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Towards Constructing Vertex Algebroids

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The notion of vertex algebroids were introduced in the late 1990's as a crucial tool for the study of chiral differential operators and chiral de Rham complex. Vertex algebroids play vital role in the study of $\mathbb{N}$-graded vertex algebra. Also, they have deep connection with representation theory of Leibniz algebras. However, the classification of irreducible modules of vertex algebroids is not completed.

The aim of this thesis is to investigate the possibility of using the simple Lie algebra G_2 and its irreducible modules to construct vertex A-algebroids B that contain G_2 as their Levi factor. Under very mild and natural assumptions, I find some exact properties on the algebraic structure of A and B that will provide precise algebraic structure of the vertex A-algebroid B.

KEYWORDS: Leibniz algebra, Lie algebra, vertex algebroid

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CONTENTS

CHAPTER I: INTRODUCTION

In the late 1990's, the notion of vertex algebroid was introduced by Gorbounov, Malikov and Schechtman for the study of chiral differential operators and chiral de Rham complex [GMS, MS1, MS2, MSV]. Since then vertex algebroids have been played a prominent role in the study of representation theory of $\mathbb{N}$-graded vertex algebras (see [BuY, GMS, MS1, MS2, MSV, JY1, JY2, LiY1, LiY2]). However, the classification of vertex algebroids is far from being completed. It is primarily important for us to attempt to classify simple vertex algebroids.

There has been some progress towards the study of representation theory of vertex algebroids. Jitjankarn and Yamskulna investigated the algebraic structure of vertex A-algebroids B such that B are semisimple Leibniz algebras that have the simple Lie algebra sl_2 as their Levi factor [JY1]. Also, Bui and Yamskulna generalized the results in [JY1] by studying the algebraic structure of any vertex A-algebroids B as modules of the simple Lie algebras, and showed that if sl_2 is contained in B then A is a local algebra. In addition, Bui and Yamskulna constructed vertex algebroids $B_{\mathfrak{g}}$ from Lie algebras $\mathfrak{g}$ of ADE-type and their modules. They showed that under a suitable condition, the vertex algebroids B are equivalent to the constructed vertex algebroids $B_{\mathfrak{g}}$ ([BuY]).

In this thesis, I investigate a possibility of constructing vertex algebroids that contain the simple Lie algebra G_2 as its Levi factor. Precisely, under the assumption that A is a commutative associative algebra and B is a Leibniz algebra that is a direct sum of two irreducible G_2-modules that are both isomorphic to the simple Lie algebra G_2, I study some necessary algebraic structure of A and B that will give the precise algebraic structure of the vertex A-algebroid B.

This thesis is divided in the following way. In Chapter II, I will provide necessary background on representation theory of Lie algebras, leibniz algebras. I introduce the notion of vertex algebroids and supply their properties. Also, I include recent developments on representation theory of vertex algebroids at the end of Chapter II. This will help lay the groundwork for the partial construction of our vertex algebroid. In Chapter III, I will first layout the necessary assumptions that we need to have in order to construct vertex A-algebroid B. Next, I state the main results in Chapter III subsection III.1.3 and provide the proofs of the stated results in subsection III.2, and subsection III.3.

It should be noted that due to the selection of the wrong choice of G_2-module, there was a lot of time attempting to construct a vertex algebroid in the beginning that did not work out. As a result, I could only obtain a partial construction of the vertex A-algebroid B.

CHAPTER II: BACKGROUND

In this section, we will provide the necessary background information in order to be able to understand the new work. First, we will look at some information on Lie algebras and modules. Then we will look at Leibniz algebras and modules. In addition, we will also cover the idea of a 1-truncated conformal algebra. Finally we will cover the idea of a vertex algebroid. The main work in this paper works very closely with the vertex algebroid and Lie algebra properties.

II.1 Lie Algebras

Definition 1. *[Hu]*

We define a Lie algebra to be a vector space $\mathfrak{g}$ *over* $\mathbb{C}$ *endowed with an operation* $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *called the bracket or commutator. The bracket satisfies the following three axioms:*

1. *Bilinearity:* $\forall a \in \mathbb{C}$ *and* $\forall x, y, z, \in \mathfrak{g}$*, we have*

$$[ax + y, z] = [ax, z] + [y, z] = a[x, z] + [y, z].$$

2. *Alternativitity:* $\forall x, y \in \mathfrak{g}$ *we have*

$$[x, x] = 0 \Rightarrow [x, y] = -[y, x].$$

3. *Jacobi Identity:* $\forall x, y, z \in \mathfrak{g}$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Definition 2. *[FLM]*

Let $\mathfrak{g}$ be a Lie algebra and let $\langle \cdot, \cdot \rangle$ be a bilinear form on $\mathfrak{g}$ a bilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathbb{C}$. Then $\langle \cdot, \cdot \rangle$ is said to be invariant or $\mathfrak{g}$-invariant if

$$\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0$$

or equivalently, for $x, y, z, \in \mathfrak{g}$ we have

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle,$$

also known as the associativity of $\langle \cdot, \cdot \rangle$.

Definition 3. *[Hu]*

A subspace I of a Lie algebra $\mathfrak{g}$ is called an ideal of $\mathfrak{g}$ if $[\mathfrak{g}, I] \subset I$.

Definition 4. *[Hu]*

A Lie algebra $\mathfrak{g}$ is said to be simple if $\mathfrak{g}$ is nonzero and has no proper ideals and if the $\dim \mathfrak{g} > 1$ *(i.e., $\mathfrak{g}$ is not abelian).*

Example 1. *[Hu] (Simple Lie algebra sl_2)*

Let sl_2 be a vector space over $\mathbb{C}$ that is generated by basis vectors E, F, H, where $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
Let $[\ ,\ ] : sl_2 \times sl_2 \to sl_2$ be a bilinear product defined by $[A, B] = AB - BA$ for all $A, B \in sl_2$. Then, $(sl_2, [\ ,\ ])$ is a Lie algebra. In particular, we have

$$[H, E] = 2E, \ [H, F] = -2F, \ [E, F] = H.$$

Moreover, $(sl_2, [\ ,\ ])$ is a simple Lie algebra over $\mathbb{C}$.

Example 2. *[HF] (Simple Lie algebra G_2)*

Let G_2 be a vector space over $\mathbb{C}$ that is generated by the following standard basis vectors $H_1, H_2, X_1, ..., X_6, Y_1, ..., Y_6$. Let $[\ ,\ ] : G_2 \times G_2 \to G_2$ be a bilinear map such that

1. *$[A, B] = -[B, A]$ for $A, B \in G_2$;*

2. $[H_i, H_j] = 0$ *for all* $i, j \in \{1, 2\}$*;*

3.

$$[H_1, X_1] = 2X_1,\ [H_1, Y_1] = -2Y_1,\ [H_1, X_2] = -3X_2,\ [H_1, Y_2] = 3Y_2,$$
$$[H_1, X_3] = -X_3,\ [H_1, Y_3] = Y_3,\ [H_1, X_4] = X_4,\ [H_1, Y_4] = -Y_4,$$
$$[H_1, X_5] = 3X_5,\ [H_1, Y_5] = -3Y_5,\ [H_1, X_6] = 0,\ [H_1, Y_6] = 0,$$
$$[H_2, X_1] = -X_1,\ [H_2, Y_1] = Y_1,\ [H_2, X_2] = 2X_2,\ [H_2, Y_2] = -2Y_2,$$
$$[H_2, X_3] = X_3,\ [H_2, Y_3] = -Y_3,\ [H_2, X_4] = 0,\ [H_2, Y_4] = 0,$$
$$[H_2, X_5] = -X_5,\ [H_2, Y_5] = Y_5,\ [H_2, X_6] = X_6,\ [H_2, Y_6] = -Y_6$$

4.

$$[X_1, Y_1] = H_1,\ [X_1, X_2] = X_3,\ [X_1, Y_2] = 0,\ [X_1, X_3] = 2X_4,\ [X_1, Y_3] = -3Y_2,$$
$$[X_1, X_4] = -3X_5,\ [X_1, Y_4] = -2Y_3,\ [X_1, X_5] = 0,\ [X_1, Y_5] = Y_4,$$
$$[X_1, X_6] = 0,\ [X_1, Y_6] = 0,$$

5.

$$[Y_1, X_2] = 0,\ [Y_1, Y_2] = -Y_3,\ [Y_1, X_3] = 3X_2,\ [Y_1, Y_3] = -2Y_4,$$
$$[Y_1, X_4] = 2X_3,\ [Y_1, Y_4] = 3Y_5,\ [Y_1, X_5] = -X_4,\ [Y_1, Y_5] = [Y_1, X_6] = [Y_1, Y_6] = 0,$$

6.

$$[X_2, Y_2] = H_2, [X_2, Y_3] = Y_1,\ [X_2, X_5] = -X_6,\ [X_2, Y_6] = Y_5,$$
$$[X_2, X_3] = [X_2, X_4] = [X_2, Y_4] = [X_2, Y_5] = [X_2, X_6] = 0,$$

7.

$$[Y_2, X_3] = -X_1,\ [Y_2, Y_5] = Y_6,\ [Y_2, X_6] = -X_5,$$
$$[Y_2, Y_3] = [Y_2, X_4] = [Y_2, Y_4] = [Y_2, X_5] = [Y_2, Y_6] = 0,$$

8.

$$[X_3, Y_3] = H_1 + 3H_2,\ [X_3, X_4] = -3X_6,\ [X_3, Y_4] = 2Y_1,\ [X_3, Y_6] = Y_4$$
$$[X_3, X_5] = [X_3, Y_5] = [X_3, X_6] = 0,$$

9.

$$[Y_3, X_4] = -2X_1,\ [Y_3, Y_4] = 3Y_6,\ [Y_3, X_6] = -X_4,$$
$$[Y_3, X_5] = [Y_3, Y_5] = [Y_3, Y_6] = 0,$$

10.

$$[X_4, Y_4] = 2H_1 + 2H_2,\ [X_4, Y_5] = -Y_1,\ [X_4, Y_6] = -Y_3,$$
$$[X_4, X_5] = [X_4, X_6] = 0,$$

11.

$$[Y_4, X_5] = X_1,\ [Y_4, X_6] = X_3,\ [Y_4, Y_5] = [Y_4, Y_6] = 0,$$
$$[X_5, Y_5] = H_1 + H_2,\ [X_5, Y_6] = -Y_2,\ [X_5, X_6] = 0,$$

$$[Y_5, X_6] = X_2, \; [Y_5, Y_6] = 0,$$

$$[X_6, Y_6] = H_1 + 2H_2.$$

Then $(G_2, [\ ,\ ])$ *is a simple Lie algebra.*

Definition 5. *[Hu] Let* $\mathfrak{g}$ *be a Lie algebra and let* V *be a vector space. Then* V *is called a* $\mathfrak{g}$*-module if there is a bilinear map*

$$\mathfrak{g} \times V \to V$$

$$(x, v) \mapsto x \cdot v$$

such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

or

$$x \cdot (y \cdot v) = y \cdot (x \cdot v) + [x, y] \cdot v$$

for $x, y \in \mathfrak{g}, v \in V$.

II.2 Leibniz Algebras

Definition 6. *[DMS, FM]*

1. *A left Leibniz algebra* $\mathfrak{L}$ *is a* $\mathbb{C}$*-vector space equipped with a bilinear map* $[\ ,\ ] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ *satisfying the Leibniz identity*

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

   for all $a, b, c \in \mathfrak{L}$.

2. *A right Leibniz algebra* $\mathbb{L}$ *is a* $\mathbb{C}$*-vector space equipped with a bilinear map*

$\{\ ,\ \} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ *satisfying the identity*

$$\{\{u, v\}, w\} = \{\{u, w\}, v\} + \{u, \{v, w\}\}$$

for all $u, v, w \in \mathbb{L}$

Example 3. *Every Lie algebra is a left Leibniz algebra and right Leibniz algebra.*

Definition 7. *[DMS]*

1. *Let* $\mathfrak{L}$ *be a left Leibniz algebra over* $\mathbb{C}$*. Let* I *be a subspace of* $\mathfrak{L}$*.* I *is a* left *(respectively,* right*)* ideal *of* $\mathfrak{L}$ *if* $[\mathfrak{L}, I] \subseteq I$ *(respectively,* $[I, \mathfrak{L}] \subseteq I$*).* I *is an* ideal *of* $\mathfrak{L}$ *if it is both a left and a right ideal.*

2. *Let* $(\mathfrak{L}, [\ ,\ ])$ *be a left Leibniz algebra. The series of ideals*

$$... \subseteq \mathfrak{L}^{(2)} \subseteq \mathfrak{L}^{(1)} \subseteq \mathfrak{L}$$

where $\mathfrak{L}^{(1)} = [\mathfrak{L}, \mathfrak{L}]$, $\mathfrak{L}^{(i+1)} = [\mathfrak{L}^{(i)}, \mathfrak{L}^{(i)}]$ *is called the* derived series *of* $\mathfrak{L}$*. A left Leibniz algebra* $\mathfrak{L}$ *is* solvable *if* $\mathfrak{L}^{(m)} = 0$ *for some integer* $m \geq 0$*. As in the case of Lie algebras, any left Leibniz algebra* $\mathfrak{L}$ *contains a unique maximal solvable ideal* $rad(\mathfrak{L})$ *called the the* radical *of* $\mathfrak{L}$ *which contains all solvable ideals.*

Example 4.

We define $Leib(\mathfrak{L}) = Span\{\ [u, u] \mid u \in \mathfrak{L}\ \} = Span\{[u, v] + [v, u] \mid u, v \in \mathfrak{L}\}$*.* $Leib(\mathfrak{L})$ *is an ideal of* $\mathfrak{L}$*. Moreover, for* $v, w \in Leib(\mathfrak{L})$, $[v, w] = 0$*.* $Leib(\mathfrak{L})$ *is a solvable ideal.*

Definition 8. *[DMS]*

1. *A left Leibniz algebra* $\mathfrak{L}$ *is simple if* $[\mathfrak{L}, \mathfrak{L}] \neq Leib(\mathfrak{L})$*, and* $\{0\}$*,* $Leib(\mathfrak{L})$*,* $\mathfrak{L}$ *are the only ideals of* $\mathfrak{L}$*.*

2. *A left Leibniz algebra* $\mathfrak{L}$ *is said to be semisimple if* $rad(\mathfrak{L}) = Leib(\mathfrak{L})$*.*

Proposition 9. *([Ba], [DMS])*

Let $\mathfrak{L}$ be a left Leibniz algebra. Then

1. *there exists a subalgebra S which is a semisimple Lie algebra of $\mathfrak{L}$ such that $\mathfrak{L} = S \dot{+} rad(\mathfrak{L})$. As in the case of a Lie algebra, we call S a Levi subalgebra or a Levi factor of $\mathfrak{L}$;*

2. *if $\mathfrak{L}$ is a semisimple Leibniz algebra then $\mathfrak{L} = (S_1 \oplus S_2 \oplus \ldots \oplus S_k) \dot{+} Leib(\mathfrak{L})$, where S_j is a simple Lie algebra for all $1 \leq j \leq k$. Moreover, $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$;*

3. *if $\mathfrak{L}$ is a simple Leibniz algebra, then there exists a simple Lie algebra S such that $Leib(\mathfrak{L})$ is an irreducible module over S and $\mathfrak{L} = S \dot{+} Leib(\mathfrak{L})$.*

Definition 10. *[JY1]*

Let $(\mathfrak{L}, [\ ,\ ])$ be a left Leibniz algebra over $\mathbb{C}$. A left $\mathfrak{L}$-module is a vector space M equipped with a $\mathbb{C}$-bilinear map $\mathfrak{L} \times M \to M; (u, m) \mapsto u \cdot m$ such that $([u, v]) \cdot m = u \cdot (v \cdot m) - v \cdot (u \cdot m)$ for all $u, v \in \mathfrak{L}, m \in M$.

Remark 11. *[JY1]*

Let $\mathfrak{L}$ is a left Leibniz algebra. For any $\mathfrak{L}$-module M, $Leib(\mathfrak{L})$ acts as a zero on M.

II.3 Vertex Algebroids

Definition 12. *[GMS]*

A 1-truncated conformal algebra is a graded vector space $C = C_0 \oplus C_1$ equipped with a linear map $\partial : C_0 \to C_1$ and bilinear operations $(u, v) \mapsto u_i v$ for $i = 0, 1$ of degree $-i-1$ on $C = C_0 \oplus C_1$ such that the following axioms hold:

1. *(Derivation) for $a \in C_0$, $u \in C_1$,*

$$(\partial a)_0 = 0, \quad (\partial a)_1 = -a_0, \quad \partial(u_0 a) = u_0 \partial a;$$

2. *(Commutativity) for $a \in C_0$, $u, v \in C_1$,*

$$u_0 a = -a_0 u, \quad u_0 v = -v_0 u + \partial(u_1 v), \quad u_1 v = v_1 u;$$

3. *(Associativity) for $\alpha, \beta, \gamma \in C_0 \oplus C_1$,*

$$\alpha_0 \beta_i \gamma = \beta_i \alpha_0 \gamma + (\alpha_0 \beta)_i \gamma.$$

Definition 13. *[Br1, Br2, GMS]*

Let $(A, *)$ be a unital commutative associative algebra over $\mathbb{C}$ with the identity 1. A vertex A-algebroid is a $\mathbb{C}$-vector space Γ equipped with

1. *a $\mathbb{C}$-bilinear map $A \times \Gamma \to \Gamma$, $\quad (a, v) \mapsto a \cdot v$ such that $1 \cdot v = v$ (i.e. a nonassociative unital A-module),*
2. *a structure of a Leibniz $\mathbb{C}$-algebra $[\ ,\ ] : \Gamma \times \Gamma \to \Gamma$,*
3. *a homomorphism of Leibniz $\mathbb{C}$-algebra $\pi : \Gamma \to Der(A)$,*
4. *a symmetric $\mathbb{C}$-bilinear pairing $\langle\ ,\ \rangle : \Gamma \otimes_{\mathbb{C}} \Gamma \to A$,*

5. *a $\mathbb{C}$-linear map $\partial : A \to \Gamma$ such that $\pi \circ \partial = 0$ which satisfy the following conditions:*

$$a \cdot (a' \cdot v) - (a * a') \cdot v = \pi(v)(a) \cdot \partial(a') + \pi(v)(a') \cdot \partial(a),$$

$$[u, a \cdot v] = \pi(u)(a) \cdot v + a \cdot [u, v],$$

$$[u, v] + [v, u] = \partial(\langle u, v \rangle),$$

$$\pi(a \cdot v) = a\pi(v),$$

$$\langle a \cdot u, v \rangle = a * \langle u, v \rangle - \pi(u)(\pi(v)(a)),$$

$$\pi(v)(\langle v_1, v_2 \rangle) = \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle,$$

$$\partial(a * a') = a \cdot \partial(a') + a' \cdot \partial(a),$$

$$[v, \partial(a)] = \partial(\pi(v)(a)),$$

$$\langle v, \partial(a) \rangle = \pi(v)(a)$$

for $a, a' \in A$, $u, v, v_1, v_2 \in \Gamma$.

Proposition 14. *[LiY1]*

Let $(A, *)$ be a unital commutative associative algebra and let B be a module for A as a nonassociative algebra . Then a vertex A-algebroid structure on B exactly amounts to a 1-truncated conformal algebra structure on $C = A \oplus B$ with

$$a_i a' = 0,$$

$$u_0 v = [u, v], \ \ u_1 v = \langle u, v \rangle,$$

$$u_0 a = \pi(u)(a), \ \ a_0 u = -u_0 a$$

for $a, a' \in A$, $u, v \in B$, $i = 0, 1$ such that

$$a \cdot (a' \cdot u) - (a * a') \cdot u = (u_0 a) \cdot \partial a' + (u_0 a') \cdot \partial a,$$

$$u_0(a \cdot v) - a \cdot (u_0 v) = (u_0 a) \cdot v,$$

$$u_0(a * a') = a * (u_0 a') + (u_0 a) * a',$$

$$a_0(a' \cdot v) = a' * (a_0 v),$$

$$(a \cdot u)_{-1} v = a * (u_{-1} v) - u_0 v_0 a,$$

$$\partial(a * a') = a \cdot \partial(a') + a' \cdot \partial(a).$$

II.3.0 Some Recent Known Results

Proposition 15. *[JY1]*

Let B be a vertex A-algebroid such that $\dim A < \infty$ and $\dim B < \infty$.

1. *Assume that B is a simple left Leibniz algebra satisfying the conditions that $Leib(B) \neq \{0\}$, and that its Levi factor $S = Span\{e, f, h\}$ such that $e_0 f = h$, $h_0 e = 2e$, $h_0 f = -2f$, and $e_1 f = k\mathfrak{e} \in (\mathbb{C}\mathfrak{e})\backslash\{0\}$. Then*

   (i) $e_1 e = f_1 f = e_1 h = f_1 h = 0$, $k = 1$, $h_1 h = 2\mathfrak{e}$.

   (ii) $Ker(\partial) = \mathbb{C}\mathfrak{e}$

   (iii) $Leib(B)$ is an irreducible sl_2-module of dimension 2. Moreover, as a sl_2-module, A is a direct sum of a trivial module and an irreducible sl_2-module of dimension 2.

   (iv) A is a local algebra. Let $A_{\neq 0}$ be an irreducible sl_2-submodule of A that has dimension 2. Let a_0 be the highest weight vector of $A_{\neq 0}$ of weight 1 and let $a_1 = f_0 a_0$. Hence, the set $\{a_0, a_1\}$ forms a basis of $A_{\neq 0}$, the set $\{\mathfrak{e}, a_0, a_1\}$ is a basis of A, and the set $\{\partial(a_0), \partial(a_1)\}$ is a basis of $Leib(B)$.

   Relationships among $a_0, a_1, e, f, h, \partial(a_0), \partial(a_1)$ are desribed below:

$$(\partial(a_0))_1 e = 0,\ (\partial(a_0))_1 f = a_1,\ (\partial(a_0))_1 h = a_0,$$

$$(\partial(a_1))_1 e = a_0,\ (\partial(a_1))_1 f = 0,\ (\partial(a_1))_1 h = -a_1,$$

$$a_0 \cdot e = 0,\ a_0 \cdot f = \partial(a_1),\ a_0 \cdot h = \partial(a_0),\ a_0 \cdot \partial(a_i) = 0 \text{ for } i \in \{0, 1\},$$

$$a_1 \cdot e = \partial(a_0),\ a_1 \cdot f = 0,\ a_1 \cdot h = -\partial(a_1),\ a_1 \cdot \partial(a_i) = 0 \text{ for } i \in \{0,1\},$$

$$a_i * a_j = 0 \text{ for all } i, j \in \{0,1\}.$$

2. *Assume that B is a semisimple left Leibniz algebra satisfying the conditions that $Leib(B) \neq \{0\}$, $Ker(\partial) = \{a \in A \mid u_0 a = 0$ for all $u \in B\}$, and its Levi factor $S = Span\{e, f, h\}$ such that $e_0 f = h, h_0 e = 2e, h_0 f = -2f$ and $e_1 f = k\mathfrak{e} \in \mathbb{C}\mathfrak{e}\backslash\{0\}$. We set $A = \mathbb{C}\mathfrak{e} \oplus_{j=1}^{l} N^j$ where each N^j is an irreducible sl_2-submodule of A. Then*

   (i) $e_1 e = f_1 f = e_1 h = f_1 h = 0$, $k = 1$, $h_1 h = 2\mathfrak{e}$;

   (ii) $Ker(\partial) = \mathbb{C}\mathfrak{e}$;

   (iii) For $j \in \{1, ..., l\}$ $\dim N^j = 2$, and $\dim Leib(B) = 2l$;

   (iv) A is a local algebra. For each j, we let $a_{j,0}$ be a highest weight vector of N^j and $a_{j,1} = f_0(a_{j,0})$. Then $\{\mathfrak{e}, a_{j,i} \mid j \in \{1, ..., l\},\ i \in \{0,1\}\}$ is a basis of A, and $\{\partial(a_{j,i}) \mid j \in \{1, ..., l\},\ i \in \{0,1\}\}$ is a basis of $Leib(B)$.

   Relations among $a_{j,i}, e, f, h, \partial(a_{j,i})$ are described below:

$$a_{j,i} * a_{j',i'} = 0,\ a_{j,0} \cdot e = 0,\ a_{j,1} \cdot e = \partial(a_{j,0}),$$

$$a_{j,0} \cdot f = \partial(a_{j,1}),\ a_{j,1} \cdot f = 0,\ a_{j,0} \cdot h = \partial(a_{j,0}),\ a_{j,1} \cdot h = -\partial(a_{j,1}),$$

$$a_{j,i} \cdot \partial(a_{j',i'}) = 0,\ \partial(a_{j,i})_1 e = e_0 a_{j,i} = (2-i)a_{j,i-1},$$

$$\partial(a_{j,i})_1 f = f_0 a_{j,i} = (i+1)a_{j,i+1},\ \partial(a_{j,i})_1 h = h_0 a_{j,i} = (1-2i)a_{j,i}.$$

Theorem 16. *[BuY]*

Let $(A, *)$ be a finite dimensional commutative associative algebra with the identity $\hat{1}$ such that $\dim A \geq 2$. Let B be a vertex A-algebroid that is not a Lie algebra. Assume that there exist $e, f, h \in B$ such that $e_0 f = h$, $h_0 e = 2e$, $h_0 f = -2f$, $h_0 h = 0$, $Span\{e, f, h\}$ is a Lie algebra that is isomorphic to sl_2, $A = \mathbb{C}\hat{1} \oplus_{i=1}^{t} N^i$ where N^i are irreducible sl_2-modules and

$(\oplus_{i=1}^{t} N^i) \cdot B \subseteq \partial(A)$.

For each $i \in \{1, ..., t\}$, we let a_0^i be the highest weight vector of N^i of weight m^i, and set $a_j^i = \frac{1}{j!}(f_0)^j a_0^i$. Note that $\{a_0^i, ..., a_{m^i}^i\}$ is a basis of N^i and $h_0 a_j^i = (m^i - 2j)a_j^i$, $f_0 a_j^i = (j+1)a_{j+1}^i$, $e_0 a_j^i = (m^i - j + 1)a_{j-1}^i$.

If $e_1 f = k\hat{1}$ where $k \neq 0 \in \mathbb{C}$, then

1. *$Ker(\partial) = \mathbb{C}\hat{1}$;*

2. *each N^i is an irreducible sl_2-module that has dimension 2 and A is a local algebra. In addition, for $i, j \in \{1, ..., t\}$, $s, r \in \{0, 1\}$, the following statements hold:*

$$a_s^i * a_r^j = 0, \; a_0^i \cdot e = 0, \; a_1^i \cdot e = \partial(a_0^i),$$
$$a_0^i \cdot f = \partial(a_1^i), \; a_1^i \cdot f = 0, \; a_0^i \cdot h = \partial(a_0^i), \; a_1^i \cdot h = -\partial(a_1^i),$$
$$a_s^i \cdot \partial(a_t^j) = 0, \; \partial(a_s^i)_1 e = e_0 a_s^i = (2-s)a_{s-1}^i,$$
$$\partial(a_s^i)_1 f = f_0 a_s^i = (s+1)a_{s+1}^i, \; \partial(a_s^i)_1 h = h_0 a_s^i = (1-2s)a_s^i, \; k = 1.$$

3. *For $a, a' \in N$, we have $a \cdot e = \partial(e_0 a)$, $a \cdot f = \partial(f_0 a)$, $a \cdot h = \partial(h_0 a)$, $a \cdot \partial(a') = 0$.*

CHAPTER III: VERTEX ALGEBROID CONSTRUCTION FROM THE SIMPLE LIE ALGEBRA G_2

In this chapter we will cover the new ideas presented in this thesis. We will investigate a possibility of constructing a vertex algebroid from a commutative associative algebra $(A, *)$ and a left Leibniz algebra B such that as G_2-modules, $A = \mathbb{C}\mathbf{1} \oplus N$ and $B = G_2 \oplus W$, where N and W are irreducible G_2-modules that are both isomorphic to G_2. In Chapter III. 1.1 and III. 1.2, we will provide natural assumptions on A and B. In Chapter III.1.3, we will state the results-Theorem 17 and Theorem 18. The proofs of Theorem 17 and Theorem 18 are in Chapter III.2, and Chapter III.3.

It is worth mentioning that the results in Theorem 17 will give exact structure of the multiplication $*$ of the commutative associative algebra A. The results in Theorem 18 will give exact structure of the symmetric bilinear function $\langle\ ,\ \rangle : B \times B \to A$. The construction is only partial, but gives a concrete foundation of how we can study vertex algebroids as modules of a Lie algebra G_2.

III.1 Assumptions and Main Results

We set $G_2 = Span\{X_i, Y_i, H_j \mid i \in \{1, .., 6\}, j \in \{1, 2\}\}$. Recall that $(G_2,\ _0)$ is a simple Lie algebra that satisfies that following properties

1. $A_0 B = -B_0 A$ for $A, B \in G_2$;

2. $(H_i)_0(H_j) = 0$ for all $i, j \in \{1, 2\}$;

3.

$$(H_1)_0(X_1) = 2X_1,\ (H_1)_0(Y_1) = -2Y_1,\ (H_1)_0(X_2) = -3X_2,\ (H_1)_0(Y_2) = 3Y_2,$$

$$(H_1)_0(X_3) = -X_3,\ (H_1)_0(Y_3) = Y_3,\ (H_1)_0(X_4) = X_4,\ (H_1)_0(Y_4) = -Y_4,$$

$$(H_1)_0(X_5) = 3X_5,\ (H_1)_0(Y_5) = -3Y_5,\ (H_1)_0(X_6) = 0,\ (H_1)_0(Y_6) = 0,$$

$$(H_2)_0(X_1) = -X_1,\ (H_2)_0(Y_1) = Y_1,\ (H_2)_0(X_2) = 2X_2,\ (H_2)_0(Y_2) = -2Y_2,$$

$$(H_2)_0(X_3) = X_3,\ (H_2)_0(Y_3) = -Y_3,\ (H_2)_0(X_4) = 0,\ (H_2)_0(Y_4) = 0,$$

$$(H_2)_0(X_5) = -X_5,\ (H_2)_0(Y_5) = Y_5,\ (H_2)_0(X_6) = X_6,\ (H_2)_0(Y_6) = -Y_6$$

$$(X_1)_0(Y_1) = H_1,\ (X_1)_0(X_2) = X_3,\ (X_1)_0(Y_2) = 0,\ (X_1)_0(X_3) = 2X_4,$$

$$(X_1)_0(Y_3) = -3Y_2,\ (X_1)_0(X_4) = -3X_5,\ (X_1)_0(Y_4) = -2Y_3,\ (X_1)_0(X_5) = 0,$$

$$(X_1)_0(Y_5) = Y_4,\ (X_1)_0(X_6) = 0,\ (X_1)_0(Y_6) = 0,$$

$$(Y_1)_0(X_2) = 0,\ (Y_1)_0(Y_2) = -Y_3,\ (Y_1)_0(X_3) = 3X_2,\ (Y_1)_0(Y_3) = -2Y_4,$$

$$(Y_1)_0(X_4) = 2X_3,\ (Y_1)_0(Y_4) = 3Y_5,\ (Y_1)_0(X_5) = -X_4,$$

$$(Y_1)_0(Y_5) = (Y_1)_0(X_6) = (Y_1)_0(Y_6) = 0,$$

4.

$$(X_2)_0(Y_2) = H_2,\ (X_2)_0(Y_3) = Y_1,\ (X_2)_0(X_5) = -X_6,\ (X_2)_0(Y_6) = Y_5,$$

$$(X_2)_0(X_3) = (X_2)_0(X_4) = (X_2)_0(Y_4) = (X_2)_0(Y_5) = (X_2)_0(X_6) = 0,$$

$$(Y_2)_0(X_3) = -X_1,\ (Y_2)_0(Y_5) = Y_6,\ (Y_2)_0(X_6) = -X_5,$$

$$(Y_2)_0(Y_3) = (Y_2)_0(X_4) = (Y_2)_0(Y_4) = (Y_2)_0(X_5) = (Y_2)_0(Y_6) = 0,$$

5.

$$(X_3)_0(Y_3) = H_1 + 3H_2,\ (X_3)_0(X_4) = -3X_6,\ (X_3)_0(Y_4) = 2Y_1,\ (X_3)_0(Y_6) = Y_4$$

$$(X_3)_0(X_5) = (X_3)_0(Y_5) = (X_3)_0(X_6) = 0,$$

$$(Y_3)_0(X_4) = -2X_1,\ (Y_3)_0(Y_4) = 3Y_6,\ (Y_3)_0(X_6) = -X_4,$$

$$(Y_3)_0(X_5) = (Y_3)_0(Y_5) = (Y_3)_0(Y_6) = 0,$$

6.

$$(X_4)_0(Y_4) = 2H_1 + 2H_2,\ (X_4)_0(Y_5) = -Y_1,\ (X_4)_0(Y_6) = -Y_3,$$

$$(X_4)_0(X_5) = (X_4)_0(X_6) = 0,$$

$$(Y_4)_0(X_5) = X_1,\ (Y_4)_0(X_6) = X_3,\ (Y_4)_0(Y_5) = (Y_4)_0(Y_6) = 0,$$

$$(X_5)_0(Y_5) = H_1 + H_2,\ (X_5)_0(Y_6) = -Y_2,\ (X_5)_0(X_6) = 0,$$

$$(Y_5)_0(X_6) = X_2,\ (Y_5)_0(Y_6) = 0,$$

$$(X_6)_0(Y_6) = H_1 + 2H_2.$$

III.1.0 **Assumption I**

We set $N = Span\{x^i, y^i, h^j \mid i \in \{1, ..., 6\}, j \in \{1, 2\}\}$. Assume that N is isomorphic to G_2 as Lie algebras. Hence, one can easily see that G_2 acts on N in the following way:

1. $(H_i) \cdot (h_j) = 0$ for all $i, j \in \{1, 2\}$;

2.

$$(H_1) \cdot (x_1) = 2x_1,\ (H_1) \cdot (y_1) = -2y_1,\ (H_1) \cdot (x_2) = -3x_2,\ (H_1) \cdot (y_2) = 3y_2,$$

$$(H_1) \cdot (x_3) = -x_3,\ (H_1) \cdot (y_3) = y_3,\ (H_1) \cdot (x_4) = x_4,\ (H_1) \cdot (y_4) = -y_4,$$

$$(H_1) \cdot (x_5) = 3x_5,\ (H_1) \cdot (y_5) = -3y_5,\ (H_1) \cdot (x_6) = 0,\ (H_1) \cdot (y_6) = 0,$$

$$(H_2) \cdot (x_1) = -x_1,\ (H_2) \cdot (y_1) = y_1,\ (H_2) \cdot (x_2) = 2x_2,\ (H_2) \cdot (y_2) = -2y_2,$$

$$(H_2) \cdot (x_3) = x_3,\ (H_2) \cdot (y_3) = -y_3,\ (H_2) \cdot (x_4) = 0,\ (H_2) \cdot (y_4) = 0,$$

$$(H_2) \cdot (x_5) = -x_5,\ (H_2) \cdot (y_5) = y_5,\ (H_2) \cdot (x_6) = x_6,\ (H_2) \cdot (y_6) = -y_6$$

$$(X_1) \cdot (y_1) = h_1,\ (X_1) \cdot (x_2) = x_3,\ (X_1) \cdot (y_2) = 0,\ (X_1) \cdot (x_3) = 2x_4,$$

$$(X_1) \cdot (y_3) = -3y_2,\ (X_1) \cdot (x_4) = -3x_5,\ (X_1) \cdot (y_4) = -2y_3,\ (X_1) \cdot (x_5) = 0,$$

$$(X_1) \cdot (y_5) = y_4,\ (X_1) \cdot (x_6) = 0,\ (X_1) \cdot (y_6) = 0,$$

$$(Y_1) \cdot (x_2) = 0,\ (Y_1) \cdot (y_2) = -Y_3,\ (Y_1) \cdot (x_3) = 3x_2,\ (Y_1) \cdot (y_3) = -2y_4,$$

$$(Y_1)\cdot(x_4)=2x_3,\ (Y_1)\cdot(y_4)=3y_5,\ (Y_1)\cdot(x_5)=-x_4,$$
$$(Y_1)\cdot(y_5)=(Y_1)\cdot(x_6)=(Y_1)\cdot(y_6)=0,$$

3.

$$(X_2)\cdot(y_2)=h_2,\ (X_2)\cdot(y_3)=y_1,\ (X_2)\cdot(x_5)=-x_6,\ (X_2)\cdot(y_6)=y_5,$$
$$(X_2)\cdot(x_3)=(X_2)\cdot(x_4)=(X_2)\cdot(y_4)=(X_2)\cdot(y_5)=(X_2)\cdot(x_6)=0,$$
$$(Y_2)\cdot(x_3)=-x_1,\ (Y_2)\cdot(y_5)=y_6,\ (Y_2)\cdot(x_6)=-x_5,$$
$$(Y_2)\cdot(y_3)=(Y_2)\cdot(x_4)=(Y_2)\cdot(y_4)=(Y_2)\cdot(x_5)=(Y_2)\cdot(y_6)=0,$$
$$(X_3)\cdot(y_3)=h_1+3h_2,\ (X_3)\cdot(x_4)=-3x_6,\ (X_3)\cdot(y_4)=2y_1,\ (X_3)\cdot(y_6)=y_4$$
$$(X_3)\cdot(x_5)=(X_3)\cdot(y_5)=(X_3)\cdot(x_6)=0,$$
$$(Y_3)\cdot(x_4)=-2x_1,\ (Y_3)_0(y_4)=3y_6,\ (Y_3)_0(x_6)=-x_4,$$
$$(Y_3)\cdot(x_5)=(Y_3)\cdot(y_5)=(Y_3)\cdot(y_6)=0,$$

4.

$$(X_4)\cdot(y_4)=2h_1+2h_2,\ (X_4)\cdot(y_5)=-y_1,\ (X_4)\cdot(y_6)=-y_3,$$
$$(X_4)\cdot(x_5)=(x_4)\cdot(x_6)=0,$$
$$(Y_4)\cdot(x_5)=x_1,\ (Y_4)\cdot(x_6)=x_3,\ (Y_4)\cdot(y_5)=(y_4)\cdot(y_6)=0,$$
$$(X_5)\cdot(y_5)=h_1+h_2,\ (X_5)\cdot(y_6)=-y_2,\ (X_5)\cdot(x_6)=0,$$
$$(Y_5)\cdot(x_6)=x_2,\ (Y_5)\cdot(y_6)=0,$$
$$(X_6)\cdot(y_6)=h_1+2h_2.$$

III.1.0 **Assumption II**

We set

$$A=\mathbb{C}\mathbf{1}\oplus N.$$

Clearly A is a G_2-module.

Next, we set

$$B = Span\{X_i, Y_i, H_j, \partial(x_i), \partial(y_i), \partial(h_j) \mid i \in \{1, ..., 6\}, j \in \{1, 2\}\} = G_2 \oplus \partial(A).$$

Here $\partial : A \to B$ is a G_2-homomorphism. Note that $(B,_0)$ is a left Leibniz algebra where

$$(u + m)_0(v + n) = u_0v + u \cdot n$$

for $u, v \in G$, $m, n \in \partial(A)$ (cf. Example 18 of [JY1]).

III.1.0 **List of Results**

Theorem 17. *Let A and B be defined as above. Hence, A and B must satisfy assumption I and assumption II. Assume that $(A, *)$ is a commutative associative algebra. Then the multiplication $*$ of A satisfies the following relations:*

$$x_1 * x_1 = 0,\ x_2 * x_2 = 0,\ x_3 * x_3 = 0,\ x_4 * x_4 = 0,\ x_5 * x_5 = 0,\ x_6 * x_6 = 0,$$

$$x_1 * x_2 = 0,\ x_1 * x_3 = 0,\ x_1 * x_4 = 0,\ x_1 * x_5 = 0,\ x_1 * x_6 = 0,$$

$$x_2 * x_3 = 0,\ x_2 * x_4 = 0,\ x_2 * x_5 = \beta_6 x_6 \textit{ for some } \beta_6 \in \mathbb{C},\ x_2 * x_6 = 0,$$

$$x_3 * x_4 = 0,\ x_3 * x_5 = 0,\ x_3 * x_6 = 0,\ x_4 * x_5 = 0,\ x_4 * x_6 = 0,\ x_5 * x_6 = 0,$$

$$y_1 * y_1 = 0,\ y_2 * y_2 = 0,\ y_3 * y_3 = 0,\ y_4 * y_4 = 0,\ y_5 * y_5 = 0,\ y_6 * y_6 = 0,$$

$$y_1 * y_2 = c_3 y_3 \textit{ for some } c_3 \in \mathbb{C},\ y_1 * y_3 = 0,\ y_1 * y_4 = 0,\ y_1 * y_5 = 0,\ y_1 * y_6 = 0,$$

$$y_2 * y_3 = 0,\ y_2 * y_4 = 0,\ y_2 * y_5 = \gamma_6 x_6 \textit{ for some } \gamma_6 \in \mathbb{C},\ y_2 * y_6 = 0,$$

$$y_3 * y_4 = 0,\ y_3 * y_5 = 0,\ y_3 * y_6 = 0,\ y_4 * y_5 = 0,\ y_4 * y_6 = 0,\ y_5 * y_6 = 0,$$

$$x_1 * y_1 = d_1 h_1 + d_2 h_2 \textit{ for some } d_1, d_2 \in \mathbb{C},\ x_2 * y_2 = d_{21} h_1 + d_{22} h_2 \textit{ for some } d_{21}, d_{22} \in \mathbb{C},$$

$x_3 * y_3 = \tau_1 h_1 + \tau_2 h_2$ *for some* $\tau_1, \tau_2 \in \mathbb{C},\ x_4 * y_4 = \tau_{46} x_6 + \chi_{46} y_6$ *for some* $\tau_{46}, \chi_{46} \in \mathbb{C}$,

$x_5 * y_5 = d_{51} h_1 + d_{52} h_2$ *for some* $d_{51}, d_{52} \in \mathbb{C},\ x_6 * y_6 = d_{61} h_1 + d_{62} h_2$ *for some* $d_{61}, d_{62} \in \mathbb{C}$,

$x_1 * y_2 = 0,\ x_1 * y_3 = 0,\ x_1 * y_4 = 0,\ x_1 * y_5 = b_{15} x_4$ *for some* $b_{15} \in \mathbb{C},\ x_1 * y_6 = 0$,

$x_2 * y_3 = c_{23} y_1$ *for some* $c_{23} \in \mathbb{C},\ x_2 * y_4 = 0,\ x_2 * y_5 = 0,\ x_2 * y_6 = \tau_{26} y_5$ *for some* $\tau_{26} \in \mathbb{C}$,

$x_3 * y_2 = b_{32} x_1$ *for some* $b_{32} \in \mathbb{C},\ x_3 * y_4 = 0,\ x_3 * y_5 = 0,\ x_3 * y_6 = b_{36} x_3$ *for some* $b_{36} \in \mathbb{C}$,

$x_4 * y_2 = 0,\ x_4 * y_5 = 0,\ x_4 * y_6 = 0,\ x_5 * y_1 = 0,\ x_5 * y_3 = 0,\ x_5 * y_4 = 0$,

$x_5 * y_6 = c_{56} y_2$ *for some* $c_{56} \in \mathbb{C},\ x_6 * y_1 = 0,\ x_6 * y_2 = \beta_{62} x_5$ *for some* $\beta_{62} \in \mathbb{C}$,

$x_6 * y_3 = \gamma_{63} y_4$ *for some* $\gamma_{63} \in \mathbb{C},\ x_6 * y_4 = 0,\ x_6 * y_5 = \gamma_{65} x_2$ *for some* $\gamma_{65} \in \mathbb{C}$,

$y_1 * x_2 = 0,\ y_1 * x_3 = 0,\ y_1 * x_4 = 0,\ y_2 * x_5 = 0,\ y_3 * x_4 = 0$,

$h_1 * x_1 = 0,\ h_1 * x_2 = b_2 x_2$ *for some* $b_2 \in \mathbb{C},\ h_1 * x_3 = b_3 x_3$ *for some* $b_3 \in \mathbb{C},\ h_1 * x_4 = 0$,

$h_1 * x_5 = b_5 x_5$ *for some* $b_5 \in \mathbb{C},\ h_1 * x_6 = b_6 x_6$ *for some* $b_6 \in \mathbb{C}$,

$h_2 * x_1 = \beta_1 x_1$ *for some* $\beta_1 \in \mathbb{C},\ h_2 * x_2 = 0,\ h_2 * x_3 = \beta_3 x_3$ *for some* $\beta_3 \in \mathbb{C}$,

$h_2 * x_4 = 0,\ h_2 * x_5 = \beta_5 x_5$ *for some* $\beta_5 \in \mathbb{C},\ h_2 * x_6 = \rho_6 x_6$ *for some* $\rho_6 \in \mathbb{C}$,

$h_1 * y_1 = 0,\ h_1 * y_2 = \rho_2 y_2$ *for some* $\rho_2 \in \mathbb{C},\ h_1 * y_3 = \rho_3 y_3$ *for some* $\rho_3 \in \mathbb{C},\ h_1 * y_4 = 0$,

$h_1 * y_5 = \rho_5 y_5$ *for some* $\rho_5 \in \mathbb{C},\ h_1 * y_6 = c_{16} x_6$ *for some* $c_{16} \in \mathbb{C}$,

$h_2 * y_1 = c_{21} y_1$ *for some* $c_{21} \in \mathbb{C},\ h_2 * y_2 = 0,\ h_2 * y_3 = 0$,

$h_2 * y_4 = 0,\ h_2 * y_5 = c_{25} y_5$ *for some* $c_{25} \in \mathbb{C},\ h_2 * y_6 = c_{26} y_6$ *for some* $c_{26} \in \mathbb{C}$,

$h_1 * h_1 = 0,\ h_1 * h_2 = 0,\ h_2 * h_2 = 0$,

$x_3 * (h_1 + 3h_2) = 0,\ x_5 * (-2h_1 - 2h_2) = 0,\ y_3 * (2h_1 + 6h_2) = 0,\ y_4 * (4h_1 + 6h_2) = 0$,

$y_5 * (2h_1 + 2h_2) = 0,\ y_6 * (2h_1 + 4h_2) = 0$.

Theorem 18. *Let A and B be defined as above. Therefore A and B satisfy assumption I and assumption II. Assume that $(A, *)$ is a commutative associative algebra. Let*

$\langle\ ,\ \rangle : B \times B \to A$ *be a symmetric* $\mathbb{C}$*-bilinear pairing such that*

$$u_0\langle v, w\rangle = \langle u_0 v, w\rangle + \langle v, u_0 w\rangle,\ \ \langle \partial(a), g\rangle = g \cdot a,\ \ \langle \partial(a), \partial(a')\rangle = 0$$

for $a, a' \in A$, $g \in \mathfrak{g}$, $u, v, w \in B$. *Then* $\langle\ ,\ \rangle$ *satisfies the following properties*

$$\begin{aligned}
&\langle X_1, X_1\rangle = 0,\ \langle X_2, X_2\rangle = 0,\ \langle X_3, X_3\rangle = 0,\ \langle X_4, X_4\rangle = 0,\ \langle X_5, X_5\rangle = 0,\ \langle X_6, X_6\rangle = 0,\\
&\langle X_1, X_2\rangle = 0,\ \langle X_1, X_3\rangle = 0,\ \langle X_1, X_4\rangle = 0,\ \langle X_1, X_5\rangle = 0,\ \langle X_1, X_6\rangle = 0,\\
&\langle X_2, X_3\rangle = 0,\ \langle X_2, X_4\rangle = 0,\ \langle X_2, X_5\rangle = b_{25}x_6 \textit{ for some } b_{25} \in \mathbb{C},\ \langle X_2, X_6\rangle = 0,\\
&\langle X_3, X_4\rangle = 0,\ \langle X_3, X_5\rangle = 0,\ \langle X_3, X_6\rangle = 0,\ \langle X_4, X_5\rangle = 0,\ \langle X_4, X_6\rangle = 0,\ \langle X_5, X_6\rangle = 0,\\
&\langle Y_1, Y_1\rangle = 0,\ \langle Y_2, Y_2\rangle = 0,\ \langle Y_3, Y_3\rangle = 0,\ \langle Y_4, Y_4\rangle = 0,\ \langle Y_5, Y_5\rangle = 0,\ \langle Y_6, Y_6\rangle = 0,\\
&\langle Y_1, Y_2\rangle = c_{12}y_3 \textit{ for some } c_{12} \in \mathbb{C},\ \langle Y_1, Y_3\rangle = 0,\ \langle Y_1, Y_4\rangle = 0,\ \langle Y_1, Y_5\rangle = 0,\ \langle Y_1, Y_6\rangle = 0,\\
&\langle Y_2, Y_3\rangle = 0,\ \langle Y_2, Y_4\rangle = 0, \langle Y_2, Y_5\rangle = c_{25}x_6 \textit{ for some } c_{25} \in \mathbb{C},\ \langle Y_2, Y_6\rangle = 0,\\
&\langle Y_3, Y_4\rangle = 0,\ \langle Y_3, Y_5\rangle = 0,\ \langle Y_3, Y_6\rangle = 0,\ \langle Y_4, Y_5\rangle = 0,\ \langle Y_4, Y_6\rangle = 0,\ \langle Y_5, Y_6\rangle = 0,\\
&\langle X_1, Y_1\rangle = d_{11}h_1 + d_{12}h_2 \textit{ for some } d_{11}, d_{12} \in \mathbb{C},\ \langle X_2, Y_2\rangle = d_{21}h_1 + d_{22}h_2 \textit{ for some } d_{21}, d_{22} \in \mathbb{C},\\
&\langle X_3, Y_3\rangle = d_{31}h_1 + d_{32}h_2 \textit{ for some } d_{31}, d_{32} \in \mathbb{C},\ \langle X_4, Y_4\rangle = \tau_{44}x_6 + \chi_{44}y_6 \textit{ for some } \tau_{44}, \chi_{44} \in \mathbb{C},\\
&\langle X_5, Y_5\rangle = d_{51}h_1 + d_{52}h_2 \textit{ for some } d_{51}, d_{52} \in \mathbb{C},\ \langle X_6, Y_6\rangle = d_{61}h_1 + d_{62}h_2 \textit{ for some } d_{61}, d_{62} \in \mathbb{C},\\
&\langle X_1, Y_2\rangle = 0,\ \langle X_1, Y_3\rangle = 0,\ \langle X_1, Y_4\rangle = 0,\ \langle X_1, Y_5\rangle = \rho_{15}x_4 \textit{ for some } \rho_{15} \in \mathbb{C},\ \langle X_1, Y_6\rangle = 0,\\
&\langle X_2, Y_3\rangle = \tau_{23}y_1 \textit{ for some } \tau_{23} \in \mathbb{C},\ \langle X_2, Y_4\rangle = 0,\ \langle X_2, Y_5\rangle = 0,\ \langle X_2, Y_6\rangle = \tau_{26}y_5 \textit{ for some } \tau_{26} \in \mathbb{C},\\
&\langle X_3, Y_2\rangle = \tau_{32}x_1 \textit{ for some } \tau_{32} \in \mathbb{C},\ \langle X_3, Y_4\rangle = 0,\ \langle X_3, Y_5\rangle = 0,\ \langle X_3, Y_6\rangle = \tau_{36}x_3 \textit{ for some } \tau_{36} \in \mathbb{C},\\
&\langle X_4, Y_2\rangle = 0,\ \langle X_4, Y_5\rangle = 0,\ \langle X_4, Y_6\rangle = 0,\ \langle X_5, Y_1\rangle = 0,\ \langle X_5, Y_3\rangle = 0,\ \langle X_5, Y_4\rangle = 0,\\
&\langle X_5, Y_6\rangle = \tau_{56}y_2 \textit{ for some } \tau_{56} \in \mathbb{C},\ \langle X_6, Y_1\rangle = 0,\ \langle X_6, Y_2\rangle = \tau_{62}x_5 \textit{ for some } \tau_{62} \in \mathbb{C},\\
&\langle X_6, Y_3\rangle = \tau_{63}y_4 \textit{ for some } \tau_{63} \in \mathbb{C},\ \langle X_6, Y_4\rangle = 0,\ \langle X_6, Y_5\rangle = \mu_{65}x_2 \textit{ for some } \mu_{65} \in \mathbb{C},\\
&\langle Y_1, X_2\rangle = 0,\ \langle Y_1, X_3\rangle = 0,\ \langle Y_1, X_4\rangle = 0,\ \langle Y_2, X_5\rangle = 0,\ \langle Y_3, X_4\rangle = 0,\\
&\langle H_1, X_1\rangle = 0,\ \langle H_1, X_2\rangle = b_{12}x_2 \textit{ for some } b_{12} \in \mathbb{C},\ \langle H_1, X_3\rangle = b_{13}x_3 \textit{ for some } b_{13} \in \mathbb{C},
\end{aligned}$$

$\langle H_1, X_4\rangle = 0,\ \langle H_1, X_5\rangle = b_{15}x_5$ *for some* $b_{15} \in \mathbb{C},\ \langle H_1, X_6\rangle = b_{16}x_6$ *for some* $b_{16} \in \mathbb{C}$,

$\langle H_2, X_1\rangle = b_{21}x_1$ *for some* $b_{21} \in \mathbb{C},\ \langle H_2, X_2\rangle = 0,\ \langle H_2, X_3\rangle = b_{23}x_3$ *for some* $b_{23} \in \mathbb{C}$,

$\langle H_2, X_4\rangle = 0,\ \langle H_2, X_5\rangle = \beta_{25}x_5$ *for some* $\beta_{25} \in \mathbb{C},\ \langle H_2, X_6\rangle = \beta_{26}x_6$ *for some* $\beta_{26} \in \mathbb{C}$,

$\langle H_1, Y_1\rangle = 0,\ \langle H_1, Y_2\rangle = \gamma_{12}y_2$ *for some* $\gamma_{12} \in \mathbb{C},\ \langle H_1, Y_3\rangle = \gamma_{13}y_3$ *for some* $\gamma_{13} \in \mathbb{C}$,

$\langle H_1, Y_4\rangle = 0,\ \langle H_1, Y_5\rangle = \gamma_{15}y_5$ *for some* $\gamma_{15} \in \mathbb{C},\ \langle H_1, Y_6\rangle = \rho_{16}x_6$ *for some* $\rho_{16} \in \mathbb{C}$,

$\langle H_2, Y_1\rangle = \rho_{21}y_1$ *for some* $\rho_{21} \in \mathbb{C},\ \langle H_2, Y_2\rangle = 0,\ \langle H_2, Y_3\rangle = 0$,

$\langle H_2, Y_4\rangle = 0,\ \langle H_2, Y_5\rangle = \rho_{25}y_5$ *for some* $\rho_{25} \in \mathbb{C},\ \langle H_2, Y_6\rangle = \rho_{26}y_6$ *for some* $\rho_{26} \in \mathbb{C}$,

$\langle H_1, H_1\rangle = 0,\ \langle H_1, H_2\rangle = 0,\ \langle H_2, H_2\rangle = 0,$

$\langle X_3, (H_1 + 3H_2)\rangle = 0,\ \langle X_5, (-2H_1 - 2H_2)\rangle = 0,\ \langle Y_3, (2H_1 + 6H_2)\rangle = 0,$

$\langle Y_4, (4H_1 + 6H_2)\rangle = 0,\ \langle Y_5, (2H_1 + 2H_2)\rangle = 0,\ \langle Y_6, (2H_1 + 4H_2\rangle) = 0.$

III.2 Proof of Theorem 17

III.2.0 **Lemma 19-Lemma 30**

Lemma 19. *We have* $x_1 * x_1 = 0$.

Proof. We set $x_1 * x_1 = a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2$ where $a, b_i, c_j, d_1, d_2 \in \mathbb{C}$. Since

$$\begin{aligned}(H_1)_0(x_1 * x_1) &= ((H_1)_0 x_1) * x_1 + x_1 * ((H_1)_0 x_1) \\ &= 2x_1 * x_1 + x_1 * 2x_1 \\ &= 4x_1 * x_1\end{aligned}$$

and

$$\begin{aligned}&(H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0)\end{aligned}$$

we have

$$\begin{aligned}&4(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).\end{aligned}$$

Consequently, we have

$a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$.

Therefore $x_1 * x_1 = 0$. □

Lemma 20. *We have* $x_2 * x_2 = 0$.

Proof. We set $x_2 * x_2 = a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2$ where $a, b_i, c_j, d_1, d_2 \in \mathbb{C}$. Because

$$\begin{aligned}
(H_1)_0(x_2 * x_2) &= ((H_1)_0 x_2) * x_2 + x_2 * ((H_1)_0 x_2) \\
&= -3x_2 * x_2 + x_2 * -3x_2 \\
&= -6x_2 * x_2
\end{aligned}$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0)
\end{aligned}$$

we have

$$\begin{aligned}
& -6(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Hence, we see that

$a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$, and

$$x_2 * x_2 = 0.$$

$\square$

Lemma 21. *We have* $x_3 * x_3 = 0$.

Proof. We set $x_3 * x_3 = a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2$ where $a, b_i, c_j, d_1, d_2 \in \mathbb{C}$. Since

$$\begin{aligned}(H_1)_0(x_3 * x_3) \quad &= ((H_1)_0 x_3) * x_3 + x_3 * ((H_1)_0 x_3) \\ &= -x_3 * x_3 + x_3 * -x_3 \\ &= -2x_3 * x_3\end{aligned}$$

and

$$\begin{aligned}&(H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad &b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ &+c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0)\end{aligned}$$

we have

$$\begin{aligned}&-2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad &b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ &+c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).\end{aligned}$$

Therefore,

$a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5 = b_6, c_2 = 0 = c_2 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. We have

$$x_3 * x_3 = c_1(y_1).$$

Now, we apply $(H_2)_0$. We have

$$\begin{aligned}(H_2)_0(x_3 * x_3) \quad &= ((H_2)_0 x_3) * x_3 + x_3 * ((H_2)_0 x_3) \\ &= x_3 * x_3 + x_3 * x_3\end{aligned}$$

$$= 2x_3 * x_3 = 2c_1y_1 \text{ and}$$

$$(H_2)_0(c_1(y_1)) \quad = c_1(y_1)$$

Now we compare our two equations. We see that $c_1 = 0$. Therefore

$$x_3 * x_3 = 0.$$

□

Lemma 22. *We have* $x_4 * x_4 = 0$.

Proof. We set $x_4 * x_4 = a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1h_1 + d_2h_2$ where $a, b_i, c_j, d_1, d_2 \in \mathbb{C}$. Observe that

$$\begin{aligned}
(H_1)_0(x_4 * x_4) &= ((H_1)_0x_4) * x_4 + x_4 * ((H_1)_0x_4) \\
&= x_4 * x_4 + x_4 * x_4 \\
&= 2x_4 * x_4
\end{aligned}$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1h_1 + d_2h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& +c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& 2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1h_1 + d_2h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0)
\end{aligned}$$

$$+c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).$$

Here we see that

$a = 0, b_2 = 0 = b_3 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. We have

$$x_4 * x_4 = b_1(x_1).$$

We apply $(H_2)_0$.

$$\begin{aligned} (H_2)_0(x_4 * x_4) &= ((H_2)_0 x_4) * x_4 + x_4 * ((H_2)_0 x_4) \\ &= 0 \\ (H_2)_0(b_1(x_1)) &= b_1(-x_1) \end{aligned}$$

Now we compare our two equations:

$$0 = b_1(-x_1)$$

We see that $b_1 = 0$. Therefore

$$x_4 * x_4 = 0.$$

□

Lemma 23. *We have* $x_5 * x_5 = 0$.

Proof. We set $x_5 * x_5 = a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2$ where $a, b_i, c_j, d_1, d_2 \in \mathbb{C}$. Since

$$\begin{aligned} (H_1)_0(x_5 * x_5) &= ((H_1)_0 x_5) * x_5 + x_5 * ((H_1)_0 x_5) \\ &= 3x_5 * x_5 + x_5 * 3x_5 \end{aligned}$$

$$= 6x_5 * x_5$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0)
\end{aligned}$$

we have

$$\begin{aligned}
& 6(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Consequently, we have

$a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2.$

Therefore

$$x_5 * x_5 = 0.$$

□

Lemma 24. *We have $x_6 * x_6 = 0$.*

Proof. We set $x_6 * x_6 = a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2$ where $a, b_i, c_j, d_1, d_2 \in \mathbb{C}$. Since

$$(H_1)_0(x_6 * x_6) \qquad = ((H_1)_0 x_6) * x_6 + x_6 * ((H_1)_0 x_6) = 0$$

and

$$
\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0),
\end{aligned}
$$

we have

$$
\begin{aligned}
& 0 \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Therefore, $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5, c_1 = 0 = c_2 = c_3 = c_4 = c_5, d_1 = 0 = d_2$. We have

$$x_6 * x_6 = b_6(x_6) + c_6(y_6).$$

Applying $(H_2)_0$ to the equation $x_6 * x_6 = b_6(x_6) + c_6(y_6)$, we have

$$
\begin{aligned}
(H_2)_0(x_6 * x_6) \quad & = ((H_2)_0 x_6) * x_6 + x_6 * ((H_2)_0 x_6) \\
& = x_6 * x_6 + x_6 * x_6 \\
& = 2x_6 * x_6 \\
& = 2(b_6(x_6) + c_6(y_6)) \\
(H_2)_0(b_6(x_6) + c_6(y_6)) \quad & = b_6(x_6) + c_6(-y_6).
\end{aligned}
$$

Now we compare the above equations:

$$2(b_6(x_6) + c_6(y_6))$$

$$= \quad b_6(x_6) + c_6(-y_6).$$

We see that $b_6 = c_6 = 0$. Therefore

$$x_6 * x_6 = 0.$$

□

Recall that

$$x_i * x_j = x_j * x_i \text{ where } i \neq j.$$

Lemma 25. *We have $x_1 * x_3 = 0$.*

Proof. Applying $(X_2)_0$ to $(x_1 * x_1)$, we have

$$\begin{aligned} 0 &= (X_2)_0(x_1 * x_1) \\ &= ((X_2)_0 x_1) * x_1 + x_1 * (X_2)_0 x_1 \\ &= -x_3 * x_1 + x_1 * (-x_3) \\ &= -2x_3 * x_1. \end{aligned}$$

Therefore

$$x_3 * x_1 = 0.$$

□

Lemma 26. *We have $x_1 * x_4 = 0$.*

Proof. Applying $(X_3)_0$ to $(x_1 * x_1)$, we have

$$\begin{aligned} 0 &= (X_3)_0(x_1 * x_1) \\ &= ((X_3)_0 x_1) * x_1 + x_1 * (X_3)_0 x_1 \\ &= -2x_4 * x_1 + x_1 * (-2x_4) \\ &= -4x_4 * x_1. \end{aligned}$$

Therefore

$$x_4 * x_1 = 0.$$

$\square$

Lemma 27. *We have* $x_1 * x_5 = 0$.

Proof. Applying $(X_4)_0$ to $(x_1 * x_1)$, we have

$$\begin{aligned}
0 &= (X_4)_0(x_1 * x_1) \\
&= ((X_4)_0 x_1) * x_1 + x_1 * (X_4)_0 x_1 \\
&= 3x_5 * x_1 + x_1 * (3x_5) \\
&= 6x_5 * x_1.
\end{aligned}$$

Therefore

$$x_5 * x_1 = 0.$$

$\square$

Lemma 28. *We have* $h_1 * x_1 = 0$.

Proof. Applying $(Y_1)_0$ to $(x_1 * x_1)$, we have

$$\begin{aligned}
0 &= (Y_1)_0(x_1 * x_1) \\
&= ((Y_1)_0 x_1) * x_1 + x_1 * (Y_1)_0 x_1 \\
&= -h_1 * x_1 + x_1 * (-h_1) \\
&= -2h_1 * x_1.
\end{aligned}$$

Therefore

$$h_1 * x_1 = 0.$$

$\square$

Lemma 29. *We have* $y_2 * x_1 = 0$.

Proof. Applying $(Y_3)_0$ to $(x_1 * x_1)$, we have

$$\begin{aligned}
0 &= (Y_3)_0(x_1 * x_1) \\
&= ((Y_3)_0 x_1) * x_1 + x_1 * (Y_3)_0 x_1 \\
&= 3y_2 * x_1 + x_1 * (3y_2) \\
&= 6y_2 * x_1.
\end{aligned}$$

Therefore

$$y_2 * x_1 = 0.$$

□

Lemma 30. *We have* $y_3 * x_1 = 0$.

Proof. Applying $(Y_4)_0$ to $(x_1 * x_1)$, we have

$$\begin{aligned}
0 &= (Y_4)_0(x_1 * x_1) \\
&= ((Y_4)_0 x_1) * x_1 + x_1 * (Y_4)_0 x_1 \\
&= 2y_3 * x_1 + x_1 * (2y_3) \\
&= 4y_3 * x_1.
\end{aligned}$$

Therefore

$$y_3 * x_1 = 0.$$

□

III.2.0 **Lemma 31-Lemma 40**

Lemma 31. *We have* $y_4 * x_1 = 0$.

Proof. Applying $(Y_5)_0$ to $(x_1 * x_1)$, we have

$$\begin{aligned} 0 &= (Y_5)_0(x_1 * x_1) \\ &= ((Y_5)_0 x_1) * x_1 + x_1 * (Y_5)_0 x_1 \\ &= -y_4 * x_1 + x_1 * (-y_4) \\ &= -2y_4 * x_1. \end{aligned}$$

Therefore

$$y_4 * x_1 = 0.$$

□

Lemma 32. *We have* $x_2 * x_3 = 0$.

Proof. Applying $(X_1)_0$ to $(x_2 * x_2)$, we have

$$\begin{aligned} 0 &= (X_1)_0(x_2 * x_2) \\ &= ((X_1)_0 x_2) * x_2 + x_2 * (X_1)_0 x_2 \\ &= x_3 * x_2 + x_2 * x_3 \\ &= 2x_3 * x_2. \end{aligned}$$

Therefore

$$x_3 * x_2 = 0.$$

□

Lemma 33. *We have* $x_2 * x_6 = 0$.

Proof. Applying $(X_5)_0$ to $(x_2 * x_2)$, we have

$$0 = (X_5)_0(x_2 * x_2)$$

$$
\begin{aligned}
&= ((X_5)_0 x_2) * x_2 + x_2 * (X_5)_0 x_2 \\
&= x_6 * x_2 + x_2 * x_6 \\
&= 2x_6 * x_2.
\end{aligned}
$$

Therefore

$$x_6 * x_2 = 0.$$

□

Lemma 34. *We have $x_2 * h_2 = 0$.*

Proof. Applying $(Y_2)_0$ to $(x_2 * x_2)$, we have

$$
\begin{aligned}
0 \quad &= (Y_2)_0(x_2 * x_2) \\
&= (Y_2)_0 x_2) * x_2 + x_2 * (Y_2)_0 x_2 \\
&= -h_2 * x_2 + x_2 * (-h_2) \\
&= -2h_2 * x_2.
\end{aligned}
$$

Therefore

$$h_2 * x_2 = 0.$$

□

Lemma 35. *We have $x_2 * y_1 = 0$.*

Proof. Applying $(Y_3)_0$ to $(x_2 * x_2)$, we have

$$
\begin{aligned}
0 \quad &= (Y_3)_0(x_2 * x_2) \\
&= ((Y_3)_0 x_2) * x_2 + x_2 * (Y_3)_0 x_2 \\
&= -y_1 * x_2 + x_2 * (-y_1) \\
&= -2y_1 * x_2.
\end{aligned}
$$

Therefore

$$y_1 * x_2 = 0.$$

$\square$

Lemma 36. *We have $x_2 * y_5 = 0$.*

Proof. Applying $(Y_6)_0$ to $(x_2 * x_2)$, we have

$$\begin{aligned} 0 &= (Y_6)_0(x_1 * x_1) \\ &= ((Y_6)_0 x_2) * x_2 + x_2 * (Y_6)_0 x_2 \\ &= -y_5 * x_2 + x_2 * (-y_5) \\ &= -2y_5 * x_2. \end{aligned}$$

Therefore

$$y_5 * x_2 = 0.$$

$\square$

Lemma 37. *We have $x_3 * x_4 = 0$.*

Proof. Applying $(X_1)_0$ to $(x_3 * x_3)$, we have

$$\begin{aligned} 0 &= (X_1)_0(x_3 * x_3) \\ &= ((X_1)_0 x_3) * x_3 + x_3 * (X_1)_0 x_3 \\ &= 2x_4 x_3 + x_3 * 2x_4 \\ &= 4x_4 * x_3. \end{aligned}$$

Therefore

$$x_4 * x_3 = 0.$$

$\square$

Lemma 38. *We have $x_3 * x_6 = 0$.*

Proof. Applying $(X_4)_0$ to $(x_3 * x_3)$, we have

$$\begin{aligned} 0 &= (X_4)_0(x_3 * x_3) \\ &= ((X_4)_0 x_3) * x_3 + x_3 * (X_4)_0 x_3 \\ &= 3x_6 * x_3 + x_3 * 3x_6 \\ &= 6x_6 * x_3. \end{aligned}$$

Therefore

$$x_6 * x_3 = 0.$$

□

Lemma 39. *We have $x_3 * (h_1 + 3h_2) = 0$.*

Proof. Applying $(Y_3)_0$ to $(x_3 * x_3)$, we have

$$\begin{aligned} 0 &= (Y_3)_0(x_3 * x_3) \\ &= ((Y_3)_0 x_3) * x_3 + x_3 * (Y_3)_0 x_3 \\ &= -(h_1 + 3h_2) * x_3 + x_3 * (-h_1 - 3h_2) \\ &= (-2h_1 - 6h_2) * x_3. \end{aligned}$$

Therefore

$$(h_1 + 3h_2) * x_3 = 0.$$

□

Lemma 40. *We have $x_3 * y_1 = 0$.*

Proof. Applying $(Y_4)_0$ to $(x_3 * x_3)$, we have

$$\begin{aligned} 0 &= (Y_4)_0(x_3 * x_3) \\ &= ((Y_4)_0 x_3) * x_3 + x_3 * (Y_4)_0 x_3 \\ &= -2y_1 * x_3 + x_3 * -2y_1 \\ &= -4y_1 * x_3. \end{aligned}$$

Therefore

$$y_1 * x_3 = 0.$$

□

III.2.0 **Lemma 41-Lemma 50**

Lemma 41. *We have* $x_3 * y_4 = 0$*.*

Proof. Applying $(Y_6)_0$ to $(x_3 * x_3)$, we have

$$\begin{aligned} 0 &= (Y_6)_0(x_3 * x_3) \\ &= ((Y_6)_0 x_3) * x_3 + x_3 * (Y_6)_0 x_3 \\ &= -y_4 * x_3 + x_3 * -y_4 \\ &= -2y_4 * x_3. \end{aligned}$$

Therefore

$$y_4 * x_3 = 0.$$

□

Lemma 42. *We have* $x_4 * x_5 = 0$*.*

Proof. Applying $(X_1)_0$ to $(x_4 * x_4)$, we have

$$\begin{aligned} 0 \quad &= (X_1)_0(x_4 * x_4) \\ &= ((X_1)_0 x_4) * x_4 + x_4 * (X_1)_0 x_4 \\ &= -3x_5 * x_4 + x_4 * -3x_5 \\ &= -6x_5 * x_4. \end{aligned}$$

Therefore

$$x_5 * x_4 = 0.$$

□

Lemma 43. *We have* $x_4 * x_6 = 0$.

Proof. Applying $(X_3)_0$ to $(x_4 * x_4)$, we have

$$\begin{aligned} 0 \quad &= (X_3)_0(x_4 * x_4) \\ &= ((X_3)_0 x_4) * x_4 + x_4 * (X_3)_0 x_4 \\ &= -3x_6 * x_4 + x_4 * -3x_6 \\ &= -6x_6 * x_4. \end{aligned}$$

Therefore

$$x_6 * x_4 = 0.$$

□

Lemma 44. *We have* $x_4 * y_1 = 0$.

Proof. Applying $(Y_5)_0$ to $(x_4 * x_4)$, we have

$$0 \quad = (Y_5)_0(x_4 * x_4)$$

$$
\begin{aligned}
&= ((Y_5)_0 x_4) * x_4 + x_4 * (Y_5)_0 x_4 \\
&= y_1 * x_4 + x_4 * y_1 \\
&= 2y_1 * x_4.
\end{aligned}
$$

Therefore

$$y_1 * x_4 = 0.$$

□

Lemma 45. *We have $x_4 * y_3 = 0$.*

Proof. Applying $(Y_6)_0$ to $(x_4 * x_4)$, we have

$$
\begin{aligned}
0 \quad &= (Y_6)_0(x_4 * x_4) \\
&= ((Y_6)_0 x_4) * x_4 + x_4 * (Y_6)_0 x_4 \\
&= y_3 * x_4 + x_4 * y_3 \\
&= 2y_3 * x_4.
\end{aligned}
$$

Therefore

$$y_3 * x_4 = 0.$$

□

Lemma 46. *We have $x_5 * (-2h_1 - 2h_2) = 0$.*

Proof. Applying $(Y_5)_0$ to $(x_5 * x_5)$, we have

$$
\begin{aligned}
0 \quad &= (Y_5)_0(x_5 * x_5) \\
&= ((Y_5)_0 x_5) * x_5 + x_5 * (Y_5)_0 x_5 \\
&= -(h_1 + h_2) * x_5 + x_5 * -(h_1 + h_2) \\
&= -2(h_1 + h_2) * x_5.
\end{aligned}
$$

Therefore

$$(-2h_1 - 2h_2) * x_5 = 0.$$

□

Lemma 47. *We have* $x_5 * y_2 = 0$.

Proof. Applying $(Y_6)_0$ to $(x_5 * x_5)$, we have

$$\begin{aligned} 0 &= (Y_6)_0(x_5 * x_5) \\ &= ((Y_6)_0 x_5) * x_5 + x_5 * (Y_6)_0 x_5 \\ &= y_2 * x_5 + x_5 * y_2 \\ &= 2y_2 * x_5. \end{aligned}$$

Therefore

$$y_2 * x_5 = 0.$$

□

Lemma 48. *We have* $x_6 * x_5 = 0$.

Proof. Applying $(Y_2)_0$ to $(x_6 * x_6)$, we have

$$\begin{aligned} 0 &= (Y_2)_0(x_6 * x_6) \\ &= ((Y_2)_0 x_6) * x_6 + x_6 * (Y_2)_0 x_6 \\ &= -x_5 * x_6 + x_6 * -x_5 \\ &= -2x_5 * x_6. \end{aligned}$$

Therefore

$$x_5 * x_6 = 0.$$

□

Lemma 49. *For $i \in \{1, ..., 6\}$, we have $y_i * y_i = 0$.*

Proof. We notice that the action of $(H_1)_0$ and $(H_2)_0$ on each respective vector in terms of $x_i's$ and $y_i's$ is just the negation of the other vector. For example,

$$(H_1)_0(x_1) = 2x_1,$$

$$(H_1)_0(y_1) = -2y_1.$$

Here we see that we just negate the respective variables when going back and forth between x and y. As a result, we can simplify the calculations with $y's$. Notice that since

$$x_i * x_j = 0 \text{ where } i = j,$$

we can conclude that

$$y_i * y_j = 0 \text{ where } i = j.$$

□

Lemma 50. *We have $y_1 * y_3 = 0$.*

Proof. Applying $(Y_2)_0$ to $(y_1 * y_1)$, we have

$$\begin{aligned} 0 \quad &= (Y_2)_0(y_1 * y_1) \\ &= ((Y_2)_0 y_1) * y_1 + y_1 * (Y_2)_0 y_1 \\ &= y_3 * y_1 + y_1 * (y_3) \\ &= 2y_3 * y_1. \end{aligned}$$

Therefore

$$y_3 * y_1 = 0.$$

□

III.2.0 **Lemma 51-Lemma 60**

Lemma 51. *We have* $y_1 * y_4 = 0$.

Proof. Applying $(Y_3)_0$ to $(y_1 * y_1)$, we have

$$\begin{aligned} 0 &= (Y_3)_0(y_1 * y_1) \\ &= ((Y_3)_0 y_1) * y_1 + y_1 * (Y_3)_0 y_1 \\ &= 2y_4 * y_1 + y_1 * (2y_4) \\ &= 4y_4 * x_y. \end{aligned}$$

Therefore

$$y_4 * y_1 = 0.$$

□

Lemma 52. *We have* $y_1 * y_5 = 0$.

Proof. Applying $(Y_4)_0$ to $(y_1 * y_1)$, we have

$$\begin{aligned} 0 &= (Y_4)_0(y_1 * y_1) \\ &= ((Y_4)_0 y_1) * y_1 + y_1 * (Y_4)_0 y_1 \\ &= -3y_5 * y_1 + y_1 * (-3y_5) \\ &= -6y_5 * y_1. \end{aligned}$$

Therefore

$$y_5 * y_1 = 0.$$

□

Lemma 53. *We have* $y_1 * h_1 = 0$.

Proof. Applying $(X_1)_0$ to $(y_1 * y_1)$, we have

$$\begin{aligned} 0 &= (X_1)_0(y_1 * y_1) \\ &= ((X_1)_0 y_1) * y_1 + y_1 * (X_1)_0 y_1 \\ &= h_1 * y_1 + y_1 * (h_1) \\ &= 2h_1 * y_1. \end{aligned}$$

Therefore

$$h_1 * y_1 = 0.$$

□

Lemma 54. *We have* $y_2 * y_3 = 0$.

Proof. Applying $(Y_1)_0$ to $(y_2 * y_2)$, we have

$$\begin{aligned} 0 &= (Y_1)_0(y_2 * y_2) \\ &= ((Y_1)_0 y_2) * y_2 + y_2 * (Y_1)_0 y_2 \\ &= -y_3 * y_2 + y_2 * -y_3 \\ &= -2y_3 * y_2. \end{aligned}$$

Therefore

$$y_3 * y_2 = 0.$$

□

Lemma 55. *We have* $y_2 * h_2 = 0$.

Proof. Applying $(X_2)_0$ to $(y_2 * y_2)$, we have

$$0 = (X_2)_0(y_2 * y_2)$$

$$
\begin{aligned}
&= ((X_2)_0 y_2) * y_2 + y_2 * (X_2)_0 y_2 \\
&= h_2 * y_2 + y_2 * h_2 \\
&= 2h_2 * y_2.
\end{aligned}
$$

Therefore

$$h_2 * y_2 = 0.$$

□

Lemma 56. *We have $y_2 * y_6 = 0$.*

Proof. Applying $(Y_5)_0$ to $(y_2 * y_2)$, we have

$$
\begin{aligned}
0 &= (Y_5)_0(y_2 * y_2) \\
&= ((Y_5)_0 y_2) * y_2 + y_2 * (Y_5)_0 y_2 \\
&= -y_6 * y_2 + y_2 * -y_6 \\
&= -2y_6 * y_2.
\end{aligned}
$$

Therefore

$$y_6 * y_2 = 0.$$

□

Lemma 57. *We have $y_3 * y_4 = 0$.*

Proof. Applying $(Y_1)_0$ to $(y_3 * y_3)$, we have

$$
\begin{aligned}
0 &= (Y_1)_0(y_3 * y_3) \\
&= ((Y_1)_0 y_3) * y_3 + y_3 * (Y_1)_0 y_3 \\
&= -2y_4 * y_3 + y_3 * -2y_4 \\
&= -4y_4 * y_3.
\end{aligned}
$$

Therefore

$$y_4 * y_3 = 0.$$

□

Lemma 58. *We have* $y_3 * (2h_1 + 6h_2) = 0$.

Proof. Applying $(X_3)_0$ to $(y_3 * y_3)$, we have

$$\begin{aligned}
0 &= (X_3)_0(y_3 * y_3) \\
&= ((X_3)_0 y_3) * y_3 + y_3 * (X_3)_0 y_3 \\
&= (h_1 + 3h_2) * y_3 + y_3 * (h_1 + 3h_2) \\
&= 2(h_1 + 3h_2) * y_3.
\end{aligned}$$

Therefore

$$(2h_1 + 6h_2) * y_3 = 0.$$

□

Lemma 59. *We have* $y_3 * y_6 = 0$.

Proof. Applying $(Y_4)_0$ to $(y_3 * y_3)$, we have

$$\begin{aligned}
0 &= (Y_4)_0(y_3 * y_3) \\
&= ((Y_4)_0 y_3) * y_3 + y_3 * (Y_4)_0 y_3 \\
&= -3y_6 * y_3 + y_3 * -3y_6 \\
&= -6y_6 * y_3.
\end{aligned}$$

Therefore

$$y_6 * y_3 = 0.$$

□

Lemma 60. *We have $y_4 * y_5 = 0$.*

Proof. Applying $(Y_1)_0$ to $(y_4 * y_4)$, we have

$$\begin{aligned} 0 &= (Y_1)_0(y_4 * y_4) \\ &= ((Y_1)_0 y_4) * y_4 + y_4 * (Y_1)_0 y_4 \\ &= 3y_5 * y_4 + y_4 * 3y_5 \\ &= 6y_5 * y_4. \end{aligned}$$

Therefore

$$y_5 * y_4 = 0.$$

□

III.2.0 Lemma 61-Lemma 70

Lemma 61. *We have $y_4 * y_6 = 0$.*

Proof. Applying $(Y_3)_0$ to $(y_4 * y_4)$, we have

$$\begin{aligned} 0 &= (Y_3)_0(y_4 * y_4) \\ &= ((Y_3)_0 y_4) * y_4 + y_4 * (Y_3)_0 y_4 \\ &= 3y_6 * y_4 + y_4 * 3y_6 \\ &= 6y_6 * y_4. \end{aligned}$$

Therefore

$$y_6 * y_4 = 0.$$

□

Lemma 62. *We have $y_4 * (4h_1 + 6h_2) = 0$.*

Proof. Applying $(X_4)_0$ to $(y_4 * y_4)$, we have

$$\begin{aligned} 0 &= (X_4)_0(y_4 * y_4) \\ &= ((X_4)_0 y_4) * y_4 + y_4 * (X_4)_0 y_4 \\ &= (2h_1 + 3h_2) * y_4 + y_4 * (2h_1 + 3h_2) \\ &= 2(2h_1 + 3h_2) * y_4. \end{aligned}$$

Therefore

$$4h_1 + 6h_2 * y_4 = 0.$$

$\square$

Lemma 63. *We have* $y_5 * y_6 = 0$.

Proof. Applying $(Y_2)_0$ to $(y_5 * y_5)$, we have

$$\begin{aligned} 0 &= (Y_2)_0(y_5 * y_5) \\ &= ((Y_2)_0 y_5) * y_5 + y_5 * (Y_2)_0 y_5 \\ &= y_6 * y_5 + y_5 * y_6 \\ &= 2y_6 * y_5. \end{aligned}$$

Therefore

$$y_6 * y_5 = 0.$$

$\square$

Lemma 64. *We have* $y_5 * (2h_1 + 2h_2) = 0$.

Proof. Applying $(X_5)_0$ to $(y_5 * y_5)$, we have

$$0 = (X_5)_0(y_5 * y_5)$$

$$= ((X_5)_0 y_5) * y_5 + y_5 * (X_5)_0 y_5$$

$$= (h_1 + h_2) * y_5 + y_5 * (h_1 + h_2)$$

$$= 2(h_1 + h_2) * y_5.$$

Therefore

$$2h_1 + 2h_2 * y_5 = 0.$$

□

Lemma 65. *We have* $y_6 * (2h_1 + 4h_2) = 0$.

Proof. Applying $(X_6)_0$ to $(y_6 * y_6)$, we have

$$\begin{aligned} 0 &= (X_6)_0(y_6 * y_6) \\ &= ((X_6)_0 y_6) * y_6 + y_6 * (X_6)_0 y_6 \\ &= (h_1 + 2h_2) * y_6 + y_6 * (h_1 + 2h_2) \\ &= 2(h_1 + 2h_2) * y_6. \end{aligned}$$

Therefore

$$2h_1 + 4h_2 * y_6 = 0.$$

□

Lemma 66. *We have* $x_1 * x_2 = 0$.

Proof. Applying $(H_1)_0$ to $x_1 * x_2$, we have

$$\begin{aligned} (H_1)_0(x_1 * x_2) &= ((H_1)_0 x_1) * x_2 + x_1 * ((H_1)_0 x_2) \\ &= 2x_1 * x_2 + x_1 * -3x_2 \\ &= -x_1 * x_2 \end{aligned}$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& -(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$x_1 * x_2 = b_3(x_3) + c_4(y_4).$$

Applying $(H_2)_0$ to $x_1 * x_2$, we have

$$\begin{aligned}
(H_2)_0(x_1 * x_2) \quad & = ((H_2)_0 x_1) * x_2 + x_1 * ((H_2)_0 x_2) \\
& = -x_1 * x_2 + x_1 * 2x_2 \\
& = x_1 * x_2, \\
(H_2)_0(b_3(x_3) + c_4(y_4)) \quad & = b_3(x_3) + 0.
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_3(x_3) + 0.
\end{aligned}$$

We see that $b_3 = c_4 = 0$. Therefore

$$x_1 * x_2 = 0.$$

$\square$

Lemma 67. *We have $x_1 * x_6 = 0$.*

Proof. Applying $(H_1)_0$ to $x_1 * x_6$, we have

$$\begin{aligned}
(H_1)_0(x_1 * x_6) \quad &= ((H_1)_0 x_1) * x_6 + x_1 * ((H_1)_0 x_6) \\
&= 2x_1 * x_6 + x_1 * 0x_6 \\
&= 2x_1 * x_6
\end{aligned}$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& 2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Here we see that

$a = 0, b_2 = 0 = b_3 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$x_1 * x_6 = b_1(x_1).$$

Applying $(H_2)_0$ to $x_1 * x_6$, we have

$$\begin{aligned}
(H_2)_0(x_1 * x_6) \quad &= ((H_2)_0 x_1) * x_6 + x_1 * ((H_2)_0 x_6) \\
&= -x_1 * x_6 + x_1 * x_6 \\
&= 0, \\
(H_2)_0(b_1(x_1)) \quad &= b_1(-x_1).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(-x_1).
\end{aligned}$$

We see that $b_1 = 0$. Therefore

$$x_1 * x_6 = 0.$$

$\square$

Lemma 68. *We have* $x_2 * x_4 = 0$.

Proof. Applying $(H_1)_0$ to $x_2 * x_4$, we have

$$\begin{aligned}
(H_1)_0(x_2 * x_4) \quad &= ((H_1)_0 x_2) * x_4 + x_2 * ((H_1)_0 x_4) \\
&= -3x_2 * x_4 + x_2 * x_4 \\
&= -2x_2 * x_4
\end{aligned}$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0)
\end{aligned}$$

$$+c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).$$

Now we compare our two equations:

$$\begin{aligned} & -2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & +c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Here we see that

$a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5 = b_6, c_2 = 0 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$x_2 * x_4 = c_1(y_1).$$

Applying $(H_2)_0$ to $x_2 * x_4$, we have

$$\begin{aligned} (H_2)_0(x_2 * x_4) \quad & = ((H_2)_0 x_2) * x_4 + x_2 * ((H_2)_0 x_4) \\ & = 2x_2 * x_4 + x_2 * 0x_4 \\ & = 2x_2 * x_4, \\ (H_2)_0(c_1(y_1)) \quad & = c_1(y_1). \end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & 2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & c_1(y_1). \end{aligned}$$

We see that $c_1 = 0$. Therefore

$$x_2 * x_4 = 0.$$

$\square$

Lemma 69. *There exists $b_6 \in \mathbb{C}$ such that $x_2 * x_5 = b_6(x_6)$.*

Proof. Applying $(H_1)_0$ to $x_2 * x_5$, we have

$$
\begin{aligned}
(H_1)_0(x_2 * x_5) \quad &= ((H_1)_0 x_2) * x_5 + x_2 * ((H_1)_0 x_5) \\
&= -3x_2 * x_5 + x_2 * 3x_5 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& 0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5, c_2 = 0 = c_3 = c_4 = c_5, d_1 = 0 = d_2$. Therefore

$$x_2 * x_5 = b_6(x_6) + c_6(y_6).$$

Applying $(H_2)_0$ to $x_2 * x_5$, we have

$$(H_2)_0(x_2 * x_5) \quad = ((H_2)_0 x_2) * x_5 + x_2 * ((H_2)_0 x_5)$$

$$
\begin{aligned}
&= 2x_2 * x_5 + x_2 * -x_5 \\
&= x_2 * x_5, \\
(H_2)_0(b_6(x_6) + c_6(y_6)) &= b_6(x_6) - c_6(y_6).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_6(x_6) - c_6(y_6).
\end{aligned}
$$

We see that $c_6 = 0$. Therefore

$$
x_2 * x_5 = b_6(x_6).
$$

□

Lemma 70. *We have* $x_3 * x_5 = 0$.

Proof. Applying $(H_1)_0$ to $x_3 * x_5$, we have

$$
\begin{aligned}
(H_1)_0(x_3 * x_5) &= ((H_1)_0 x_3) * x_5 + x_3 * ((H_1)_0 x_5) \\
&= -x_3 * x_5 + x_3 * 3x_5 \\
&= 2x_3 * x_5
\end{aligned}
$$

and

$$
\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$\begin{aligned} & 2(a1+\sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Here we see that

$a=0, b_2=0=b_3=b_4=b_5=b_6, c_1=0=c_2=c_3=c_4=c_5=c_6, d_1=0=d_2$. Therefore

$$x_3 * x_5 = b_1(x_1).$$

Applying $(H_2)_0$ to $x_3 * x_5$, we have

$$\begin{aligned} (H_2)_0(x_3 * x_5) \quad & = ((H_2)_0 x_3) * x_5 + x_3 * ((H_2)_0 x_5) \\ & = x_3 * x_5 + x_3 * -x_5 \\ & = 0, \\ (H_2)_0(b_1(x_1)) \quad & = b_1(-x_1). \end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & 0(a1+\sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(-x_1). \end{aligned}$$

We see that $b_1 = 0$. Therefore

$$x_3 * x_5 = 0.$$

□

III.2.0 **Lemma 71-Lemma 80**

Lemma 71. *There exists $c_3 \in \mathbb{C}$ such that $y_1 * y_2 = c_3(y_3)$.*

Proof. Applying $(H_1)_0$ to $y_1 * y_2$, we have

$$\begin{aligned} (H_1)_0(y_1 * y_2) \quad &= ((H_1)_0 y_1) * y_2 + y_1 * ((H_1)_0 y_2) \\ &= -2y_1 * y_2 + y_1 * 3y_2 \\ &= y_1 * y_2 \end{aligned}$$

and

$$\begin{aligned} & (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_5 = b_6, c_1 = 0 = c_2 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$y_1 * y_2 = b_4(x_4) + c_3(y_3).$$

Applying $(H_2)_0$ to $y_1 * y_2$, we have .

$$(H_2)_0(y_1 * y_2) \quad = ((H_2)_0 y_1) * y_2 + y_1 * ((H_2)_0 y_2)$$

$$
\begin{aligned}
& && = y_1 * y_2 + y_1 * -2y_2 \\
& && = -y_1 * y_2, \\
& (H_2)_0(b_4(x_4) + c_3(y_3)) && = 0 + c_3(-y_3).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& && -(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
& = && c_3(-y_3).
\end{aligned}
$$

We see that $b_4 = 0$. Therefore

$$
y_1 * y_2 = c_3(y_3).
$$

□

Lemma 72. *We have* $y_1 * y_6 = 0$.

Proof. Applying $(H_1)_0$ to $y_1 * y_6$, we have

$$
\begin{aligned}
(H_1)_0(y_1 * y_6) &= ((H_1)_0 y_1) * y_6 + y_1 * ((H_1)_0 y_6) \\
&= -2y_1 * y_2 + y_1 * 0y_2 \\
&= -2y_1 * y_2
\end{aligned}
$$

and

$$
\begin{aligned}
& && (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
& = && b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& && + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$\begin{aligned}
& -2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Here we see that

$a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5 = b_6, c_2 = 0 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$y_1 * y_6 = c_1(y_1).$$

Applying $(H_2)_0$ to $y_1 * y_6$, we have

$$\begin{aligned}
(H_2)_0(y_1 * y_6) \quad & = ((H_2)_0 y_1) * y_6 + y_1 * ((H_2)_0 y_6) \\
& = y_1 * y_6 + y_1 * -y_6 \\
& = 0, \\
(H_2)_0(c_1(y_1)) \quad & = c_1(y_1).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& 0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & c_1(y_1).
\end{aligned}$$

We see that $c_1 = 0$. Therefore

$$y_1 * y_6 = 0.$$

$\square$

Lemma 73. *We have $y_2 * y_4 = 0$.*

Proof. Applying $(H_1)_0$ to $y_2 * y_4$, we have

$$
\begin{aligned}
(H_1)_0(y_2 * y_4) \quad &= ((H_1)_0 y_2) * y_4 + y_2 * ((H_1)_0 y_4) \\
&= 3y_2 * y_4 + y_2 * -y_4 \\
&= 2y_2 * y_4
\end{aligned}
$$

and

$$
\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& 2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Here we see that

$a = 0, b_2 = 0 = b_3 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$y_2 * y_4 = b_1(x_1).$$

Applying $(H_2)_0$ to $y_2 * y_4$, we have

$$
\begin{aligned}
(H_2)_0(y_2 * y_4) \quad &= ((H_2)_0 y_2) * y_4 + y_2 * ((H_2)_0 y_4) \\
&= -2y_2 * y_4 + y_2 * 0y_6
\end{aligned}
$$

$$
\begin{aligned}
&= -2y_2 * y_4, \\
(H_2)_0(b_1(x_1)) \quad &= b_1(-x_1).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& -2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(-x_1).
\end{aligned}
$$

We see that $b_1 = 0$. Therefore

$$y_2 * y_4 = 0.$$

□

Lemma 74. *There is $b_6 \in \mathbb{C}$ such that $y_2 * y_5 = b_6(x_6)$.*

Proof. Applying $(H_1)_0$ to $y_2 * y_5$, we have

$$
\begin{aligned}
(H_1)_0(y_2 * y_5) \quad &= ((H_1)_0 y_2) * y_5 + y_2 * ((H_1)_0 y_5) \\
&= 3y_2 * y_5 + y_2 * -3y_5 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& +c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$\begin{aligned} & 0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5, c_1 = 0 = c_2 = c_3 = c_4 = c_5, d_1 = 0 = d_2$. Therefore

$$y_2 * y_5 = b_6(x_6) + c_6(y_6).$$

Applying $(H_2)_0$ to $y_2 * y_5$, we have

$$\begin{aligned} (H_2)_0(y_2 * y_5) \quad & = ((H_2)_0 y_2) * y_5 + y_2 * ((H_2)_0 y_5) \\ & = -2y_2 * y_5 + y_2 * y_5 \\ & = -y_2 * y_5, \\ (H_2)_0(b_6(x_6) + c_6(y_6)) \quad & = b_6(x_6) + c_6(-y_6). \end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & -(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_6(x_6) + c_6(-y_6). \end{aligned}$$

We see that $c_6 = 0$. Therefore

$$y_2 * y_5 = b_6(x_6).$$

$\square$

Lemma 75. *We have* $y_3 * y_5 = 0$.

Proof. Applying $(H_1)_0$ to $y_3 * y_5$, we have

$$\begin{aligned}
(H_1)_0(y_3 * y_5) \quad &= ((H_1)_0 y_3) * y_5 + y_3 * ((H_1)_0 y_5) \\
&= y_3 * y_5 + y_3 * -3y_5 \\
&= -2y_3 * y_5
\end{aligned}$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& +c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& -2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& +c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Here we see that

$a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5 = b_6, c_2 = 0 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$y_3 * y_5 = c_1(y_1).$$

Applying $(H_2)_0$ to $y_3 * y_5$, we have

$$\begin{aligned}
(H_2)_0(y_3 * y_5) \quad &= ((H_2)_0 y_3) * y_5 + y_3 * ((H_2)_0 y_5) \\
&= -y_3 * y_5 + y_3 * y_5
\end{aligned}$$

$$
\begin{aligned}
&= 0, \\
(H_2)_0(c_1(y_1)) \qquad &= c_1(y_1).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& 0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & c_1(y_1).
\end{aligned}
$$

We see that $c_1 = 0$. Therefore

$$y_3 * y_5 = 0.$$

□

Lemma 76. *There exists* $b_2 \in \mathbb{C}$ *such that* $h_1 * x_2 = b_2(x_2)$.

Proof. Applying $(H_1)_0$ to $h_1 * x_2$, we have

$$
\begin{aligned}
(H_1)_0(h_1 * x_2) \qquad &= ((H_1)_0 h_1) * x_2 + h_1 * ((H_1)_0 x_2) \\
&= 0h_1 * x_2 + h_1 * -3x_2 \\
&= -3h_1 * x_2
\end{aligned}
$$

and

$$
\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$\begin{aligned} & -3(a1+\sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & +c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_3 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_4 = c_6, d_1 = 0 = d_2$. Therefore

$$h_1 * x_2 = b_2(x_2) + c_5(y_5).$$

Applying $(H_2)_0$ to $h_1 * x_2$, we have

$$\begin{aligned} (H_2)_0(h_1 * x_2) \quad & = ((H_2)_0 h_1) * x_2 + h_1 * ((H_2)_0 x_2) \\ & = 0h_1 * x_2 + h_1 * 2x_2 \\ & = 2h_1 * x_2, \\ (H_2)_0(b_2(x_2) + c_5(y_5)) \quad & = b_2(2x_2) + c_5(y_5). \end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & 2(a1+\sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_2(2x_2) + c_5(y_5). \end{aligned}$$

We see that $c_5 = 0$. Therefore

$$h_1 * x_2 = b_2(x_2).$$

$\square$

Lemma 77. *There exists $b_3 \in \mathbb{C}$ such that $h_1 * x_3 = b_3(x_3)$.*

Proof. Applying $(H_1)_0$ to $h_1 * x_3$, we have

$$\begin{aligned}
(H_1)_0(h_1 * x_3) \quad &= ((H_1)_0 h_1) * x_3 + h_1 * ((H_1)_0 x_3) \\
&= 0h_1 * x_3 + h_1 * -x_3 \\
&= -h_1 * x_3
\end{aligned}$$

and

$$\begin{aligned}
&(H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
&-(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$h_1 * x_3 = b_3(x_3) + c_4(y_4).$$

Applying $(H_2)_0$ to $h_1 * x_3$, we have

$$\begin{aligned}
(H_2)_0(h_1 * x_3) \quad &= ((H_2)_0 h_1) * x_3 + h_1 * ((H_2)_0 x_3) \\
&= 0h_1 * x_3 + h_1 * x_3 \\
&= h_1 * x_3, \\
(H_2)_0(b_3(x_3) + c_4(y_4)) \quad &= b_3(x_3) + 0.
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_3(x_3) + 0. \end{aligned}$$

We see that $c_4 = 0$. Therefore

$$h_1 * x_3 = b_3(x_3).$$

□

Lemma 78. *We have* $h_1 * x_4 = 0$.

Proof. Applying $(H_1)_0$ to $h_1 * x_4$, we have

$$\begin{aligned} (H_1)_0(h_1 * x_4) \quad &= ((H_1)_0 h_1) * x_4 + h_1 * ((H_1)_0 x_4) \\ &= 0h_1 * x_4 + h_1 * x_4 \\ &= h_1 * x_4 \end{aligned}$$

and

$$\begin{aligned} & (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_5 = b_6, c_1 = 0 = c_2 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$h_1 * x_3 = b_4(x_4) + c_3(y_3).$$

Applying $(H_2)_0$ to $h_1 * x_4$, we have

$$\begin{aligned}
(H_2)_0(h_1 * x_4) &= ((H_2)_0 h_1) * x_4 + h_1 * ((H_2)_0 x_4) \\
&= 0h_1 * x_4 + h_1 * 0x_4 \\
&= 0, \\
(H_2)_0(b_4(x_4) + c_3(y_3)) &= b_4(x_4) + c_3(-y_3).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& 0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_4(x_4) + c_3(-y_3).
\end{aligned}$$

We see that $b_4 = c_3 = 0$. Therefore

$$h_1 * x_4 = 0.$$

$\square$

Lemma 79. *There is $b_5 \in \mathbb{C}$ such that $h_1 * x_5 = b_5(x_5)$.*

Proof. Applying $(H_1)_0$ to $h_1 * x_5$, we have

$$\begin{aligned}
(H_1)_0(h_1 * x_5) &= ((H_1)_0 h_1) * x_5 + h_1 * ((H_1)_0 x_5) \\
&= 0h_1 * x_5 + h_1 * 3x_5 \\
&= 3h_1 * x_5
\end{aligned}$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& 3(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_6, c_1 = 0 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$h_1 * x_4 = b_5(x_5) + c_2(y_2).$$

Applying $(H_2)_0$ to $h_1 * x_5$, we have

$$\begin{aligned}
(H_2)_0(h_1 * x_5) \quad & = ((H_2)_0 h_1) * x_5 + h_1 * ((H_2)_0 x_5) \\
& = 0h_1 * x_5 + h_1 * -x_5 \\
& = -h_1 * x_5, \\
(H_2)_0(b_5(x_5) + c_2(y_2)) \quad & = b_5(-x_5) + c_2(-2y_2).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& -(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_5(-x_5) + c_2(-2y_2).
\end{aligned}$$

We see that $c_2 = 0$. Therefore

$$h_1 * x_5 = b_5(x_5).$$

$\square$

Lemma 80. *There is $b_6 \in \mathbb{C}$ such that $h_1 * x_6 = b_6(x_6)$.*

Proof. Applying $(H_1)_0$ to $h_1 * x_6$, we have

$$\begin{aligned}(H_1)_0(h_1 * x_6) \quad &= ((H_1)_0 h_1) * x_6 + h_1 * ((H_1)_0 x_6)\\ &= 0h_1 * x_6 + h_1 * 0x_6\\ &= 0h_1 * x_6\end{aligned}$$

and

$$\begin{aligned}&(H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2)\\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0)\\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}&0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2)\\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0)\\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).\end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5, c_1 = 0 = c_2 = c_3 = c_4 = c_5, d_1 = 0 = d_2$. Therefore

$$h_1 * x_5 = b_6(x_6) + c_6(y_6).$$

Applying $(H_2)_0$ to $h_1 * x_6$, we have

$$\begin{aligned}(H_2)_0(h_1 * x_6) \quad &= ((H_2)_0 h_1) * x_6 + h_1 * ((H_2)_0 x_6) \\ &= 0h_1 * x_6 + h_1 * x_6 \\ &= h_1 * x_6, \\ (H_2)_0(b_6(x_6) + c_6(y_6)) \quad &= b_6(x_6) + c_6(-y_6).\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}&(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad &b_6(x_6) + c_6(-y_6).\end{aligned}$$

We see that $c_6 = 0$. Therefore

$$h_1 * x_6 = b_6(x_6).$$

□

III.2.0 **Lemma 81-Lemma 90**

Lemma 81. *There is $b_1 \in \mathbb{C}$ such that $h_2 * x_1 = b_1(x_1)$.*

Proof. Applying $(H_1)_0$ to $h_2 * x_1$, we have

$$\begin{aligned}(H_1)_0(h_2 * x_1) \quad &= ((H_1)_0 h_2) * x_1 + h_2 * ((H_1)_0 x_1) \\ &= 0h_2 * x_1 + h_2 * 2x_1 \\ &= 2h_2 * x_1\end{aligned}$$

and

$$(H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2)$$

$$
\begin{aligned}
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Here we see that

$a = 0, b_2 = 0 = b_3 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$h_2 * x_1 = b_1(x_1).$$

Applying $(H_2)_0$ to $h_2 * x_1$, we have

$$
\begin{aligned}
(H_2)_0(h_2 * x_1) \quad & = ((H_2)_0 h_2) * x_1 + h_2 * ((H_2)_0 x_1) \\
& = 0h_2 * x_1 + h_2 * -x_1 \\
& = -h_2 * x_1, \\
(H_2)_0(b_1(x_1)) \quad & = b_1(-x_1).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& -(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(-x_1).
\end{aligned}
$$

Therefore

$$h_2 * x_1 = b_1(x_1).$$

$\square$

Lemma 82. *We have $h_2 * x_3 = 0$.*

Proof. Applying $(H_1)_0$ to $h_2 * x_3$, we have

$$\begin{aligned}(H_1)_0(h_2 * x_3) \quad &= ((H_1)_0 h_2) * x_3 + h_2 * ((H_1)_0 x_3) \\ &= 0h_2 * x_3 + h_2 * -x_3 \\ &= -h_2 * x_3\end{aligned}$$

and

$$\begin{aligned} & (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & -(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).\end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_4 = b_5 = b_6, c_1 = 0 = c_2 = c_3 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$h_2 * x_3 = b_3(x_3) + c_4(y_4).$$

Applying $(H_2)_0$ to $h_2 * x_3$, we have

$$(H_2)_0(h_2 * x_3) \quad = ((H_2)_0 h_2) * x_3 + h_2 * ((H_2)_0 x_3)$$

$$
\begin{aligned}
& = 0h_2 * x_3 + h_2 * x_3 \\
& = h_2 * x_3, \\
(H_2)_0(b_3(x_3) + c_4(y_4)) & = b_3(x_3) + 0.
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_3(x_3) + 0.
\end{aligned}
$$

We see that $b_3 = c_4 = 0$. Therefore

$$h_2 * x_3 = 0.$$

$\square$

Lemma 83. *We have* $h_2 * x_4 = 0$.

Proof. Applying $(H_1)_0$ to $h_2 * x_4$, we have

$$
\begin{aligned}
(H_1)_0(h_2 * x_4) & = ((H_1)_0 h_2) * x_4 + h_2 * ((H_1)_0 x_4) \\
& = 0h_2 * x_4 + h_2 * 0x_4 \\
& = 0
\end{aligned}
$$

and

$$
\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$\begin{aligned} & 0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5, c_1 = 0 = c_2 = c_3 = c_4 = c_5, d_1 = 0 = d_2$. Therefore

$$h_2 * x_4 = b_6(x_6) + c_6(y_6).$$

Applying $(H_2)_0$ to $h_2 * x_4$, we have

$$\begin{aligned} (H_2)_0(h_2 * x_4) \quad & = ((H_2)_0 h_2) * x_4 + h_2 * ((H_2)_0 x_4) \\ & = 0h_2 * x_4 + h_2 * 0x_4 \\ & = 0, \\ (H_2)_0(b_6(x_6) + c_6(y_6)) \quad & = b_6(x_6) + c_6(-y_6). \end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & 0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_6(x_6) + c_6(-y_6). \end{aligned}$$

We see that $b_6 = c_6 = 0$. Therefore

$$h_2 * x_4 = 0.$$

$\square$

Lemma 84. *There is $b_5 \in \mathbb{C}$ such that $h_2 * x_5 = b_5(x_5)$.*

Proof. Applying $(H_1)_0$ to $h_2 * x_5$, we have

$$\begin{aligned}(H_1)_0(h_2 * x_5) &= ((H_1)_0 h_2) * x_5 + h_2 * ((H_1)_0 x_5)\\ &= 0h_2 * x_5 + h_2 * 3x_5\\ &= 3h_2 * x_5\end{aligned}$$

and

$$\begin{aligned}&(H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2)\\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0)\\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}&3(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2)\\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0)\\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).\end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_6, c_1 = 0 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$h_2 * x_5 = b_5(x_5) + c_2(y_2).$$

Applying $(H_2)_0$ to $h_2 * x_5$, we have

$$\begin{aligned}(H_2)_0(h_2 * x_5) &= ((H_2)_0 h_2) * x_5 + h_2 * ((H_2)_0 x_5)\\ &= 0h_2 * x_5 + h_2 * -x_5\\ &= -h_2 * x_5,\\ (H_2)_0(b_5(x_5) + c_2(y_2)) &= b_5(-x_5) + c_2(-2y_2).\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & -(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_5(-x_5) + c_2(-2y_2). \end{aligned}$$

We see that $c_2 = 0$. Therefore

$$h_2 * x_5 = b_5(x_5).$$

□

Lemma 85. *There is $b_6 \in \mathbb{C}$ such that $h_2 * x_6 = b_6(x_6)$.*

Proof. Applying $(H_1)_0$ to $h_2 * x_6$, we have

$$\begin{aligned} (H_1)_0(h_2 * x_6) \quad & = ((H_1)_0 h_2) * x_6 + h_2 * ((H_1)_0 x_6) \\ & = 0h_2 * x_6 + h_2 * 0x_6 \\ & = 0 \end{aligned}$$

and

$$\begin{aligned} & (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & 0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\ & + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0). \end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_5, c_1 = 0 = c_2 = c_3 = c_4 = c_5, d_1 = 0 = d_2$. Therefore

$$h_2 * x_6 = b_6(x_6) + c_6(y_6).$$

Applying $(H_2)_0$ to $h_2 * x_6$, we have

$$\begin{aligned}
(H_2)_0(h_2 * x_6) \quad &= ((H_2)_0 h_2) * x_6 + h_2 * ((H_2)_0 x_6) \\
&= 0h_2 * x_6 + h_2 * x_6 \\
&= h_2 * x_6, \\
(H_2)_0(b_6(x_6) + c_6(y_6)) \quad &= b_6(x_6) + c_6(-y_6).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_6(x_6) + c_6(-y_6).
\end{aligned}$$

We see that $c_6 = 0$. Therefore

$$h_2 * x_6 = b_6(x_6).$$

□

Lemma 86. *There is $c_2 \in \mathbb{C}$ such that $h_1 * y_2 = c_2(y_2)$.*

Proof. Applying $(H_1)_0$ to $h_1 * y_2$, we have

$$\begin{aligned}
(H_1)_0(h_1 * y_2) \quad &= ((H_1)_0 h_1) * y_2 + h_1 * ((H_1)_0 y_2) \\
&= 0h_1 * y_2 + h_1 * 3y_2 \\
&= 3h_1 * y_2
\end{aligned}$$

and

$$
\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& 3(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}
$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_4 = b_6, c_1 = 0 = c_3 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$h_1 * y_2 = b_5(x_5) + c_2(y_2).$$

Applying $(H_2)_0$ to $h_1 * y_2$, we have

$$
\begin{aligned}
(H_2)_0(h_1 * y_2) \quad & = ((H_2)_0 h_1) * y_2 + h_1 * ((H_2)_0 y_2) \\
& = 0h_1 * y_2 + h_1 * -2y_2 \\
& = -2h_1 * y_2, \\
(H_2)_0(b_5(x_5) + c_2(y_2)) \quad & = b_5(-x_5) + c_2(-2y_2).
\end{aligned}
$$

Now we compare our two equations:

$$
\begin{aligned}
& -2(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_5(-x_5) + c_2(-2y_2).
\end{aligned}
$$

We see that $b_5 = 0$. Therefore

$$h_1 * y_2 = c_2(y_2).$$

$\square$

Lemma 87. *There exists $c_3 \in \mathbb{C}$ such that $h_1 * y_3 = c_3(y_3)$.*

Proof. Applying $(H_1)_0$ to $h_1 * y_3$, we have

$$\begin{aligned}
(H_1)_0(h_1 * y_3) \quad &= ((H_1)_0 h_1) * y_3 + h_1 * ((H_1)_0 y_3) \\
&= 0h_1 * y_3 + h_1 * y_3 \\
&= h_1 * y_3
\end{aligned}$$

and

$$\begin{aligned}
& (H_1)_0(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned}
& (a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\
= \quad & b_1(2x_1) + b_2(-3x_2) + b_3(-x_3) + b_4(x_4) + b_5(3x_5) + b_6(0) \\
& + c_1(-2y_1) + c_2(3y_2) + c_3(y_3) + c_4(-y_4) + c_5(-3y_5) + c_6(0).
\end{aligned}$$

Here we see that $a = 0, b_1 = 0 = b_2 = b_3 = b_5 = b_6, c_1 = 0 = c_2 = c_4 = c_5 = c_6, d_1 = 0 = d_2$. Therefore

$$h_1 * y_3 = b_4(x_4) + c_3(y_3).$$

Applying $(H_2)_0$ to $h_1 * y_3$, we have

$$\begin{aligned}(H_2)_0(h_1 * y_3) \quad &= ((H_2)_0 h_1) * y_3 + h_1 * ((H_2)_0 y_3) \\ &= 0h_1 * y_3 + h_1 * -y_3 \\ &= -h_1 * y_3, \\ (H_2)_0(b_4(x_4) + c_3(y_3)) \quad &= 0 + c_3(-y_3).\end{aligned}$$

Now we compare our two equations:

$$\begin{aligned} & -(a1 + \sum_{i=1}^{6} b_i x_i + \sum_{j=1}^{6} c_j y_j + d_1 h_1 + d_2 h_2) \\ = \quad & 0 + c_3(-y_3).\end{aligned}$$

We see that $b_4 = 0$. Therefore

$$h_1 * y_3 = c_3(y_3).$$

$\square$

After noticing a pattern between $h_1 * x_i$ and $h_1 * y_i$, we can say the following:

Lemma 88.

$$\begin{aligned} h_1 * y_4 \quad &= 0 \\ h_1 * y_5 \quad &= c_5(y_5) \\ h_1 * y_6 \quad &= c_6(y_6)\end{aligned}$$

After noticing a pattern between $h_2 * x_i$ and $h_2 * y_i$, we can say the following:

Lemma 89.

$$h_2 * y_1 \quad = c_1(y_1)$$

$$
\begin{aligned}
h_2 * y_3 &= 0 \\
h_2 * y_4 &= 0 \\
h_2 * y_5 &= c_5(y_5) \\
h_2 * y_6 &= c_6(y_6)
\end{aligned}
$$

for some $c_1, c_5, c_6 \in \mathbb{C}$*.*

Lemma 90. *By properties of Lie bracket, we know that*

$$h_1 * h_1 = h_1 * h_2 = h_2 * h_2 = 0.$$

III.2.0 **Lemma 91-Lemma 100** We can use the properties of eigenvectors with their associated eigenvalues to calculate $x_i * y_i$ in a more simplified way. We will calculate the action of H_1 and H_2 on x_i and y_i where $i = \{1, ..., 6\}$. We will look at the coefficient on these eigenvectors after H_i, $i = \{1, 2\}$, has been applied. We will then compare coefficients to find out what we can eliminate. Observe the following example:

Example 5. *We want to calculate* $h_1 * x_6$*. We will first apply* $(H_1)_0$ *then* $(H_2)_0$*. We see that*

$$
\begin{aligned}
(H_1)_0(h_1 * x_6) &= (0 + 0)(h_1 * x_6) = 0h_1 * x_6, \\
h_1 * x_6 &= b_6 x_6 + c_6 y_6 + d_1 h_1 + d_2 h_2, \\
(H_2)_0(h_1 * x_6) &= (0 + 1)h_1 * x_6 = 1x_1 * y_1 \\
(H_2)_0(b_6 x_6 + c_6 y_6 + d_1 h_1 + d_2 h_2) &= (1 + (-1) + 0 + 0), \\
h_1 * x_6 &= b_6 x_6.
\end{aligned}
$$

Notice this result exactly matches that of Lemma 80.

Lemma 91. *There are* $a, d_1, d_2 \in \mathbb{C}$ *such that* $x_1 * y_1 = a\mathbb{1} + d_1 h_1 + d_2 h_2$*.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_1 * y_1$, we have

$$\begin{aligned}
(H_1)_0(x_1 * y_1) &= (2 + (-2))x_1 * y_1 = 0x_1 * y_1, \\
x_1 * y_1 &= a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2, \\
(H_2)_0(x_1 * y_1) &= (-1 + 1)x_1 * y_1 = 0x_1 * y_1, \\
(H_2)_0(a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2) &= (0 + 1 + (-1) + 0 + 0), \\
x_1 * y_1 &= a\mathbb{1} + d_1h_1 + d_2h_2.
\end{aligned}$$

□

Lemma 92. *There is $b_4 \in \mathbb{C}$ such that $x_1 * y_5 = b_4x_4$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_1 * y_5$, we have

$$\begin{aligned}
(H_1)_0(x_1 * y_5) &= (-1)x_1 * y_5, \\
x_1 * y_5 &= b_3x_3 + b_4x_4, \\
(H_2)_0(x_1 * y_5) &= (-1 + 1)x_1 * y_5 = 0x_1 * y_5, \\
(H_2)_0(b_3x_3 + b_4x_4) &= (1 + 0), \\
x_1 * y_5 &= b_4x_4.
\end{aligned}$$

□

Lemma 93. *We have $x_1 * y_6 = 0$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_1 * y_6$, we have

$$\begin{aligned}
(H_1)_0(x_1 * y_6) &= (2 + 0)x_1 * y_6, \\
x_1 * y_6 &= b_1x_1, \\
(H_2)_0(x_1 * y_6) &= (-1 + -1)x_1 * y_6 = -2x_1 * y_6, \\
(H_2)_0(b_1x_1) &= (-1),
\end{aligned}$$

$$x_1 * y_6 \qquad = 0.$$

$\square$

Lemma 94. *There are $a, d_1, d_2 \in \mathbb{C}$ such that $x_2 * y_2 = a\mathbb{1} + d_1h_1 + d_2h_2$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_2 * y_2$, we have

$$\begin{aligned}
(H_1)_0(x_2 * y_2) \quad &= (-3+3)x_2 * y_2 = 0x_2 * y_2,\\
x_2 * y_2 \quad &= a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2,\\
(H_2)_0(x_2 * y_2) \quad &= (2+-2)x_2 * y_2 = 0x_2 * y_2,\\
(H_2)_0(a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2) \quad &= (0+1+-1+0+0),\\
x_2 * y_2 \quad &= a\mathbb{1} + d_1h_1 + d_2h_2.
\end{aligned}$$

$\square$

Lemma 95. *There is $c_1 \in \mathbb{C}$ such that $x_2 * y_3 = c_1y_1$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_2 * y_3$, we have

$$\begin{aligned}
(H_1)_0(x_2 * y_3) \quad &= (-3+1)x_2 * y_3 = -2x_2 * y_3,\\
x_2 * y_3 \quad &= c_1y_1,\\
(H_2)_0(x_2 * y_3) \quad &= (2+-1)x_2 * y_3 = 1x_2 * y_3,\\
(H_2)_0(c_1y_1) \quad &= 1,\\
x_2 * y_3 \quad &= c_1y_1.
\end{aligned}$$

$\square$

Lemma 96. *We have $x_2 * y_4 = 0$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_2 * y_4$, we have

$$
\begin{aligned}
(H_1)_0(x_2 * y_4) &= (-3+1)x_2 * y_4 = -2x_2 * y_4, \\
x_2 * y_4 &= c_1 y_1, \\
(H_2)_0(x_2 * y_4) &= (2+0)x_2 * y_4 = 2x_2 * y_4, \\
(H_2)_0(c_1 y_1) &= 1, \\
x_2 * y_4 &= 0.
\end{aligned}
$$

□

Lemma 97. *There is $c_5 \in \mathbb{C}$ such that $x_2 * y_6 = c_5 y_5$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $y_2 * y_6$, we have

$$
\begin{aligned}
(H_1)_0(x_2 * y_6) &= (-3+0)x_2 * y_6 = -3x_2 * y_6, \\
x_2 * y_6 &= b_2 x_2 + c_5 y_5, \\
(H_2)_0(x_2 * y_6) &= (2+-1)x_2 * y_6 = 1x_2 * y_6, \\
(H_2)_0(b_2 x_2 + c_5 y_5) &= (2+1), \\
x_2 * y_6 &= c_5 y_5.
\end{aligned}
$$

□

Lemma 98. *There are $a, d_1, d_2 \in \mathbb{C}$ such that $x_3 * y_3 = a\mathbb{1} + d_1 h_1 + d_2 h_2$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_3 * y_3$, we have

$$
\begin{aligned}
(H_1)_0(x_3 * y_3) &= (-1+1)x_3 * y_3 = 0x_3 * y_3, \\
x_3 * y_3 &= a\mathbb{1} + b_6 x_6 + c_6 y_6 + d_1 h_1 + d_2 h_2, \\
(H_2)_0(x_3 * y_3) &= (1+-1)x_3 * y_3 = 0x_3 * y_3, \\
(H_2)_0(a\mathbb{1} + b_6 x_6 + c_6 y_6 + d_1 h_1 + d_2 h_2) &= (0+1+-1+0+0),
\end{aligned}
$$

$$x_3 * y_3 \qquad = a\mathbb{1} + d_1h_1 + d_2h_2.$$

□

Lemma 99. *We have* $x_3 * y_5 = 0$.

Proof. Applying $(H_1)_0$ to $x_3 * y_5$, we have

$$(H_1)_0(x_3 * y_5) \qquad = (-1 + -3)x_3 * y_5 = -4x_3 * y_5,$$

$$x_3 * y_5 \qquad = 0.$$

□

Lemma 100. *There is* $b_3 \in \mathbb{C}$ *such that* $x_3 * y_6 = b_3x_3$.

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_3 * y_6$, we have

$$(H_1)_0(x_3 * y_6) \qquad = (-1 + 0)x_3 * y_6 = -1x_3 * y_6,$$

$$x_3 * y_6 \qquad = b_3x_3 + b_4x_4,$$

$$(H_2)_0(x_3 * y_6) \qquad = (1 + -1)x_3 * y_6 = 0x_3 * y_6,$$

$$(H_2)_0(b_3x_3 + b_4x_4) \qquad = (1 + 0),$$

$$x_3 * y_6 \qquad = b_3x_3.$$

□

III.2.0 **Lemma 101-Lemma 110**

Lemma 101. *There are* $a, b_6, c_6 \in \mathbb{C}$ *such that* $x_4 * y_4 = a\mathbb{1} + b_6x_6 + c_6y_6$.

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_4 * y_4$, we have

$$(H_1)_0(x_4 * y_4) \qquad = (-1 + 1)x_4 * y_4 = 0x_4 * y_4,$$

$$
\begin{aligned}
x_4 * y_4 &= a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2,\\
(H_2)_0(x_4 * y_4) &= (0+0)x_4 * y_4 = 0x_4 * y_4,\\
(H_2)_0(a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2) &= (0 + 1 + -1 + 0 + 0),\\
x_4 * y_4 &= a\mathbb{1} + b_6x_6 + c_6y_6.
\end{aligned}
$$

□

Lemma 102. *We have* $x_4 * y_5 = 0$.

Proof. Applying $(H_1)_0$ to $x_4 * y_5$, we have

$$
\begin{aligned}
(H_1)_0(x_4 * y_5) &= (-1 + -3)x_4 * y_5 = -4x_4 * y_5,\\
x_4 * y_5 &= 0
\end{aligned}
$$

□

Lemma 103. *We have* $x_4 * y_6 = 0$.

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_4 * y_6$, we have

$$
\begin{aligned}
(H_1)_0(x_4 * y_6) &= (-1 + 0)x_4 * y_6 = -1x_4 * y_6,\\
x_4 * y_6 &= b_3x_3 + b_4x_4,\\
(H_2)_0(x_4 * y_6) &= (0 + -1)x_4 * y_6 = -1x_4 * y_6,\\
(H_2)_0(b_3x_3 + b_4x_4) &= (1 + 0),\\
x_4 * y_6 &= 0.
\end{aligned}
$$

□

Lemma 104. *There are* $a, d_1, d_2 \in \mathbb{C}$ *such that* $x_5 * y_5 = a\mathbb{1} + d_1h_1 + d_2h_2$.

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_5 * y_5$, we have

$$\begin{aligned}
(H_1)_0(x_5 * y_5) &= (3 + -3)x_5 * y_5 = 0x_5 * y_5, \\
x_5 * y_5 &= a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2, \\
(H_2)_0(x_5 * y_5) &= (-1 + 1)x_5 * y_5 = 0x_5 * y_5, \\
(H_2)_0(a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2) &= (0 + 1 + -1 + 0 + 0), \\
x_5 * y_5 &= a\mathbb{1} + d_1h_1 + d_2h_2.
\end{aligned}$$

□

Lemma 105. *There is $c_2 \in \mathbb{C}$ such that $x_5 * y_6 = c_2y_2$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_5 * y_6$, we have

$$\begin{aligned}
(H_1)_0(x_5 * y_6) &= (3 + 0)x_5 * y_6 = 3x_5 * y_6, \\
x_5 * y_6 &= b_5x_5 + c_2y_2, \\
(H_2)_0(x_5 * y_6) &= (-1 + -1)x_5 * y_6 = -2x_5 * y_6, \\
(H_2)_0(b_5x_5 + c_2y_2) &= (-1 + -2), \\
x_5 * y_6 &= c_2y_2.
\end{aligned}$$

□

Lemma 106. *There are $a, d_1, d_2 \in \mathbb{C}$ such that $x_6 * y_6 = a\mathbb{1} + d_1h_1 + d_2h_2$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_6 * y_6$, we have

$$\begin{aligned}
(H_1)_0(x_6 * y_6) &= (0 + 0)x_6 * y_6 = 0x_6 * y_6, \\
x_6 * y_6 &= a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2, \\
(H_2)_0(x_6 * y_6) &= (1 + -1)x_6 * y_6 = 0x_6 * y_6, \\
(H_2)_0(a\mathbb{1} + b_6x_6 + c_6y_6 + d_1h_1 + d_2h_2) &= (0 + 1 + -1 + 0 + 0),
\end{aligned}$$

$$x_6 * y_6 \quad = a\mathbb{1} + d_1h_1 + d_2h_2.$$

$\square$

Lemma 107. *There exists $c_4 \in \mathbb{C}$ such that $x_5 * y_1 = c_4y_4$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_5 * y_1$, we have

$$\begin{aligned}
(H_1)_0(x_5 * y_1) \quad &= (3 + -2)x_5 * y_1 = 1x_5 * y_1, \\
x_5 * y_1 \quad &= c_3y_3 + c_4y_4, \\
(H_2)_0(x_5 * y_1) \quad &= (-1 + 1)x_5 * y_1 = 0x_5 * y_1, \\
(H_2)_0(c_3y_3 + c_4y_4) \quad &= (-1 + 0), \\
x_5 * y_1 \quad &= c_4y_4.
\end{aligned}$$

$\square$

Lemma 108. *We have $x_6 * y_1 = 0$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_6 * y_1$, we have

$$\begin{aligned}
(H_1)_0(x_6 * y_1) \quad &= (0 + -2)x_6 * y_1 = -2x_6 * y_1, \\
x_6 * y_1 \quad &= c_1y_1, \\
(H_2)_0(x_6 * y_1) \quad &= (1 + 1)x_6 * y_1 = 2x_6 * y_1, \\
(H_2)_0(c_1y_1) \quad &= 1, \\
x_6 * y_1 \quad &= 0.
\end{aligned}$$

$\square$

Lemma 109. *There exists $b_1 \in \mathbb{C}$ such that $x_3 * y_2 = b_1x_1$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_3 * y_2$, we have

$$\begin{aligned}
(H_1)_0(x_3 * y_2) &= (-1+3)x_3 * y_2 = 2, \\
x_3 * y_2 &= b_1 x_1, \\
(H_2)_0(x_3 * y_2) &= (1+-2) = -1, \\
(H_2)_0(b_1 x_1) &= -1, \\
x_3 * y_2 &= b_1 x_1.
\end{aligned}$$

□

Lemma 110. *We have* $x_4 * y_2 = 0$.

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_4 * y_2$, we have

$$\begin{aligned}
(H_1)_0(x_4 * y_2) &= (-1+3)x_4 * y_2 = 2x_4 * y_2, \\
x_4 * y_2 &= b_1 x_1, \\
(H_2)_0(x_4 * y_2) &= (0+-2)x_4 * y_2 = -2x_4 * y_2, \\
(H_2)_0(b_1 x_1) &= -1, \\
x_4 * y_2 &= 0.
\end{aligned}$$

□

III.2.0 **Lemma 111-Lemma 116**

Lemma 111. *There is* $b_5 \in \mathbb{C}$ *such that* $x_6 * y_2 = b_5 x_5$.

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_6 * y_2$, we have

$$\begin{aligned}
(H_1)_0(x_6 * y_2) &= (0+3)x_6 * y_2 = 3x_6 * y_2, \\
x_6 * y_2 &= b_5 x_5 + c_2 y_2,
\end{aligned}$$

$$\begin{aligned}
(H_2)_0(x_6 * y_2) &= (1+-2)x_6 * y_2 = -1x_6 * y_2, \\
(H_2)_0(b_5x_5 + c_2y_2) &= (-1+-2), \\
x_6 * y_2 &= b_5x_5.
\end{aligned}$$

□

Lemma 112. *We have* $x_5 * y_3 = 0$.

Proof. Applying $(H_1)_0$ to $x_5 * y_3$, we have

$$\begin{aligned}
(H_1)_0(x_5 * y_3) &= (3+1)x_5 * y_3 = 4x_5 * y_3, \\
x_5 * y_3 &= 0.
\end{aligned}$$

□

Lemma 113. *There is* $c_4 \in \mathbb{C}$ *such that* $x_6 * y_3 = c_4y_4$.

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_6 * y_3$, we have

$$\begin{aligned}
(H_1)_0(x_6 * y_3) &= (0+1)x_6 * y_3 = 1x_6 * y_3, \\
x_6 * y_3 &= c_3y_3 + c_4y_4, \\
(H_2)_0(x_6 * y_3) &= (1+-1)x_6 * y_3 = 0x_6 * y_3, \\
(H_2)_0(c_3y_3 + c_4y_4) &= (-1+0), \\
x_6 * y_3 &= c_4y_4.
\end{aligned}$$

□

Lemma 114. *We have* $x_5 * y_4 = 0$.

Proof. Applying $(H_1)_0$ to $x_5 * y_4$, we have

$$(H_1)_0(x_5 * y_4) = (3+1)x_5 * y_4 = 4x_5 * y_4,$$

$$x_5 * y_4 \qquad = 0.$$

$\square$

Lemma 115. *We have $x_6 * y_4 = 0$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_6 * y_4$, we have

$$\begin{aligned}
(H_1)_0(x_6 * y_4) &= (0+1)x_6 * y_4 = 1, \\
x_6 * y_4 &= c_3 y_3 + c_4 y_4, \\
(H_2)_0(x_6 * y_4) &= (1+0)x_6 * y_4 = 1x_6 * y_4, \\
(H_2)_0(c_3 y_3 + c_4 y_4) &= (-1+0), \\
x_6 * y_4 &= 0.
\end{aligned}$$

$\square$

Lemma 116. *There is $b_2 \in \mathbb{C}$ such that $x_6 * y_5 = b_2 x_2$.*

Proof. Applying $(H_1)_0$ and then $(H_2)_0$ to $x_6 * y_5$, we have

$$\begin{aligned}
(H_1)_0(x_6 * y_5) &= (0+-3)x_6 * y_5 = -3x_6 * y_5, \\
x_6 * y_5 &= b_2 x_2 + c_5 y_5, \\
(H_2)_0(x_6 * y_5) &= (1+1)x_6 * y_5 = 2x_6 * y_5, \\
(H_2)_0(b_2 x_2 + c_5 y_5) &= (2+1), \\
x_6 * y_5 &= b_2 x_2.
\end{aligned}$$

$\square$

III.3 Proof of Theorem 18

The proof of Theorem 18 is almost identical with the proof of Theorem 17. First, in the proof of Theorem 17, we must replace

1. x_i by X_i, y_j by Y_j, h_t by H_t and
2. $a * a'$ by $\langle u, v\rangle$. Here, $a, a' \in \{x_1, ..., x_6, y_1, ..., y_6, h_1, h_2\}$,
   $u, v \in \{X_1, ..., X_6, Y_1, ..., Y_6, H_1, H_2\}$.

Next, to obtain the desired results for Theorem 18, we only need to following the proof of Theorem 17 step by step.

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