\mathcal{G}_1$ is r_i-regular for $i \in A$.

Let $\mathcal{F}$ be a hypergraph whose vertex set is $\{\alpha\}$ and has $\lambda\binom{m}{h}$ copies of an edge of the form $\{\alpha^h\}$. We color the edges of $\mathcal{F}$ such that

$$\mathbf{mult}_{\mathcal{F}(i)}(\alpha^h) \leq \begin{cases} \left\lfloor \dfrac{(r_i - s_i)m}{h} \right\rfloor & \text{for } i \in [q] \backslash (A \cap B), \\ \left\lfloor \dfrac{(r_i - s_i - 1)m}{h} \right\rfloor & \text{for } i \in A \cap B, \\ \left\lfloor \dfrac{r_i m}{h} \right\rfloor & \text{for } i \in ([k] \backslash [q]) \backslash A, \\ \left\lfloor \dfrac{(r_i - 1)m}{h} \right\rfloor & \text{for } i \in A \cap ([k] \backslash [q]), \end{cases}$$

By [5, Theorem 4.1], there exists an m-vertex hypergraph $\mathcal{H}$ obtained by replacing the vertex α of $\mathcal{F}$ by m new vertices $\alpha_1, \ldots, \alpha_m$ in $\mathcal{H}$, and replacing each α^h-edge by an edge of the form U where $U \subseteq \{\alpha_1, \ldots, \alpha_m\}, |U| = h$ such that the edges incident with α (in each color class of $\mathcal{F}$) are shared as evenly as possible among $\alpha_1, \ldots, \alpha_m$ in $\mathcal{H}$ in the following way.

$$\mathbf{mult}_{\mathcal{H}}(U) = \mathbf{mult}_{\mathcal{F}}(\alpha^h) \Big/ \binom{m}{h} = \lambda \binom{m}{h} \Big/ \binom{m}{h} = \lambda \qquad \forall U \subseteq \{\alpha_1, \ldots, \alpha_n\}, |U| = h;$$

$$\forall i \in [m] : \mathbf{deg}_{\mathcal{H}(j)}(\alpha_i) \approx \frac{\mathbf{deg}_{\mathcal{F}(j)}(\alpha)}{m} \leq \frac{h\, \mathbf{mult}_{\mathcal{F}(j)}(\alpha^h)}{m}$$
$$\leq \begin{cases} r_j - s_j & \text{for } j \in [q] \backslash (A \cap B), \\ r_j - s_j - 1 & \text{for } j \in A \cap B, \\ r_j & \text{for } j \in ([k] \backslash [q]) \backslash A, \\ r_j - 1 & \text{for } j \in A \cap ([k] \backslash [q]). \end{cases}$$

To obtain a partial $\mathbf{r}$-factorization of $\mathcal{G}_1$, we color each edge of $\mathcal{G}_1 \backslash \mathcal{G}$ with color of the corresponding edge in $\mathcal{H}$. Recall that no component of color class i of the partial $\mathbf{s}$-factorization of $\mathcal{G}$ is r_i-regular for $i \in A$. Hence, $\mathcal{G}_1(i)$ is r_i-irregular for $i \in A$, and we are done.

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