7-16-2021

On the Spectrum Problem for a Class of 3-Uniform Hypergraphs with 5 Edges

Kaitlin Elizabeth Shoukry
Illinois State University, Keshouk@ilstu.edu

ON THE SPECTRUM PROBLEM FOR A CLASS OF 3-UNIFORM HYPERGRAPHS WITH 5 EDGES

KAITLIN ELIZABETH SHOUKRY

22 Pages

The complete 3-uniform hypergraph of order v has a set V of size v as its vertex set and the set of all 3-element subsets of V as its edge set. The degree of a vertex is the number of edges in its edge set that contain it. We consider a class of 3-uniform hypergraphs with 5 edges and 10 vertices such that: every vertex has degree either 1 or 2 and any two edges intersect in at most one vertex. There are 5 such hypergraphs. For $k \in \{1,2,3,4,5\}$, let H_k denote the hypergraphs with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and edge sets $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_4, v_6\}, \{v_3, v_7, v_8\}, \{v_5, v_9, v_{10}\}\big\}$, $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_4, v_6\}, \{v_3, v_7, v_8\}, \{v_7, v_9, v_{10}\}\big\}$, $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}, \{v_4, v_6, v_{10}\}\big\}$, $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_4, v_8, v_9\}, \{v_6, v_8, v_{10}\}\big\}$, and $\big\{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_7, v_8, v_9\}, \{v_9, v_{10}, v_0\}\big\}$, respectively. We give necessary and sufficient conditions for the existence of a decomposition of the complete 3-uniform hypergraph of order v into isomorphic copies of each H_k.

KEYWORDS: 3-Uniform Hypergraphs, Hypergraph Decompositions

ON THE SPECTRUM PROBLEM FOR A CLASS OF 3-UNIFORM HYPERGRAPHS WITH 5 EDGES

KAITLIN ELIZABETH SHOUKRY

A Thesis Submitted in Partial
Fulfillment of the Requirements
for the Degree of

MASTER OF SCIENCE

Department of Mathematics

ILLINOIS STATE UNIVERSITY

2021

ON THE SPECTRUM PROBLEM FOR A CLASS OF 3-UNIFORM HYPERGRAPHS WITH 5 EDGES

KAITLIN ELIZABETH SHOUKRY

COMMITTEE MEMBERS:

Saad I. El-Zanati, Chair

Ryan C. Bunge

Papa A. Sissokho

ACKNOWLEDGMENTS

Part of this work was initiated in MAT461, *Advanced Topics in Discrete Mathematics*, taught by Professor Saad El-Zanati (assisted by Dr. Ryan Bunge) in Spring 2019. The template used for this work was shared by Drs. El-Zanati and Bunge.

K. E. S.

CONTENTS

	Page
ACKNOWLEDGMENTS	i
CONTENTS	ii
FIGURES	iii
CHAPTER I: INTRODUCTION	1
CHAPTER II: NOTATION AND TERMINOLOGY	6
CHAPTER III: EXAMPLES OF H_k-DECOMPOSITIONS	7
CHAPTER IV: MAIN RESULTS	19
REFERENCES	21

FIGURES

Figure	Page
1. H_1 denoted $H_1[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$	4
2. H_2 denoted $H_2[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$	4
3. H_3 denoted $H_3[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$	4
4. H_4 denoted $H_4[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$	4
5. H_5 denoted $H_4[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$	5

CHAPTER I: INTRODUCTION

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite set of elements called the *vertices* of G and $E(G)$ is a set of 2-element subsets of $V(G)$, called the *edges* of G. If $e = \{u, v\}$ is an edge in $E(G)$, then we say edge e is *incident* with vertices u and v and call u and v the *end-vertices* of e. In this case, we also say vertices u and v are *incident* with edge e. Two vertices u and v in $V(G)$ are *adjacent* in G if $\{u, v\} \in E(G)$. Similarly, edges e and e' are *adjacent* in G if e and e' share a common end-vertex. The *degree* of a vertex $v \in V(G)$ is the number of edges in $E(G)$ that contain v. We call $|V(G)|$ the *order* of G and $|E(G)|$ its *size*.

Two graphs $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$ are said to be *isomorphic* if there exists a one-to-one and onto map $f\colon V(G) \mapsto V(G')$ that preserves adjacency. Thus in this case, two vertices u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in G'.

A commonly studied problem in combinatorics concerns decompositions of graphs into edge-disjoint subgraphs. A *decomposition* of a graph K is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of pairwise edge-disjoint subgraphs of K such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_s) = E(K)$. If each element of Δ is isomorphic to a fixed graph G, then Δ is called a *G-decomposition* of K. A G-decomposition of K_v is also known as a *G-design of order v*. A K_k-design of order v is an *$S(2, k, v)$-design* or a *Steiner system*. An $S(2, k, v)$-design is also known as a *balanced incomplete block design of index 1* or a *$(v, k, 1)$-BIBD*. The problem of determining all v for which there exists a G-design of order v is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to decompositions of uniform hypergraphs. A *hypergraph* H consists of a finite nonempty set V of *vertices* and a set $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of V called *hyperedges*. If for each $e \in E$ we have $|e| = t$, then H is said to be *t-uniform*. Thus graphs are 2-uniform hypergraphs. The complete t-uniform hypergraph on the vertex set V has the set of all t-element subsets of V as its edge set and is denoted by $K_V^{(t)}$. If $v = |V|$, then $K_v^{(t)}$ is called the *complete t-uniform* *hypergraph of order* v and is used to denote any hypergraph isomorphic to $K_V^{(t)}$.

A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of pairwise edge-disjoint subgraphs of K such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s) = E(K)$. If each element H_i of Δ is isomorphic to a fixed hypergraph H, then each H_i is called an *H-block*, and Δ is called an *H-decomposition* of K. If there exists an H-decomposition of K, then we may simply state that H *decomposes* K. An H-decomposition of the complete t-uniform hypergraph of order v is also called an *H-design of order v*. The problem of determining all v for which there exists an H-design of order v is called the *spectrum problem for H-designs*.

A $K_k^{(t)}$-design of order v is a generalization of Steiner systems and is equivalent to an $S(t, k, v)$-design. A summary of results on $S(t, k, v)$-designs appears in [8]. Keevash [14] has recently shown that for all t and k the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of v. Similar results were obtained by Glock, Kühn, Lo, and Osthus [9, 10] and extended to include the corresponding asymptotic results for H-designs of order v for all uniform hypergraphs H. These results for t-uniform hypergraphs mirror the celebrated results of Wilson [19] for graphs. Although these asymptotic results assure the existence of H-designs for sufficiently large values of v for any uniform hypergraph H, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on G-decompositions of K_v where G is a graph with a relatively small number of edges (see [1] and [6] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [4], the spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the spectrum problem was settled in [5] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [5], they also settle the spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered H-designs where H is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let T, O, and I denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph T is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [11]. In another paper [12], Hanani settled the spectrum problem for O-designs and gave necessary conditions for the existence of I-designs.

Perhaps the best known general result on decompositions of complete t-uniform hypergraphs is Baranyai's result [3] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers m. There are, however, several articles on decompositions of complete t-uniform hypergraphs (see [2] and [17]) and of t-uniform t-partite hypergraphs (see [15] and [18]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [13] and [16]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this work, we considered the spectrum problem for the class of 3-uniform hypergraphs with 5 edges and 10 vertices, where the minimum vertex degree is 1, the maximum vertex degree is 2, and any two edges intersect in at most one vertex. There are 5 such hypergraphs as shown in Figures 1–2 below. For $k \in \{1,2,3,4,5\}$, let H_k denote the hypergraphs with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and edge sets $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_4, v_6\}, \{v_3, v_7, v_8\}, \{v_5, v_9, v_{10}\}\big\}$, $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_4, v_6\}, \{v_3, v_7, v_8\}, \{v_7, v_9, v_{10}\}\big\}$, $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}, \{v_4, v_6, v_{10}\}\big\}$, $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_4, v_8, v_9\}, \{v_6, v_8, v_{10}\}\big\}$, and $\big\{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_7, v_8, v_9\}, \{v_9, v_{10}, v_0\}\big\}$, respectively.

The graph H_5 is known as a *loose* 5-*cycle*. It is shown in [7] that there exists an H_5-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1 \text{ or } 2 \pmod 5$, and $v \geq 10$. We settle the spectrum problem for the remaining 4 hypergraphs.

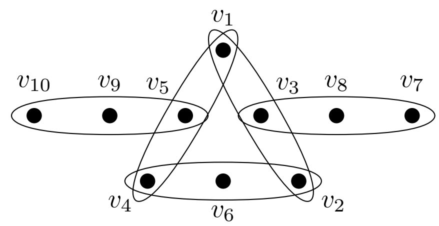

Figure 1: H_1 denoted $H_1[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$

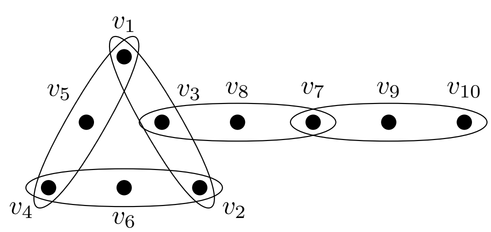

Figure 2: H_2 denoted $H_2[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$

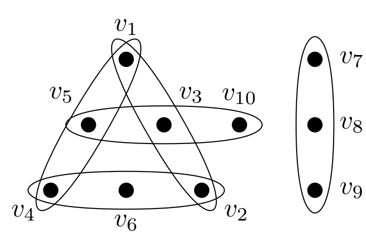

Figure 3: H_3 denoted $H_3[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$

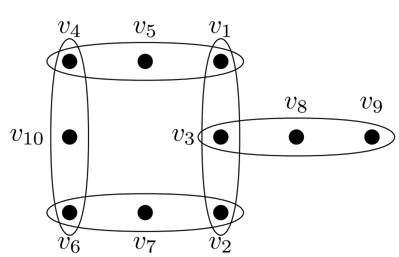

Figure 4: H_4 denoted $H_4[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$

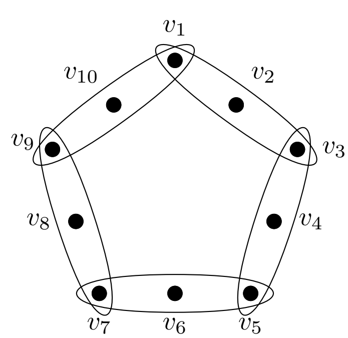

Figure 5: H_5 denoted $H_4[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$

CHAPTER II: NOTATION AND TERMINOLOGY

If a and b are integers, we define $[a, b]$ to be $\{r \in \mathbb{Z} : a \leq r \leq b\}$. Let $\mathbb{Z}_n$ denote the group of integers modulo n.

We will often describe our hypergraphs by giving their edge set only. Since the hypergraphs we consider will never contain isolated vertices, this is enough to uniquely define them. The complete k-uniform hypergraph with vertex set V has the set of all k-element subsets of V as its edge set and is denoted by $K_V^{(k)}$. If $v = |V|$, then $K_v^{(k)}$ is used to denote any hypergraph isomorphic to $K_V^{(k)}$. If H' is a subhypergraph of H, then $H \setminus H'$ denotes the hypergraph obtained from H by deleting the edges of H'.

We need to define some notation for certain types of multipartite hypergraphs. Let $U_1, U_2, \ldots, U_m$ be pairwise disjoint sets. The hypergraph with vertex set $V = U_1 \cup U_2 \cup \cdots \cup U_m$ and edge set consisting of all k-element subsets of V having at most one vertex in each of $U_1, U_2, \ldots, U_m$ is denoted by $K_{U_1, U_2, \ldots, U_m}^{(k)}$. If $|U_i| = u_i$ for $i \in [1, m]$, we may use $K_{u_1, u_2, \ldots, u_m}^{(k)}$ to denote any hypergraph that is isomorphic to $K_{U_1, U_2, \ldots, U_m}^{(k)}$, and if $u_1 = u_2 = \cdots = u_m = u$, then the notation $K_{m \times u}^{(k)}$ may be used instead of $K_{u_1, u_2, \ldots, u_m}^{(k)}$.

For pairwise disjoint sets $U_1, U_2, \ldots, U_m$, $1 \leq m \leq k-1$, the hypergraph with vertex set $V = U_1 \cup U_2 \cup \cdots \cup U_m$ and edge set consisting of all k-element subsets of V having at least one element in each of $U_1, U_2, \ldots, U_m$ is denoted by $L_{U_1, U_2, \ldots, U_m}^{(k)}$. If $|U_i| = u_i$ for $i \in [1, m]$, we may use $L_{u_1, u_2, \ldots, u_m}^{(k)}$ to denote any hypergraph that is isomorphic to $L_{U_1, U_2, \ldots, U_m}^{(k)}$. If $k_1, k_2, \ldots, k_m$ are positive integers with $k_1 + k_2 + \cdots + k_m = k$, then $L_{U_1, U_2, \ldots, U_m}^{(k_1, k_2, \ldots, k_m)}$ is the subgraph of $L_{U_1, U_2, \ldots, U_m}^{(k)}$ where each hyperedge contains exactly k_i elements from each U_i. We define $L_{u_1, u_2, \ldots, u_m}^{(k_1, k_2, \ldots, k_m)}$ similarly.

CHAPTER III: EXAMPLES OF H_k-DECOMPOSITIONS

We give several examples of H_k-decompositions, $k \in \{1, 2, 3, 4\}$, that are used in proving our main result.

Example 1. Let $V\left(K_{10}^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty_1, \infty_2\}$ and let

$$\begin{aligned}
B_1 &= \big\{H_1[2,1,4,\infty_2,5,3,0,6,7,\infty_1],\ H_1[0,5,2,\infty_1,3,\infty_2,1,6,4,7]\big\},\\
B_1' &= \big\{H_1[\infty_1,0,7,4,3,\infty_2,5,6,1,2],\ H_1[\infty_1,1,0,5,4,\infty_2,6,7,2,3],\\
&\qquad H_1[\infty_1,2,1,6,5,\infty_2,7,0,3,4],\ H_1[\infty_1,3,2,7,6,\infty_2,0,1,4,5],\\
&\qquad H_1[\infty_2,0,7,4,3,\infty_1,1,2,5,6],\ H_1[\infty_2,1,0,5,4,\infty_1,2,3,6,7],\\
&\qquad H_1[\infty_2,2,1,6,5,\infty_1,3,4,7,0],\ H_1[\infty_2,3,2,7,6,\infty_1,4,5,0,1]\big\},\\
B_2 &= \big\{H_2[5,2,0,7,6,1,4,3,\infty_1,\infty_2],\ H_2[6,0,4,3,2,\infty_1,1,7,5]\big\},\\
B_2' &= \big\{H_2[4,1,2,\infty_2,5,0,7,6,3,\infty_1],\ H_2[5,2,3,\infty_2,6,1,0,7,4,\infty_1],\\
&\qquad H_2[6,3,4,\infty_2,7,2,1,0,5,\infty_1],\ H_2[7,4,5,\infty_2,0,3,2,1,6,\infty_1],\\
&\qquad H_2[5,0,6,\infty_1,4,1,3,2,7,\infty_2],\ H_2[6,1,7,\infty_1,5,2,4,3,0,\infty_2],\\
&\qquad H_2[7,2,0,\infty_1,6,3,5,4,1,\infty_2],\ H_2[0,3,1,\infty_1,7,4,6,5,2,\infty_2]\big\},\\
B_3 &= \big\{H_3[0,\infty_1,3,2,\infty_2,4,1,7,5,6], H_3[0,5,2,1,7,6,\infty_1,\infty_2,4,3]\big\},\\
B_3' &= \big\{H_3[\infty_1,1,0,4,5,3,2,6,\infty_2,7],\ H_3[\infty_1,2,1,5,6,4,3,7,\infty_2,0],\\
&\qquad H_3[\infty_1,3,2,6,7,5,4,0,\infty_2,1],\ H_3[\infty_1,4,3,7,0,6,5,1,\infty_2,2],\\
&\qquad H_3[\infty_2,0,1,5,4,6,3,7,\infty_1,2],\ H_3[\infty_2,1,2,6,5,7,4,0,\infty_1,3],\\
&\qquad H_3[\infty_2,2,3,7,6,0,5,1,\infty_1,4],\ H_3[\infty_2,3,4,0,7,1,6,2,\infty_1,5]\big\},\\
B_4 &= \big\{H_4[\infty_2,0,2,3,6,\infty_1,5,4,7,1],\ H_4[6,2,0,4,5,7,1,\infty_1,\infty_2,3]\big\},\\
B_4' &= \big\{H_4[1,7,6,4,5,\infty_1,0,2,\infty_2,3],\ H_4[2,0,7,5,6,\infty_1,1,3,\infty_2,4],\\
&\qquad H_4[3,1,0,6,7,\infty_1,2,4,\infty_2,5],\ H_4[4,2,1,7,0,\infty_1,3,5,\infty_2,6],
\end{aligned}$$

$$H_4[5,3,2,0,1,\infty_2,4,6,\infty_1,7],\ H_4[6,4,3,1,2,\infty_2,5,7,\infty_1,0],$$
$$H_4[7,5,4,2,3,\infty_2,6,0,\infty_1,1],\ H_4[0,6,5,3,4,\infty_2,7,1,\infty_1,2]\big\}.$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $K_{10}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$ and $j \mapsto j+1 \pmod 8$ along with the H_k-blocks in B'_k.

Example 2. Let $V\left(K_{11}^{(3)}\right) = \mathbb{Z}_{11}$ and let

$$B_1 = \big\{H_1[8,3,0,10,1,4,6,7,2,5],\ H_1[4,0,7,10,6,2,9,1,3,8],\ H_1[2,0,7,10,3,5,8,9,1,4]\big\},$$
$$B_2 = \big\{H_2[3,8,0,5,1,6,9,10,2,7],\ H_2[6,0,5,2,7,8,3,10,1,4],\ H_2[4,7,0,1,5,8,9,3,10,2]\big\},$$
$$B_3 = \big\{H_3[5,4,10,6,9,1,0,3,8,7],\ H_3[1,10,8,2,6,0,3,4,7,5],\ H_3[7,0,2,3,1,9,4,5,6,8]\big\},$$
$$B_4 = \big\{H_4[5,0,6,3,8,4,1,7,9,10],\ H_4[8,0,3,4,1,5,9,6,7,2],\ H_4[9,2,0,8,7,10,4,1,5,3]\big\}.$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $K_{11}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $j \mapsto j+1 \pmod{11}$.

Example 3. Let $V\left(K_{12}^{(3)}\right) = \mathbb{Z}_{11} \cup \{\infty\}$ and let

$$B_1 = \big\{H_1[5,0,2,10,6,7,4,8,9,\infty],\ H_1[0,3,8,4,1,9,7,10,6,\infty],$$
$$H_1[4,7,0,8,2,3,9,10,6,\infty],\ H_1[0,2,9,1,\infty,10,4,7,3,6]\big\},$$
$$B_2 = \big\{H_2[4,0,2,10,5,7,6,3,9,\infty],\ H_2[8,3,0,6,5,1,4,9,2,\infty],$$
$$H_2[7,4,0,8,5,3,9,6,2,\infty],\ H_2[0,7,2,5,4,6,\infty,8,9,10]\big\},$$
$$B_3 = \big\{H_3[0,5,2,6,10,9,1,3,7,\infty],\ H_3[3,8,0,7,2,4,1,6,10,\infty],$$
$$H_3[0,4,7,9,2,3,5,6,8,\infty],\ H_3[0,1,\infty,2,8,10,5,6,7,4]\big\},$$
$$B_4 = \big\{H_4[4,0,2,8,5,6,1,10,\infty,3],\ H_4[6,3,0,7,2,8,5,9,\infty,1],$$
$$H_4[4,8,0,10,6,9,5,7,\infty,1],\ H_4[2,3,\infty,7,0,5,4,1,6,8]\big\}.$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $K_{12}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{11}$.

Example 4. Let $V\left(K_{15}^{(3)}\right) = \mathbb{Z}_{13} \cup \{\infty_1, \infty_2\}$ and let

$$
\begin{aligned}
B_1 = \big\{ & H_1[0,1,\infty_2,12,\infty_1,3,4,6,7,9],\ H_1[0,1,3,12,10,6,4,8,5,9],\\
& H_1[0,1,4,12,9,7,5,10,3,8],\ H_1[0,3,7,6,10,1,4,12,5,8],\\
& H_1[0,3,\infty_2,10,\infty_1,1,2,6,7,11],\ H_1[0,5,\infty_2,8,\infty_1,11,1,7,6,12],\\
& H_1[0,4,8,2,6,3,1,7,\infty_1,\infty_2]\big\},\\
B_2 = \big\{ & H_2[0,1,3,12,10,6,4,\infty_2,5,\infty_1],\ H_2[0,1,4,12,9,3,6,\infty_2,8,\infty_1],\\
& H_2[0,1,5,12,8,4,2,\infty_2,6,\infty_1],\ H_2[0,1,6,12,7,5,2,\infty_2,10,\infty_1],\\
& H_2[0,3,7,6,10,9,12,\infty_2,2,\infty_1],\ H_2[0,4,9,5,8,3,\infty_2,2,6,\infty_1],\\
& H_2[0,6,7,8,2,3,5,1,11,\infty_1]\big\},\\
B_3 = \big\{ & H_3[0,3,1,12,10,\infty_1,4,6,8,\infty_2],\ H_3[0,4,1,12,9,\infty_1,2,6,10,\infty_2],\\
& H_3[0,5,1,12,8,\infty_1,4,7,10,\infty_2],\ H_3[0,1,6,12,7,\infty_1,2,5,10,\infty_2],\\
& H_3[0,5,2,8,11,\infty_1,4,6,\infty_2,9],\ H_3[0,7,2,6,11,\infty_1,1,4,\infty_2,8],\\
& H_3[0,6,7,2,1,9,11,\infty_1,\infty_2,5]\big\},\\
B_4 = \big\{ & H_4[0,3,1,10,12,6,\infty_2,4,7,\infty_1],\ H_4[0,4,1,9,12,6,\infty_2,3,5,\infty_1],\\
& H_4[0,5,1,8,12,6,\infty_2,4,9,\infty_1],\ H_4[0,5,2,11,8,12,\infty_2,1,3,\infty_1],\\
& H_4[0,6,1,7,12,2,\infty_2,5,9,\infty_1],\ H_4[0,6,2,7,11,1,\infty_2,3,9,\infty_1],\\
& H_4[0,2,7,10,6,5,9,\infty_1,\infty_2,12]\big\}.
\end{aligned}
$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $K_{15}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$, and $j \mapsto j+1$ $\pmod{13}$.

Example 5. Let $V\left(K_{16}^{(3)}\right) = \mathbb{Z}_{16}$ and let

$$
\begin{aligned}
B_1 = \big\{ & H_1[0,1,3,15,13,8,5,9,7,11],\ H_1[0,1,4,15,12,13,6,11,5,10], \\
& H_1[0,5,1,11,15,8,3,6,10,13],\ H_1[0,6,1,10,15,2,4,9,7,12], \\
& H_1[0,7,1,9,15,8,5,10,6,11],\ H_1[0,7,3,9,13,15,4,11,5,12], \\
& H_1[0,3,9,13,7,5,4,15,1,11]\big\}, \\
B_2 = \big\{ & H_2[0,3,1,13,15,8,9,4,2,12],\ H_2[0,4,1,12,15,8,6,3,9,13], \\
& H_2[0,5,1,11,15,8,9,3,4,12],\ H_2[0,6,1,10,15,8,7,3,2,9], \\
& H_2[0,7,1,9,15,8,3,11,5,10],\ H_2[0,4,9,12,7,5,3,13,1,10], \\
& H_2[0,10,3,9,13,2,15,5,1,12]\big\}, \\
B_3 = \big\{ & H_3[0,3,1,13,15,8,6,9,14,4],\ H_3[0,4,1,12,15,8,6,9,13,5], \\
& H_3[0,5,1,11,15,8,4,7,13,6],\ H_3[0,6,1,10,15,8,2,5,12,7], \\
& H_3[0,7,1,9,15,8,2,5,13,12],\ H_3[0,9,4,7,12,1,3,6,15,11], \\
& H_3[0,4,10,2,12,13,6,14,15,3]\big\}, \\
B_4 = \big\{ & H_4[0,3,1,13,15,8,5,4,9,11],\ H_4[0,1,4,15,12,8,3,7,11,13], \\
& H_4[0,1,5,15,11,7,3,8,14,13],\ H_4[0,1,6,15,10,9,3,2,11,13], \\
& H_4[0,1,7,9,15,11,4,3,13,2],\ H_4[0,1,14,13,3,12,7,1,9,4], \\
& H_4[0,1,15,12,4,13,8,2,11,5]\big\}.
\end{aligned}
$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $K_{16}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map and $j \mapsto j+1 \pmod{16}$.

Example 6. Let $V\left(K_{17}^{(3)}\right) = \mathbb{Z}_{17}$ and let

$$
\begin{aligned}
B_1 = \big\{ & H_1[0,3,1,14,16,8,4,10,7,13],\ H_1[0,4,1,13,16,9,3,7,10,14], \\
& H_1[0,5,1,12,16,8,3,10,9,14],\ H_1[0,6,1,11,16,9,3,12,5,14],
\end{aligned}
$$

$$H_1[0,7,1,10,16,5,3,11,8,13],\ H_1[0,8,1,16,9,3,5,12,2,13],$$

$$H_1[0,16,1,9,8,13,4,11,10,15],\ H_1[0,5,12,3,14,7,8,16,2,11]\big\},$$

$$B_2 = \big\{H_2[0,3,1,14,16,6,4,13,8,15],\ H_2[0,4,1,13,16,7,6,3,2,8],$$

$$H_2[0,5,1,12,16,8,4,14,6,15],\ H_2[0,1,6,11,16,4,8,12,5,10],$$

$$H_2[0,1,7,16,10,3,5,12,9,15],\ H_2[0,1,8,9,16,4,6,14,2,11],$$

$$H_2[0,1,16,9,8,5,14,6,7,12],\ H_2[0,5,12,11,6,8,16,7,1,9]\big\},$$

$$B_3 = \big\{H_3[0,3,1,14,16,6,5,9,15,4],\ H_3[0,4,1,13,16,7,6,10,15,5],$$

$$H_3[0,5,1,12,16,8,2,7,11,6],\ H_3[0,1,6,11,16,4,5,7,14,8],$$

$$H_3[0,1,7,16,10,3,4,8,15,2],\ H_3[0,1,8,9,16,4,3,5,15,10],$$

$$H_3[0,1,16,9,8,5,7,10,12,14],\ H_3[0,5,12,11,6,8,9,13,16,10]\big\},$$

$$B_4 = \big\{H_4[0,3,1,14,16,8,5,4,9,12],\ H_4[0,4,1,13,16,6,10,9,15,11],$$

$$H_4[0,1,5,16,12,9,3,8,15,13],\ H_4[0,1,6,16,11,8,3,9,13,14],$$

$$H_4[0,1,7,16,10,11,3,4,13,14],\ H_4[0,1,8,16,9,10,5,2,14,13],$$

$$H_4[0,4,10,13,7,5,3,1,14,15],\ H_4[0,5,12,2,15,1,13,4,9,10]\big\}.$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $K_{17}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map and $j \mapsto j+1 \pmod{17}$.

Example 7. Let $V\left(L_{5,5}^{(3)}\right) = \mathbb{Z}_{10}$ with vertex partition $\big\{\{0,2,4,6,8\},\ \{1,3,5,7,9\}\big\}$ and let

$$B_1 = \big\{H_1[5,3,2,7,8,0,4,9,1,6],\ H_1[0,4,1,6,9,5,2,7,3,8]\big\},$$

$$B_2 = \big\{H_2[0,7,3,2,5,6,8,4,1,9],\ H_2[7,2,0,6,4,3,9,1,5,8]\big\},$$

$$B_3 = \big\{H_3[0,2,7,3,8,9,4,5,6,1],\ H_3[5,2,4,6,8,1,0,3,7,9]\big\},$$

$$B_4 = \big\{H_4[0,3,7,1,9,6,4,5,8,2],\ H_4[5,0,2,9,6,7,1,3,8,4]\big\}.$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $L_{5,5}^{(3)}$ consists of the orbits of the

H_k-blocks in B_k under the action of the map $j \mapsto j+1 \pmod{10}$.

Example 8. Let $V\left(L^{(3)}_{5,5} \cup K^{(3)}_{1,5,5}\right) = \mathbb{Z}_{10} \cup \{\infty\}$ with vertex partition $\big\{\{\infty\}, \{0,2,4,6,8\}, \{1,3,5,7,9\}\big\}$ and let

$$
\begin{aligned}
B_1 &= \big\{H_1[0,1,3,7,9,4,2,8,6,\infty],\ H_1[2,0,5,9,4,1,6,\infty,3,8]\big\},\\
B_2' &= \big\{H_1[2,8,1,3,9,\infty,0,4,5,6],\ H_1[4,0,3,5,1,\infty,2,6,7,8],\ H_1[6,2,5,7,3,\infty,4,8,9,0],\\
&\qquad H_1[8,4,7,9,5,\infty,6,0,1,2],\ H_1[0,6,9,1,7,\infty,8,2,3,4]\big\},\\
B_2 &= \big\{H_2[0,1,3,7,9,4,8,2,5,\infty],\ H_2[2,9,4,0,5,1,8,3,7,\infty]\big\},\\
B_2' &= \big\{H_2[0,1,7,6,9,\infty,4,3,5,8],\ H_2[2,3,9,8,1,\infty,6,5,7,0],\ H_2[4,5,1,0,3,\infty,8,7,9,2],\\
&\qquad H_2[6,7,3,2,5,\infty,0,9,1,4],\ H_2[8,9,5,4,7,\infty,2,1,3,6]\big\},\\
B_3 &= \big\{H_3[2,0,5,9,4,3,1,8,\infty,7],\ H_3[0,6,1,9,4,8,2,3,\infty,7]\big\},\\
B_3' &= \big\{H_3[4,0,3,5,1,\infty,6,7,8,2],\ H_3[6,2,5,7,3,\infty,8,9,0,4],\ H_3[8,4,7,9,5,\infty,0,1,2,6],\\
&\qquad H_3[0,6,9,1,7,\infty,2,3,4,8],\ H_3[2,8,1,3,9,\infty,4,5,6,0]\big\},\\
B_4 &= \big\{H_4[0,4,1,6,9,3,\infty,2,7,8],\ H_4[0,1,3,7,9,4,8,6,\infty,2]\big\},\\
B_4' &= \big\{H_4[6,1,2,5,4,0,9,3,7,\infty],\ H_4[8,3,4,7,6,2,1,5,9,\infty],\ H_4[0,5,6,9,8,4,3,7,1,\infty],\\
&\qquad H_4[2,7,8,1,0,6,5,9,3,\infty],\ H_4[4,9,0,3,2,8,7,1,5,\infty]\big\}.
\end{aligned}
$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $L^{(3)}_{5,5} \cup K^{(3)}_{1,5,5}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{10}$ along with the H_k-blocks in B_k'.

Example 9. Let $V\left(L^{(3)}_{5,5} \cup K^{(3)}_{2,5,5}\right) = \mathbb{Z}_{10} \cup \{\infty_1, \infty_2\}$ with vertex partition $\big\{\{\infty_1, \infty_2\},$

$\{0,2,4,6,8\},\ \{1,3,5,7,9\}\big\}$ and let

$$
\begin{aligned}
B_1 = \big\{&H_1[0,1,6,4,9,7,3,\infty_1,8,\infty_2],\ H_1[5,2,0,4,7,9,3,\infty_2,6,8]\big\},\\
B_1' = \big\{&H_1[2,8,1,3,9,\infty_1,0,4,5,6],\ H_1[4,0,3,5,1,\infty_1,2,6,7,8],\\
&H_1[6,2,5,7,3,\infty_1,4,8,9,0],\ H_1[8,4,7,9,5,\infty_1,6,0,1,2],\\
&H_1[0,6,9,1,7,\infty_1,8,2,3,4],\ H_1[0,1,\infty_1,2,3,9,5,6,8,\infty_2],\\
&H_1[2,3,\infty_1,4,5,1,7,8,0,\infty_2],\ H_1[4,5,\infty_1,6,7,3,9,0,2,\infty_2],\\
&H_1[6,7,\infty_1,8,9,5,1,2,4,\infty_2],\ H_1[8,9,\infty_1,0,1,7,3,4,6,\infty_2]\big\},\\
B_2 = \big\{&H_2[0,1,3,9,7,4,\infty_2,2,5,8],\ H_2[3,7,0,8,1,9,5,6,2,\infty_1]\big\},\\
B_2' = \big\{&H_2[0,1,7,6,9,\infty_1,4,3,5,8],\ H_2[2,3,9,8,1,\infty_1,6,5,7,0],\\
&H_2[4,5,1,0,3,\infty_1,8,7,9,2],\ H_2[6,7,3,2,5,\infty_1,0,9,1,4],\\
&H_2[8,9,5,4,7,\infty_1,2,1,3,6],\ H_2[\infty_1,0,1,5,6,\infty_2,2,7,3,8],\\
&H_2[\infty_1,2,3,7,8,\infty_2,4,9,5,0],\ H_2[\infty_1,4,5,9,0,\infty_2,6,1,7,2],\\
&H_2[\infty_1,6,7,1,2,\infty_2,8,3,9,4],\ H_2[\infty_1,8,9,3,4,\infty_2,0,5,1,6]\big\},\\
B_3 = \big\{&H_3[0,1,3,7,9,2,5,8,\infty_1,6],\ H_3[1,0,\infty_2,3,6,5,2,7,8,9]\big\},\\
B_3' = \big\{&H_3[4,0,3,5,1,\infty_1,6,7,8,2],\ H_3[6,2,5,7,3,\infty_1,8,9,0,4],\\
&H_3[8,4,7,9,5,\infty_1,0,1,2,6],\ H_3[0,6,9,1,7,\infty_1,2,3,4,8],\\
&H_3[2,8,1,3,9,\infty_1,4,5,6,0],\ H_3[\infty_1,1,0,5,6,2,4,9,\infty_2,7],\\
&H_3[\infty_1,3,2,7,8,4,6,1,\infty_2,9],\ H_3[\infty_1,5,4,9,0,6,8,3,\infty_2,1],\\
&H_3[\infty_1,7,6,1,2,8,0,5,\infty_2,3],\ H_3[\infty_1,9,8,3,4,0,2,7,\infty_2,5]\big\},\\
B_4 = \big\{&H_4[2,0,5,4,9,\infty_1,6,8,7],\ H_4[0,1,6,7,3,\infty_2,2,8,9,4]\big\},\\
B_4' = \big\{&H_4[6,1,2,5,4,0,9,3,7,\infty_1],\ H_4[8,3,4,7,6,2,1,5,9,\infty_1],\\
&H_4[0,5,6,9,8,4,3,7,1,\infty_1],\ H_4[2,7,8,1,0,6,5,9,3,\infty_1],\\
&H_4[4,9,0,3,2,8,7,1,5,\infty_1],\ H_4[5,9,8,1,2,0,6,3,\infty_2,4],
\end{aligned}
$$

$$H_4[7,1,0,3,4,2,8,5,\infty_2,6],\ H_4[9,3,2,5,6,4,0,7,\infty_2,8],$$
$$H_4[1,5,4,7,8,6,2,9,\infty_2,0],\ H_4[3,7,6,9,0,8,4,1,\infty_2,2]\big\}.$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $L_{5,5}^{(3)} \cup K_{2,5,5}^{(3)}$ consists of the orbits of the H_k-blocks in B_k under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$, and $j \mapsto j+1 \pmod{10}$ along with the H_k-blocks in B'_k.

Example 10. Let $V\Big(K_{3,5,5}^{(3)}\Big) = \mathbb{Z}_{10} \cup \{\infty_1,\infty_2,\infty_3\}$ with vertex partition $\big\{\{\infty_1,\infty_2,\infty_3\}$, $\{0,2,4,6,8\}$, $\{1,3,5,7,9\}\big\}$ and let

$$\begin{aligned}
B_1 = \big\{&H_1[1,\infty_1,0,4,\infty_2,5,3,\infty_3,2,7],\ H_1[1,\infty_2,0,4,\infty_3,5,3,\infty_1,2,7],\\
&H_1[1,\infty_3,0,4,\infty_1,5,3,\infty_2,2,7],\ H_1[9,\infty_1,4,2,\infty_2,5,3,\infty_3,1,6],\\
&H_1[9,\infty_2,4,2,\infty_3,5,3,\infty_1,1,6],\ H_1[9,\infty_3,4,2,\infty_1,5,3,\infty_2,1,6],\\
&H_1[6,\infty_1,3,7,\infty_2,8,2,\infty_3,0,5],\ H_1[6,\infty_2,3,7,\infty_3,8,2,\infty_1,0,5],\\
&H_1[6,\infty_3,3,7,\infty_1,8,2,\infty_2,0,5],\ H_1[8,9,\infty_1,\infty_2,1,0,4,7,2,\infty_3],\\
&H_1[8,9,\infty_2,\infty_3,1,0,4,7,2,\infty_1],\ H_1[8,9,\infty_3,\infty_1,1,0,4,7,2,\infty_2],\\
&H_1[5,\infty_1,8,6,\infty_2,9,3,\infty_3,0,7],\ H_1[5,\infty_2,8,6,\infty_3,9,3,\infty_1,0,7],\\
&H_1[5,\infty_3,8,6,\infty_1,9,3,\infty_2,0,7]\big\},\\
B_2 = \big\{&H_2[\infty_2,0,5,3,4,\infty_1,\infty_3,8,2,9],\ H_2[\infty_3,0,5,3,4,\infty_2,\infty_1,8,2,9],\\
&H_2[\infty_1,0,5,3,4,\infty_3,\infty_2,8,2,9],\ H_2[\infty_2,1,6,4,5,\infty_1,\infty_3,9,7,8],\\
&H_2[\infty_1,1,6,4,5,\infty_2,\infty_1,9,7,8],\ H_2[\infty_3,1,6,4,5,\infty_3,\infty_2,9,7,8],\\
&H_2[\infty_2,2,7,5,6,\infty_1,\infty_3,0,8,9],\ H_2[\infty_3,2,7,5,6,\infty_2,\infty_1,0,8,9],\\
&H_2[\infty_1,2,7,5,6,\infty_3,\infty_2,0,8,9],\ H_2[1,\infty_2,8,0,\infty_1,9,\infty_3,3,6,7],\\
&H_2[1,\infty_3,8,0,\infty_2,9,\infty_1,3,6,7],\ H_2[1,\infty_1,8,0,\infty_3,9,\infty_2,3,6,7],\\
&H_2[2,3,\infty_1,\infty_2,1,6,4,9,7,\infty_3],\ H_2[2,3,\infty_2,\infty_3,1,6,4,9,7,\infty_1],\\
&H_2[2,3,\infty_3,\infty_1,1,6,4,9,7,\infty_2]\big\},
\end{aligned}$$

$$
\begin{aligned}
B_3 = \big\{ & H_3[1,\infty_1,0,4,\infty_2,7,8,9,\infty_3,5],\ H_3[1,\infty_2,0,4,\infty_3,7,8,9,\infty_1,5],\\
& H_3[1,\infty_3,0,4,\infty_1,7,8,9,\infty_2,5],\ H_3[2,\infty_1,9,5,\infty_2,4,8,1,\infty_3,6],\\
& H_3[2,\infty_2,9,5,\infty_3,4,8,1,\infty_1,6],\ H_3[2,\infty_3,9,5,\infty_1,4,8,1,\infty_2,6],\\
& H_3[0,\infty_1,3,7,\infty_2,2,5,8,\infty_3,6],\ H_3[0,\infty_2,3,7,\infty_3,2,5,8,\infty_1,6],\\
& H_3[0,\infty_3,3,7,\infty_1,2,5,8,\infty_2,6],\ H_3[2,\infty_1,1,3,\infty_2,4,0,9,\infty_3,6],\\
& H_3[2,\infty_2,1,3,\infty_3,4,0,9,\infty_1,6],\ H_3[2,\infty_3,1,3,\infty_1,4,0,9,\infty_2,6],\\
& H_3[7,\infty_1,6,8,\infty_2,3,4,9,\infty_3,5],\ H_3[7,\infty_2,6,8,\infty_3,3,4,9,\infty_1,5],\\
& H_3[7,\infty_3,6,8,\infty_1,3,4,9,\infty_2,5]\big\},\\
B_4 = \big\{ & H_4[0,1,\infty_1,7,\infty_3,\infty_2,2,3,8,4],\ H_4[0,1,\infty_2,7,\infty_1,\infty_3,2,3,8,4],\\
& H_4[0,1,\infty_3,7,\infty_2,\infty_1,2,3,8,4],\ H_4[2,3,\infty_1,9,\infty_3,\infty_2,4,5,0,6],\\
& H_4[2,3,\infty_2,9,\infty_1,\infty_3,4,5,0,6],\ H_4[2,3,\infty_3,9,\infty_2,\infty_1,4,5,0,6],\\
& H_4[4,5,\infty_1,1,\infty_3,\infty_2,6,7,2,8],\ H_4[4,5,\infty_2,1,\infty_1,\infty_3,6,7,2,8],\\
& H_4[4,5,\infty_3,1,\infty_2,\infty_1,6,7,2,8],\ H_4[6,7,\infty_1,3,\infty_3,\infty_2,8,9,4,0],\\
& H_4[6,7,\infty_2,3,\infty_1,\infty_3,8,9,4,0],\ H_4[6,7,\infty_3,3,\infty_2,\infty_1,8,9,4,0],\\
& H_4[8,9,\infty_1,5,\infty_3,\infty_2,0,1,6,2],\ H_4[8,9,\infty_2,5,\infty_1,\infty_3,0,1,6,2],\\
& H_4[8,9,\infty_3,5,\infty_2,\infty_1,0,1,6,2]\big\}.
\end{aligned}
$$

Then, for $k \in \{1,2,3,4\}$, B_k is an H_k-decomposition of $K^{(3)}_{3,5,5}$.

Example 11. Let $V\Big(K^{(3)}_{4,5,5}\Big) = \mathbb{Z}_{10} \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$ with vertex partition $\big\{\{\infty_1,\infty_2,\infty_3,\infty_4\},\ \{0,2,4,6,8\},\ \{1,3,5,7,9\}\big\}$ and let

$$
\begin{aligned}
B_1 = \big\{ & H_1[0,\infty_1,1,3,\infty_2,6,2,\infty_3,4,9],\ H_1[0,\infty_2,1,3,\infty_3,6,2,\infty_4,4,9],\\
& H_1[0,\infty_3,1,3,\infty_4,6,2,\infty_1,4,9],\ H_1[0,\infty_4,1,3,\infty_1,6,2,\infty_2,4,9],\\
& H_1[4,\infty_1,1,7,\infty_2,0,6,\infty_3,2,3],\ H_1[4,\infty_2,1,7,\infty_3,0,6,\infty_4,2,3],\\
& H_1[4,\infty_3,1,7,\infty_4,0,6,\infty_1,2,3],\ H_1[4,\infty_4,1,7,\infty_1,0,6,\infty_2,2,3],
\end{aligned}
$$

$H_1[9,0,\infty_1,\infty_2,8,5,3,4,7,\infty_3]$, $H_1[9,0,\infty_2,\infty_3,8,5,3,4,7,\infty_4]$,

$H_1[9,0,\infty_3,\infty_4,8,5,3,4,7,\infty_1]$, $H_1[9,0,\infty_4,\infty_1,8,5,3,4,7,\infty_2]$,

$H_1[5,4,\infty_1,\infty_2,8,2,0,7,3,\infty_3]$, $H_1[5,4,\infty_2,\infty_3,8,2,0,7,3,\infty_4]$,

$H_1[5,4,\infty_3,\infty_4,8,2,0,7,3,\infty_1]$, $H_1[5,4,\infty_4,\infty_1,8,2,0,7,3,\infty_2]$,

$H_1[7,8,\infty_1,\infty_2,2,1,5,6,9,\infty_3]$, $H_1[7,8,\infty_2,\infty_3,2,1,5,6,9,\infty_4]$,

$H_1[7,8,\infty_3,\infty_4,2,1,5,6,9,\infty_1]$, $H_1[7,8,\infty_4,\infty_1,2,1,5,6,9,\infty_2]\big\}$,

$B_2 = \big\{H_2[1,\infty_1,0,4,\infty_2,5,\infty_3,7,8,9]$, $H_2[1,\infty_2,0,4,\infty_3,5,\infty_4,7,8,9]$,

$H_2[1,\infty_3,0,4,\infty_4,5,\infty_1,7,8,9]$, $H_2[1,\infty_4,0,4,\infty_1,5,\infty_2,7,8,9]$,

$H_2[0,\infty_1,5,3,\infty_2,8,2,\infty_3,7,\infty_4]$, $H_2[0,\infty_2,5,3,\infty_3,8,2,\infty_4,7,\infty_1]$,

$H_2[0,\infty_3,5,3,\infty_4,8,2,\infty_1,7,\infty_2]$, $H_2[0,\infty_4,5,3,\infty_1,8,2,\infty_2,7,\infty_3]$,

$H_2[9,\infty_1,2,6,\infty_2,7,\infty_3,3,1,8]$, $H_2[9,\infty_2,2,6,\infty_3,7,\infty_4,3,1,8]$,

$H_2[9,\infty_3,2,6,\infty_4,7,\infty_1,3,1,8]$, $H_2[9,\infty_4,2,6,\infty_1,7,\infty_2,3,1,8]$,

$H_2[6,\infty_1,1,5,\infty_2,8,\infty_3,2,4,9]$, $H_2[6,\infty_2,1,5,\infty_3,8,\infty_4,2,4,9]$,

$H_2[6,\infty_3,1,5,\infty_4,8,\infty_1,2,4,9]$, $H_2[6,\infty_4,1,5,\infty_1,8,\infty_2,2,4,9]$,

$H_2[4,\infty_1,3,7,\infty_2,8,\infty_3,6,0,9]$, $H_2[4,\infty_2,3,7,\infty_3,8,\infty_4,6,0,9]$,

$H_2[4,\infty_3,3,7,\infty_4,8,\infty_1,6,0,9]$, $H_2[4,\infty_4,3,7,\infty_1,8,\infty_2,6,0,9]\big\}$,

$B_3 = \big\{H_3[1,\infty_1,0,4,\infty_2,7,8,9,\infty_3,5]$, $H_3[1,\infty_2,0,4,\infty_3,7,8,9,\infty_4,5]$,

$H_3[1,\infty_3,0,4,\infty_4,7,8,9,\infty_1,5]$, $H_3[1,\infty_4,0,4,\infty_1,7,8,9,\infty_2,5]$,

$H_3[2,\infty_1,9,5,\infty_2,4,8,1,\infty_3,6]$, $H_3[2,\infty_2,9,5,\infty_3,4,8,1,\infty_4,6]$,

$H_3[2,\infty_3,9,5,\infty_4,4,8,1,\infty_1,6]$, $H_3[2,\infty_4,9,5,\infty_1,4,8,1,\infty_2,6]$,

$H_3[0,\infty_1,3,7,\infty_2,2,5,8,\infty_3,6]$, $H_3[0,\infty_2,3,7,\infty_3,2,5,8,\infty_4,6]$,

$H_3[0,\infty_3,3,7,\infty_4,2,5,8,\infty_1,6]$, $H_3[0,\infty_4,3,7,\infty_1,2,5,8,\infty_2,6]$,

$H_3[2,\infty_1,1,3,\infty_2,4,0,9,\infty_3,6]$, $H_3[2,\infty_2,1,3,\infty_3,4,0,9,\infty_4,6]$,

$H_3[2,\infty_3,1,3,\infty_4,4,0,9,\infty_1,6]$, $H_3[2,\infty_4,1,3,\infty_1,4,0,9,\infty_2,6]$,

$$
\begin{aligned}
&H_3[7,\infty_1,6,8,\infty_2,3,4,9,\infty_3,5],\ H_3[7,\infty_2,6,8,\infty_3,3,4,9,\infty_4,5],\\
&H_3[7,\infty_3,6,8,\infty_4,3,4,9,\infty_1,5],\ H_3[7,\infty_4,6,8,\infty_1,3,4,9,\infty_2,5]\big\},\\
B_4 = \big\{&H_4[1,\infty_1,0,6,\infty_2,5,2,7,\infty_4,\infty_3],\ H_4[1,\infty_2,0,6,\infty_3,5,2,7,\infty_1,\infty_4],\\
&H_4[1,\infty_3,0,6,\infty_4,5,2,7,\infty_2,\infty_1],\ H_4[1,\infty_4,0,6,\infty_1,5,2,7,\infty_3,\infty_2],\\
&H_4[2,\infty_1,1,7,\infty_2,6,3,8,\infty_4,\infty_3],\ H_4[2,\infty_2,1,7,\infty_3,6,3,8,\infty_1,\infty_4],\\
&H_4[2,\infty_3,1,7,\infty_4,6,3,8,\infty_2,\infty_1],\ H_4[2,\infty_4,1,7,\infty_1,6,3,8,\infty_3,\infty_2],\\
&H_4[3,\infty_1,2,8,\infty_2,7,4,9,\infty_4,\infty_3],\ H_4[3,\infty_2,2,8,\infty_3,7,4,9,\infty_1,\infty_4],\\
&H_4[3,\infty_3,2,8,\infty_4,7,4,9,\infty_2,\infty_1],\ H_4[3,\infty_4,2,8,\infty_1,7,4,9,\infty_3,\infty_2],\\
&H_4[4,\infty_1,3,9,\infty_2,8,5,0,\infty_4,\infty_3],\ H_4[4,\infty_2,3,9,\infty_3,8,5,0,\infty_1,\infty_4],\\
&H_4[4,\infty_3,3,9,\infty_4,8,5,0,\infty_2,\infty_1],\ H_4[4,\infty_4,3,9,\infty_1,8,5,0,\infty_3,\infty_2],\\
&H_4[5,\infty_1,4,0,\infty_2,9,6,1,\infty_4,\infty_3],\ H_4[5,\infty_2,4,0,\infty_3,9,6,1,\infty_1,\infty_4],\\
&H_4[5,\infty_3,4,0,\infty_4,9,6,1,\infty_2,\infty_1],\ H_4[5,\infty_4,4,0,\infty_1,9,6,1,\infty_3,\infty_2]\big\}.
\end{aligned}
$$

Then, for $k \in \{1,2,3,4\}$, B_k is an H_k-decomposition of $K^{(3)}_{4,5,5}$.

Example 12. Let $V\left(K^{(3)}_{5,5,5}\right) = \mathbb{Z}_{15}$ with vertex partition $\big\{\{0,3,6,9,12\}, \{1,4,7,10,13\},$ $\{2,5,8,11,14\}\big\}$ and let

$$
\begin{aligned}
B_1 = \big\{&H_1[2,1,0,12,10,5,7,8,9,14],\ H_1[3,2,1,13,11,6,8,9,10,0],\\
&H_1[4,3,2,14,12,7,9,10,11,1],\ H_1[0,11,1,10,2,6,9,14,3,13],\\
&H_1[13,3,8,11,9,1,4,6,5,7]\big\},\\
B_2 = \big\{&H_2[2,0,1,4,6,8,11,3,7,12,\ H_2[3,1,2,5,7,9,12,4,8,13],\\
&H_2[4,2,3,6,8,10,13,5,9,14],\ H_2[1,8,3,9,2,13,7,14,5,12],\\
&H_2[1,2,6,0,5,7,4,11,9,14]\big\},
\end{aligned}
$$

$$
\begin{aligned}
B_3 = \big\{ & H_3[0,1,5,14,10,9,2,7,12,3],\ H_3[1,2,6,0,11,10,5,7,9,4],\\
& H_3[2,3,7,1,12,11,0,4,8,5],\ H_3[0,8,1,4,2,12,5,6,13,9],\\
& H_3[1,2,3,0,14,13,9,10,11,7]\big\},\\
B_4 = \big\{ & H_4[0,1,5,14,10,9,2,4,6,7],\ H_4[1,2,6,0,11,10,3,5,7,8],\\
& H_4[2,3,7,1,12,11,4,6,8,9],\ H_4[4,0,2,12,8,10,5,9,13,14],\\
& H_4[4,11,0,6,2,1,3,5,7,8]\big\}.
\end{aligned}
$$

Then, for $k \in \{1,2,3,4\}$, an H_k-decomposition of $K^{(3)}_{5,5,5}$ consists of the orbits of the H_k-blocks in B_k under the action of the map and $j \mapsto j+3 \pmod{15}$.

CHAPTER IV: MAIN RESULTS

We begin by giving necessary conditions for the existence of an H_k-decomposition of $K_v^{(3)}$. An obvious necessary condition is that 5 must divide the number of edges in $K_v^{(3)}$, and thus we must have $v \equiv 0$, 1, or 2 $\pmod 5$. Also since H_k has 10 vertices, we must also have $n \geq 10$ for a non-trivial H_k-decomposition of $K_v^{(3)}$. Thus we have the following.

Lemma 1. *There exists an H_k-decomposition of $K_v^{(3)}$ only if $v \equiv 0$, 1, or 2 $\pmod 5$ and $v \geq 10$.*

We show that the above conditions are sufficient by showing how to construct H_k-decompositions of $K_v^{(3)}$ for all $v \equiv 0$, 1, or 2 $\pmod 5$ with $v \geq 10$. Our constructions are dependent on the many small examples given in Chapter III.

We begin by proving a lemma that is fundamental to our constructions.

Lemma 2. *For $r \in \{0,1,2\}$ and all positive integers x and y, there exists a decomposition of $K_{r,5x,5y}^{(3)} \cup L_{5x,5y}^{(3)}$ into copies of $K_{5,5,5}^{(3)}$ and $K_{r,5,5}^{(3)} \cup L_{5,5}^{(3)}$.*

Proof. Let $r \in \{0,1,2\}$ and let x and y be positive integers. The vertices of $K_{r,5x,5y}^{(3)} \cup L_{5x,5y}^{(3)}$ can be partitioned into sets V_i, W_j, and R where $1 \leq i \leq x$, $1 \leq j \leq y$, $|V_i| = 5 = |W_j|$, and $|R| = r$ such that every edge $\{a,b,c\}$ is of exactly one of the following types:

Type 1: there exist i, j with $a \in R$, $b \in V_i$, and $c \in W_j$;

Type 2: there exist i, j, k with $i \neq j$, $a \in V_i$, $b \in V_j$, and $c \in W_k$;

Type 3: there exist i, j, k with $j \neq k$, $a \in V_i$, $b \in W_j$, and $c \in W_k$;

Type 4: there exist i, j with $a, b \in V_i$ and $c \in W_j$; or

Type 5: there exist i, j with $a \in V_i$ and $b, c \in W_j$.

For every choice of i and j we can put together the edges of Types 1, 4, and 5 to form a copy of $K_{r,5,5}^{(3)} \cup L_{5,5}^{(3)}$. For every choice of i, j, and k the edges of Types 2 and 3 form copies of $K_{5,5,5}^{(3)}$. Since all edges are accounted for by exactly one of the aforementioned choices of subscripts, we have the desired decomposition into copies of $K_{5,5,5}^{(3)}$ and $K_{r,5,5}^{(3)} \cup L_{5,5}^{(3)}$. □

Theorem 3. *There exists an H_k-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0$, 1, or 2 (mod 5) and $v \geq 10$.*

Proof. The necessary conditions for the existence of an H_k-decomposition of $K_v^{(3)}$ are established in Lemma 1. Thus we need only to establish their sufficiency. Let $v = 5x + r$ where $x \geq 2$ and $r \in \{0, 1, 2\}$. We will consider two cases depending on the parity of x.

When x is even we can write $K_v^{(3)}$ as the edge-disjoint union of copies of $K_{10+r}^{(3)}$, $K_{r,10,10}^{(3)} \cup L_{10,10}^{(3)}$, and $K_{10,10,10}^{(3)}$, where the group of r vertices is common to every applicable copy. By Examples 1, 2, and 3 we have that an H_k-decomposition of $K_{10+r}^{(3)}$ exists. By Lemma 2 we have that $K_{r,10,10}^{(3)} \cup L_{10,10}^{(3)}$ can be decomposed into copies of $K_{5,5,5}^{(3)}$ and $K_{r,5,5}^{(3)} \cup L_{5,5}^{(3)}$. By Example 12 an H_k-decomposition of $K_{5,5,5}^{(3)}$ exists. When $r = 0$ $K_{0,5,5}^{(3)} \cup L_{5,5}^{(3)}$ is isomorphic to $L_{5,5}^{(3)}$, which admits an H_k-decomposition by Example 7. When $r \in \{1, 2\}$ an H_k-decomposition of $K_{r,5,5}^{(3)} \cup L_{5,5}^{(3)}$ exists by Examples 8 and 9. Finally, it is straightforward to see that $K_{10,10,10}^{(3)}$ can be decomposed into copies of $K_{5,5,5}^{(3)}$; thus, by Example 12 an H_k-decomposition of $K_{10,10,10}^{(3)}$ exists.

When x is odd the construction is similar, and we can write $K_v^{(3)}$ as the edge disjoint union of copies of $K_{15+r}^{(3)}$, $K_{10+r}^{(3)}$, $K_{r,15,10}^{(3)} \cup L_{15,10}^{(3)}$, $K_{r,10,10}^{(3)} \cup L_{10,10}^{(3)}$, $K_{15,10,10}^{(3)}$, and $K_{10,10,10}^{(3)}$, where again the group of r vertices is common to every applicable copy. There exist H_k-decompositions of each of these hypergraphs exist by using the ingredients listed in the previous case along with Examples 4, 5, and 6. $\square$

REFERENCES

[1] P. Adams, D. Bryant, and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[2] R. F. Bailey and B. Stevens, Hamilton decompositions of complete k-uniform hypergraphs, *Discrete Math.* **310** (2010), 3088–3095.

[3] Zs. Baranyai, On the factorization of the complete uniform hypergraph, in: Infinite and finite sets, *Colloq. Math. Soc. János Bolyai* **10**, North-Holland, Amsterdam, 1975, 91–108.

[4] J.-C. Bermond, A. Germa, and D. Sotteau, Hypergraph-designs, *Ars Combinatoria* **3** (1977), 47–66.

[5] D. Bryant, S. Herke, B. Maenhaut, and W. Wannasit, Decompositions of complete 3-uniform hypergraphs into small 3-uniform hypergraphs, *Australas. J. Combin.* **60** (2014), 227–254.

[6] D. E. Bryant and T. A. McCourt, Existence results for G-designs, http://wiki.smp.uq.edu.au/G-designs/

[7] R. C. Bunge, S. I. El-Zanati, J. Jetton, M. Juarez, A. J. Netz, D. Roberts, and P. Ward, On loose 5-cycle decompositions, packings, and coverings of complete lambda-fold 3-uniform hypergraphs, *Ars Combinatoria*, to appear.

[8] C. J. Colbourn and R. Mathon, Steiner systems, in *The CRC Handbook of Combinatorial Designs*, 2nd edition, (Eds. C. J. Colbourn and J. H. Dinitz), CRC Press, Boca Raton (2007), 102–110.

[9] S. Glock, D. Kühn, A. Lo, and D. Osthus, The existence of designs via iterative absorption, arXiv:1611.06827v2, (2017), 63 pages.

[10] S. Glock, D. Kühn, A. Lo, and D. Osthus, Hypergraph F-designs for arbitrary F, arXiv:1706.01800, (2017), 72 pages.

[11] H. Hanani, On quadruple systems, *Canad. J. Math.*, **12** (1960), 145–157.

[12] H. Hanani, Decomposition of hypergraphs into octahedra, Second International Conference on Combinatorial Mathematics (New York, 1978), pp. 260–264, *Ann. New York Acad. Sci.*, 319, New York Acad. Sci., New York, 1979.

[13] H. Jordon and G. Newkirk, 4-cycle decompositions of complete 3-uniform hypergraphs, *Australas. J. Combin.* **71** (2018), 312–323.

[14] P. Keevash, The existence of designs, arXiv:1401.3665v2, (2018), 39 pages.

[15] J. Kuhl and M. W. Schroeder, Hamilton cycle decompositions of k-uniform k-partite hypergraphs, *Australas. J. Combin.* **56** (2013), 23–37.

[16] D. Kühn and D. Osthus, Decompositions of complete uniform hypergraphs into Hamilton Berge cycles, *J. Combin. Theory Ser. A* **126** (2014), 128–135.

[17] M. Meszka and A. Rosa, Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles, *Australas. J. Combin.* **45** (2009), 291–302.

[18] M. W. Schroeder, On Hamilton cycle decompositions of r-uniform r-partite hypergraphs, *Discrete Math.* **315** (2014), 1–8.

[19] R. M. Wilson, Decompositions of Complete Graphs into Subgraphs Isomorphic to a Given Graph, in "Proc. Fifth British Combinatorial Conference" (C. St. J. A. Nash-Williams and J. Sheehan, Eds.), pp. 647–659, *Congressus Numerantium* XV, 1975.