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Hamiltonian Cycles and 2-Factors in Tough Graphs without Forbidden Subgraphs

Elizabeth Veronica Grimm
Illinois State University, elizabethbanks1998@gmail.com

HAMILTONIAN CYCLES AND 2-FACTORS IN TOUGH GRAPHS WITHOUT FORBIDDEN SUBGRAPHS

ELIZABETH V. GRIMM

42 Pages

A Hamiltonian cycle in a graph G is a cycle which contains every vertex of G. The study of Hamiltonian cycle problem has a long history in graph theory and is a central theme. In general, it is NP-complete to decide whether a graph contains a Hamiltonian cycle. Thus researchers have been investigating sufficient conditions that guarantee the existence of a Hamiltonian cycle in a graph. There are many classic results along this line. For example, in 1952, Dirac showed that an n-vertex graph G with $n \geq 3$ is Hamiltonian if $\delta(G) \geq \frac{n}{2}$.

Chvátal studied Hamiltonian cycles by considering graph toughness, a measure of resilience under the removal of vertices. Let $t \geq 0$ be a real number and denote by $c(G)$ the number of components of G. We say a graph G is t-tough if for each cut set S of G we have $t \cdot c(G - S) \leq |S|$. The toughness of a graph G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough if G is non-complete, and is defined to be ∞ if G is complete. Chvátal conjectured in 1973 the existence of some constant t such that all t-tough graphs with at least three vertices are Hamiltonian. While the conjecture has been proven for some special classes of graphs, it remains open in general. Supporting this conjecture of Chvátal's, in the first part of this thesis, we show that every 3-tough $(P_2 \cup 3P_1)$-free graph with at least three vertices is Hamiltonian, where $P_2 \cup 3P_1$ is the disjoint union of an edge and three isolated vertices.

The notion of a 2-factor is a generalization of a Hamiltonian cycle, which consists of vertex disjoint cycles which together cover the vertices of G. Thus, a Hamiltonian cycle is just a 2-factor with exactly one cycle. It is known that every 2-tough graph with at least three vertices has a 2-factor. In graphs with restricted structures the toughness bound 2 can be improved. For example, it was shown that every $2K_2$-free 3/2-tough graph with at least three vertices has a 2-factor, and the toughness bound 3/2 is best possible. In viewing $2K_2$, the disjoint union of two edges, as a linear forest, in this thesis, for any linear forest R on 5, 6, or 7 vertices, we find the sharp toughness bound t such that every t-tough R-free graph on at least three vertices has a 2-factor.

KEYWORDS: Hamiltonian cycle, 2-factor, toughness, forbidden subgraphs

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ELIZABETH V. GRIMM

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COMMITTEE MEMBERS:

Songling Shan, Chair

Amin Bahmanian

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E. V. G.

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CHAPTER I: INTRODUCTION

We begin with some general notation. Then, we will introduce some background into Hamiltonian cycles, Chvátal's toughness conjecture, and 2-factors. In Chapter 2, we prove Theorem 1 and, in Chapter 3, we prove Theorems 2, 3, 4, 5, and 6. Finally, in Chapter 4, we discuss further research questions.

Let G be a simple, undirected graph and let $E(G), V(G)$ denote its edge and vertex set respectively. For $v \in V(G)$, denote by $N_G(v)$ the set of neighbors of v in G. The closed neighborhood of a vertex v in G, denoted by $N_G[v]$, is the set $\{v\} \cup N_G(v)$. For two disjoint subgraphs H_1, H_2 of G, $N_{H_1}(H_2)$ denotes the set of neighbors of vertices of H_2 in G that are contained in $V(H_1)$. Let $d_G(v) = |N_G(v)|$ be the degree of v in G. For any subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced on S, $G - S$ denotes the subgraph $G[V(G) \setminus S]$, and $N(S) = \cup_{v \in S} N_G(v)$. Given disjoint subsets S and T of $V(G)$, we denote by $E_G(S,T)$ the set of edges which have one end vertex in S and the other end vertex in T, and let $e_G(S,T) = |E_G(S,T)|$. If $S = \{s\}$ is a singleton, we write $e_G(s,T)$ for $e_G(\{s\},T)$. If $H \subseteq G$ is a subgraph of G, and $T \subseteq V(G)$ with $T \cap V(H) = \emptyset$, we write $E_G(H,T)$ and $e_G(H,T)$ for notational simplicity.

If u and v are adjacent in G, we write $u \sim v$. A path P connecting two vertices u and v is called a (u,v)-*path*, and we write uPv or vPu in order to specify the two endvertices of P. Let uPv and xQy be two disjoint paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx. The independence number of a graph G, denoted $\alpha(G)$, is the size of a largest independent set of G. For a given graph R, we say that G is R-free if there does not exist an induced copy of R in G.

In this thesis, we study two types of problems: the Hamiltonian cycle problem and the 2-factor problem.

I.1 Hamiltonicity and Chvátal's Toughness Conjecture

A Hamiltonian cycle in a graph G is a cycle which contains every vertex of G. It was named after Irish mathematician Sir William Rowan Hamilton who invented a game in the 1850s where such a cycle was sought after in the polyhedron edges of a dodecahedron. Historical evidence suggests, however, that such cycles were studied long before the 1850s. In the 9th century, Hamiltonian cycles were studied in graphs corresponding to the moves of a knight on a chessboard.

Deciding whether a graph contains a Hamiltonian cycle is NP-complete. Thus, research has been done to find sufficient conditions which guarantee the existence of a Hamiltonian cycle. In his 1952 paper [9], Dirac showed that if G is an n-vertex graph with $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian. Ore showed in 1960 [16] that if G is an n-vertex graph with $n \geq 3$ and $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian. Twelve years later, in 1972, Chvátal and Erdős [8] showed that if G is a graph with connectivity k such that $\alpha(G) \leq k$, then G is Hamiltonian. It was shown by Häggkvist and Nicoghossian [12] nine years later that if G is a 2-connected graph of order n, connectivity k, and $\delta(G) \geq \frac{n+k}{3}$, then G is Hamiltonian. We refer the reader to [11] for a survey on Hamiltonian cycles.

Observe that these conditions are sufficient but not necessary as there exist graphs which satisfy none of the above conditions but are Hamiltonian. For example, C_m, a cycle on m vertices, is clearly Hamiltonian but does not satisfy any of the above conditions. Because of this, we are interested in finding a property P such that if G is Hamiltonian then G is "close to" having the property P.

In search of such a property, we study graph toughness. Graph toughness is a measure of resilience under the removal of vertices, introduced by Chvátal in his 1973 paper [7]. Denote by $c(G)$ the number of components of G. Let $t \geq 0$ be a real number. We say a graph G is t-tough if for each cut set S of G we have $t \cdot c(G - S) \leq |S|$. The toughness of a graph G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough if

G is non-complete, and is defined to be ∞ if G is complete.

It follows from the definition that every non-complete t-tough graph is $2\lceil t \rceil$-connected, which implies that $\kappa(G) \geq 2\tau(G)$ for non-complete graphs G. Therefore, the minimum degree of any t-tough non-complete graph is at least $2\lceil t \rceil$. It is interesting to note that high connectivity does not imply high toughness. For example, the complete bipartite graph $K_{n,n}$ is n-connected but only 1-tough.

We know every Hamiltonian graph is 1-tough, but what about the converse? In the same 1973 paper [7] where Chvátal introduced the notion of toughness, he also conjectured the existence of a constant t_0 such that every t_0-tough graph on at least three vertices is Hamiltonian. It was shown in [3] that $t_0 \geq \frac{9}{4}$. The conjecture has since been verified for certain special classes of graphs, but remains open in general. Recent work has proven the conjecture for $2K_2$-free graphs [6, 19, 17], $(P_2 \cup P_3)$-free graphs [20], $(K_2 \cup 2K_1)$-free graphs [15], and planar chordal graphs. We refer the reader to [1] for a survey on more related results. It is of interest to note that, for any positive integer k, there exist graphs which are k-connected and still have no Hamiltonian cycle. Therefore, an analogous conjecture for connectivity cannot exist.

In this thesis, we support Chvátal's conjecture by proving the following result:

Theorem 1. *If G is a 3-tough $(P_2 \cup 3P_1)$-free graph on at least 3 vertices, then G is Hamiltonian.*

It is not known whether 3-tough is best possible. That being said, the graph H_1 depicted in Figure 1 is $(P_2 \cup 3P_1)$-free, 1-tough, and not Hamiltonian. Therefore, we must have $1 < \tau(G) \leq 3$.

I.2 2-Factors

For integers a and b with $a \geq 0$ and $b \geq 1$, we denote by aP_b the graph consisting of a disjoint copies of the path P_b. When $a = 1$, $1P_b$ is just P_b, and when $a = 0$, $0P_b$ is the null graph. For two integers p and q, let $[p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$.

For an integer $k \geq 1$, we say a k-regular spanning subgraph is a *k-factor* of G. A Hamiltonian cycle then, can be viewed as a 2-regular spanning subgraph − a special 2-factor which only has a single cycle. It is well known, according to a theorem by Enomoto, Jackson, Katerinis, and Saito [10] from 1998, that every k-tough graph with at least three vertices has a k-factor if $k|V(G)|$ is even and $|V(G)| \geq k+1$. In terms of a sharp toughness bound, particular research interest has been taken when $k = 2$ for graphs with restricted structures. For example, it was shown that every 3/2-tough 5-chordal graph (graphs with no induced cycle of length at least 5) on at least three vertices has a 2-factor [4] and that every 3/2-tough $2K_2$-free graph on at least three vertices has a 2-factor [17]. The toughness bound 3/2 is best possible in both results.

A *linear forest* is a graph consisting of disjoint paths. In viewing $2K_2$ as a linear forest on 4 vertices and the result by Ota and Sanka [17] that every 3/2-tough $2K_2$-free graph on at least three vertices has a 2-factor, we investigate the existence of 2-factors in R-free graphs when R is a linear forest on 5, 6, or 7 vertices. These graphs R are listed below, where the unions are vertex disjoint unions.

1. P_5 $\quad P_4 \cup P_1$ $\quad P_3 \cup P_2$ $\quad P_3 \cup 2P_1$ $\quad 2P_2 \cup P_1$ $\quad P_2 \cup 3P_1$ $\quad 5P_1$;
2. P_6 $\quad P_5 \cup P_1$ $\quad P_4 \cup P_2$ $\quad P_4 \cup 2P_1$ $\quad 2P_3$ $\quad P_3 \cup P_2 \cup P_1$ $\quad P_3 \cup 3P_1$ $\quad 3P_2$ $\quad 2P_2 \cup 2P_1$ $\quad P_2 \cup 4P_1$ $\quad 6P_1$;
3. P_7 $\quad P_6 \cup P_1$ $\quad P_5 \cup P_2$ $\quad P_5 \cup 2P_1$ $\quad P_4 \cup P_3$ $\quad P_4 \cup P_2 \cup P_1$ $\quad P_4 \cup 3P_1$ $\quad 2P_3 \cup P_1$ $\quad P_3 \cup 2P_2$ $\quad P_3 \cup P_2 \cup 2P_1$ $\quad P_3 \cup 4P_1$ $\quad 3P_2 \cup P_1$ $\quad 2P_2 \cup 3P_1$ $\quad P_2 \cup 5P_1$ $\quad 7P_1$.

Our results are the following:

Theorem 2. *Let $t > 0$ be a real number, R be any linear forest on 5 vertices, and G be a t-tough R-free graph on at least 3 vertices. Then G has a 2-factor provided that*

(1) $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ and $t = 1$ unless

(a) $R = P_2 \cup 3P_1$, *and* $G \cong H_0$ *or* G *contains* H_1, H_2 *or* H_3 *as a spanning subgraph such that* $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ *for each* $i \in [1, 3]$, *where* H_i, S *and* T *are defined in Figure 1.*

(b) $R = P_3 \cup 2P_1$ *and* G *contains* H_1 *as a spanning subgraph such that* $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$.

(2) $R = 5P_1$ *and* $t > 1$.

(3) $R \in \{P_5, P_3 \cup P_2, 2P_2 \cup P_1\}$ *and* $t = 3/2$.

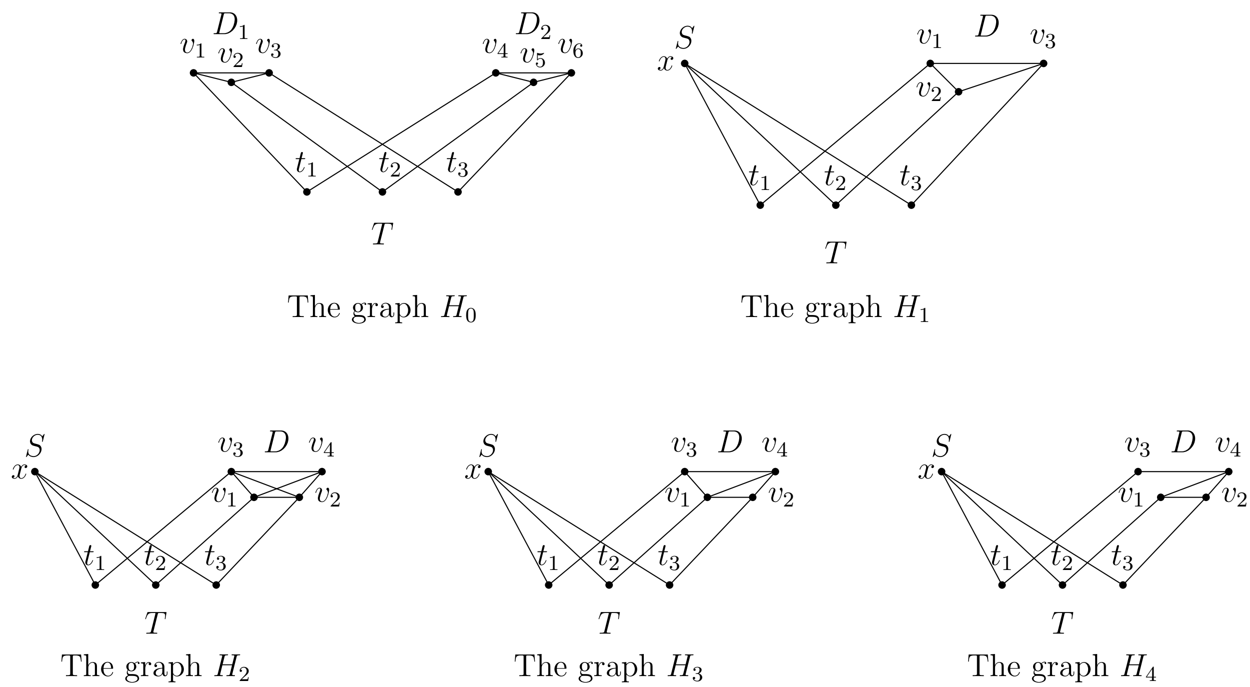
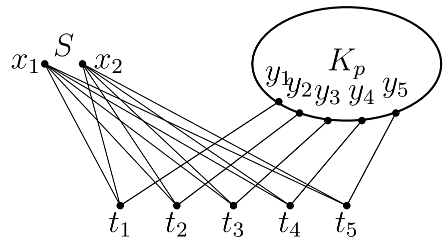


Figure 1: The four exceptional graphs for Theorem 2(1), where $S = \{x\}$ and $T = \{t_1, t_2, t_3\}$.

Theorem 3. *Let* $t > 0$ *be a real number,* R *be any linear forest on* 6 *vertices, and* G *be a* t*-tough* R*-free graph on at least 3 vertices. Then* G *has a 2-factor provided that*

(1) $R \in \{P_4 \cup 2P_1, P_3 \cup 3P_1, P_2 \cup 4P_1, 6P_1\}$ *and* $t > 1$ *unless* $R = 6P_1$ *and* G *contains* H_5 *with* $p = 5$ *as a spanning subgraph such that* $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S))$, *where* H_5, S *and* T *are defined in Figure 2.*

(2) $R \in \{P_6, P_5 \cup P_1, P_4 \cup P_2, 2P_3, P_3 \cup P_2 \cup P_1, 3P_2, 2P_2 \cup 2P_1\}$ *and* $t = 3/2$.



The graph H_5

Figure 2: The exceptional graph for Theorem 3(1), where $S = \{x_1, x_2\}$, $T = \{t_1, \ldots, t_5\}$, and $p = 5$.

Theorem 4. *Let* $t > 0$ *be a real number,* R *be any linear forest on* 7 *vertices, and* G *be a* t*-tough* R*-free graph on at least 3 vertices. Then* G *has a 2-factor provided that*

(1) $R \in \{P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1\}$ *and* $t > 1$ *unless*

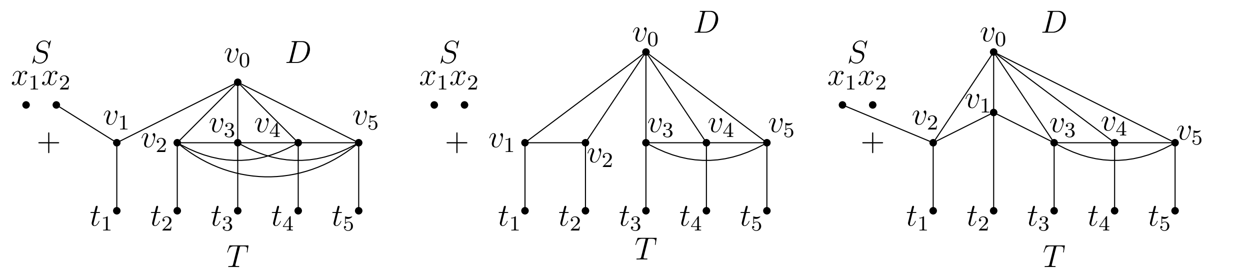
(a) *when* $R \neq P_4 \cup 3P_1$, G *contains* H_5 *with* $p = 5$ *as a spanning subgraph such that* $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$, *where* H_5, S *and* T *are defined in Figure 2.*

(b) $R = P_2 \cup 5P_1$ *and* G *contains one of* $H_6, \ldots, H_{11}$ *as a spanning subgraph such that* $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S]) \cup E(G[V(G) \setminus (T \cup S)])$, *where* H_i, S *and* T *are defined in Figure 3 for each* $i \in [6, 11]$.

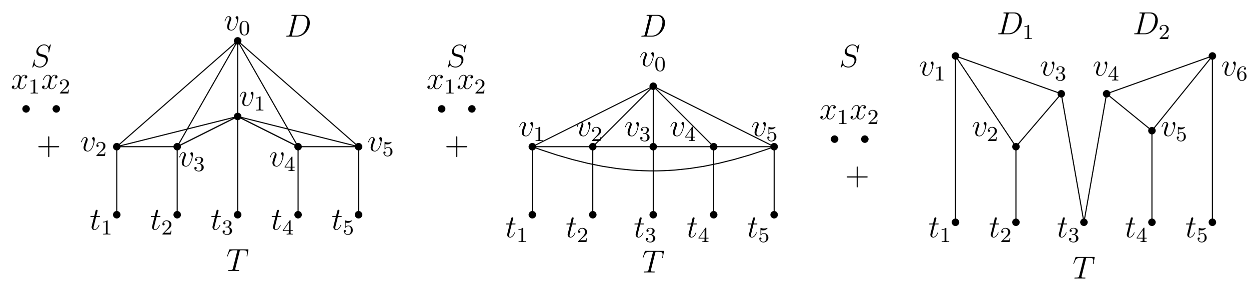
(2) $R = 7P_1$ *and* $t > \frac{7}{6}$ *unless* G *contains* H_5 *with* $p = 5$ *as a spanning subgraph such that* $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$.

(3) $R \in \{P_7, P_6 \cup P_1, P_5 \cup P_2, P_5 \cup 2P_1, P_4 \cup P_2 \cup P_1, 2P_3 \cup P_1, P_4 \cup P_3, P_3 \cup 2P_2, P_3 \cup P_2 \cup 2P_1, 3P_2 \cup P_1, 2P_2 \cup 3P_1\}$ *and* $t = 3/2$.

Remark 1. *[Examples demonstrating sharp toughness bounds] The toughness bounds in Theorems 2 to 4 are all sharp.*



The graph H_6 The graph H_7 The graph H_8



The graph H_9 The graph H_{10} The graph H_{11}

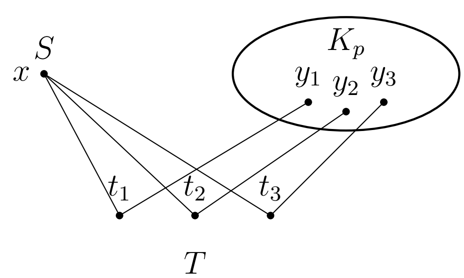
Figure 3: The five exceptional graphs for Theorem 4(1)(b), where $S = \{x_1, x_2\}$, $T = \{t_1, t_2, t_3, t_4, t_5\}$, and "+" represents the join of $H_i[S]$ and $H_i[T]$, $i \in [6, 11]$.

(1) *Theorem 2(1) when $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ and $t = 1$. The graph showing that the toughness 1 is best possible is the complete bipartite $K_{n-1,n}$ for any integer $n \geq 2$. The graph $K_{n,n-1}$ is P_4-free and so is R-free, with $\lim_{n\to\infty} \tau(K_{n,n-1}) = \lim_{n\to\infty} \frac{n-1}{n} = 1$, but contains no 2-factor.*

(2) *Theorem 2(2), Theorem 3(1) and Theorem 4(1) and $t > 1$. The graph showing that the toughness is best possible is the graph H_{12}, which is constructed as below: let $p \geq 3$, K_p be a complete graph, and $y_1, y_2, y_3 \in V(K_p)$ be distinct, $S = \{x\}$, and $T = \{t_1, t_2, t_3\}$, then H_{12} is obtained from K_p, S and T by adding edges $t_i x$ and $t_i y_i$ for each $i \in [1, 3]$. See Figure 4 for a depiction. By inspection, the graph is $5P_1$-free and $(P_4 \cup 2P_1)$-free. So the graph is R-free for any $R \in \{5P_1, P_4 \cup 2P_1, P_3 \cup 3P_1, P_2 \cup 4P_1, 6P_1, P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1\}$. For any given $p \geq 3$, the graph H_{12} does not contain a 2-factor, as any 2-factor has to contain the edges $t_1 x, t_2 x$ and $t_3 x$. We will show $\tau(H_{12}) = 1$ in the last section of Chapter III.*

(3) For Theorem 2(3), Theorem 3(2) and Theorem 4(3) and $t = \frac{3}{2}$: note that all the graphs R in these cases contain $2K_2$ as an induced subgraph. Chvátal [7] constructed a sequence $\{G_k\}_{k=1}^{\infty}$ of split graphs (graphs whose vertex set can be partitioned into a clique and an independent set) having no 2-factors and $\tau(G_k) = \frac{3k}{2k+1}$ for each positive integer k. As the class of $2K_2$-free graphs is a superclass of split graphs, $\frac{3}{2}$-tough is the best possible toughness bound for a $2K_2$-free graph to have a 2-factor.

(4) Theorem 4(2) and $t > \frac{7}{6}$. The graph showing that the toughness is best possible is the graph H_5 with $p \geq 6$, which is constructed as below: let $p \geq 5$, K_p be a complete graph, and $y_1, y_2, y_3, y_4, y_5 \in V(K_p)$ be distinct, $S = \{x_1, x_2\}$, and $T = \{t_1, t_2, t_3, t_4, t_5\}$. Then H_5 is obtained from K_p, S and T by adding edges $t_i x_j$ and $t_i y_i$ for each $i \in [1, 5]$ and each $j \in [1, 2]$. See Figure 2 for a depiction. By inspection, the graph is $7P_1$-free. For any given $p \geq 5$, the graph H_5 does not contain a 2-factor, as any 2-factor has to contain at least three edges from one of x_1 and x_2 to at least three vertices of T. We will show $\tau(H_5) = \frac{7}{6}$ when $p \geq 6$ in the last section of Chapter III.



The graph H_{12}

Figure 4: Sharpness example for Theorem 2(2), Theorem 3(1) and Theorem 4(1), where $S = \{x\}$ and $T = \{t_1, t_2, t_3\}$.

To supplement Theorems 2 to 4, we show that the exceptional graphs in Figures 1 to 3 satisfy the corresponding conditions below.

Theorem 5. *The following statements hold.*

(1) The graph H_i is $(P_2 \cup 3P_1)$-free, contains no 2-factor, and $\tau(H_i) = 1$ for each $i \in [0, 4]$, the graph H_1 is also $(P_3 \cup 2P_1)$-free.

(2) The graph H_i is $(P_2 \cup 5P_1)$-free and contains no 2-factor for each $i \in [5, 11]$, H_5 with $p = 5$ is $(P_3 \cup 4P_1)$-free and $6P_1$-free. Furthermore, $\tau(H_5) = \frac{6}{5}$ when $p = 5$ and $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 11]$.

We have explained that H_5 and H_{12} are R-free for the corresponding linear forests R and contain no 2-factor in Remark 1(2) and (4). The Theorem below is to verify the toughness of the graphs H_5 with $p \geq 6$ and H_{12}.

Theorem 6. *The following statements hold.*

(1) $\tau(H_5) = \frac{7}{6}$ when $p \geq 6$;

(2) $\tau(H_{12}) = 1$.

CHAPTER II: PROOF OF THEOREM 1

II.1 Preliminaries

In this section, we give results necessary to complete the proof of Theorem 1.

Lemma 1 (Dirac [9], Ore [16]). *Let G be a graph on n vertices such that $\delta(G) \geq \frac{n+1}{2}$. Then G is Hamiltonian-connected.*

Lemma 2 (Bauer et al. [2]). *Let G be a t-tough graph on $n \geq 3$ vertices with $\delta(G) > n/(t+1) - 1$. Then G is Hamiltonian.*

Lemma 3 (Li et al. [15]). *Let R be an induced subgraph of P_4, $P_1 \cup P_3$ or $P_2 \cup 2P_1$. Then every R-free 1-tough graph on at least three vertices is Hamiltonian.*

The following lemma is a consequence of Menger's theorem, which can be found in [5]. For a positive integer k, define $[1, k] = \{1, 2, \cdots, k\}$.

Lemma 4. *Let G be a k-connected graph and X_1, X_2 be distinct subsets of $V(G)$. Then there exist k internally disjoint paths $P_1, \ldots, P_k$ such that*

(a) $|V(P_i) \cap X_1| = |V(P_i) \cap X_2| = 1$, and P_i is internally disjoint from each X_1 and X_2.

(b) if $|X_i| \geq k$ for some $i \in [1, 2]$, then $V(P_j) \cap X_i \neq V(P_\ell) \cap X_i$ for all distinct $j, \ell \in [1, k]$.

(c) if $|X_i| < k$ for some $i \in [1, 2]$, then every vertex of X_i is an end-vertex of some path P_j for $j \in [1, k]$.

The following lemma provides some structural properties of $(P_2 \cup 3P_1)$-free graphs.

Lemma 5. *Let G be a connected $(P_2 \cup 3P_1)$-free graph, and $S \subseteq V(G)$ be a cut set such that $G - S$ has at least three components. Then we have the following statements:*

(a) If $G - S$ has a nontrivial component, then $G - S$ has exactly three components.

(b) If $G - S$ has a nontrivial component, then the component is $(P_2 \cup P_1)$-free.

Proof. For part (a), let D denote a nontrivial component of $G - S$. Assume for the sake of contradiction that $G - S$ has more than three components. Taking an edge from D and a single vertex from three other components, respectively, gives an induced copy of $P_2 \cup 3P_1$. This gives a contradiction to the $(P_2 \cup 3P_1)$-freeness of G.

For part (b), $G - S$ must have exactly three components by part (a). Assume for the sake of contradiction that the nontrivial component is not $(P_2 \cup P_1)$-free. Then taking an induced copy of $P_2 \cup P_1$ from this component and one vertex each from the other two components gives an induced copy of $P_2 \cup 3P_1$, which contradicts the $(P_2 \cup 3P_1)$-freeness of G. □

Note that in any $(P_2 \cup 3P_1)$-free graph G, the components yielded by any cut set S such that $c(G - S) \geq 3$ must be $(P_2 \cup P_1)$-free. The following lemmas deal with the structure of $(P_2 \cup P_1)$-free graphs. Lemma 6 is used in the proof of Lemma 7.

Lemma 6. *If G is a $(P_2 \cup P_1)$-free graph and S is a cut set of G, then every component of $G - S$ is trivial.*

Proof. Assume there exists some nontrivial component of $G - S$. Since S is a cut set, it must disconnect G into at least two components. Taking an edge from a nontrivial component and a vertex from another component gives an induced copy of $P_2 \cup P_1$, contradicting the $(P_2 \cup P_1)$-freeness of G. □

The independence number of a graph G, denoted $\alpha(G)$, is the size of a largest independent set of G.

Lemma 7. *Let $t > 0$ be real and G be a $(P_2 \cup P_1)$-free graph on n vertices with $\alpha(G) \leq \frac{n}{t+1}$. Then $\delta(G) \geq n - \frac{n}{t+1}$.*

Proof. Assume $\delta(G) < n - \frac{n}{t+1}$. Let $v \in V(G)$ with $d_G(v) = \delta(G)$, and let $W = V(G) \setminus N_G(v)$. Then $|W| = |V(G)| - \delta(G) > \frac{n}{t+1}$. As G is $(P_2 \cup P_1)$-free, and $N_G(v)$ is a cut set of G, every component of $G - N_G(v)$ is trivial by Lemma 6. Since $W = G - N_G(v)$, W is an independent set of G. However, $|W| > \frac{n}{t+1} \geq \alpha(G)$, giving a contradiction. $\square$

II.2 Proof of Theorem 1

Let $\overrightarrow{C}$ be a cycle with some fixed orientation. For any $u, v \in V(C)$, we denote by $u\overrightarrow{C}v$ the path from u to v following the orientation of C. Similarly, we denote by $u\overleftarrow{C}v$ the inverse path from u to v. The immediate successor of u on $\overrightarrow{C}$ is denoted by u^+.

Theorem 1. *If G is a 3-tough $(P_2 \cup 2P_1)$-free graph on at least 3 vertices, then G is Hamiltonian.*

Proof. Let G be a 3-tough $(P_2 \cup 3P_1)$-free graph. We may assume that G is not complete, otherwise there exists a Hamiltonian cycle. Therefore, G is 6-connected. By Theorem 2, we may assume $\delta(G) \leq \frac{n}{4} - 1$. Since $\delta(G) \geq 6$, we get $n \geq 28$. Let C be a longest cycle of G.

Claim 1. $|V(C)| \geq \frac{3n}{4}$.

Proof. We assume first that there exist $u, v \in V(G)$ with $u \not\sim v$ such that $|N(u) \cup N(v)| \leq \frac{n}{4}$. Let $S = N(u) \cup N(v)$.

Note that the components of $G - S$ cannot all be trivial, as this would imply $\frac{|S|}{c(G-S)} \leq \frac{\frac{n}{4}}{\frac{3n}{4}} < 3$, but G is 3-tough. Thus, there must exist a nontrivial component D in $G - S$. Since each of u, v are components of $G - S$ and $G - S$ has a nontrivial component, it follows that $c(G - S) \geq 3$. Thus $G - S$ has exactly 3 components by Lemma 5. Since G is 3-tough, it follows that $\alpha(G) \leq \frac{n}{4}$.

Also, we know that $\delta(D) \geq |V(D)| - \frac{n}{4} \geq \frac{|V(D)|+1}{2}$, as $n \geq 12$ and $|V(D)| \geq \frac{3n}{4} - 2$ by Lemma 7. Thus D is Hamiltonian-connected by Lemma 1. Since G is 2-connected, by Lemma 4 we can find in G two disjoint paths P_1 from u to some $x_1 \in V(D)$, and P_2 from u to some $x_2 \in V(D)$ where $x_1 \neq x_2$, and each P_i is internally disjoint from D. As D is Hamiltonian-connected, we can find a Hamiltonian path Q in D from x_1 to x_2. Define

$C' = uP_1x_1Qx_2P_2u$. Then $|V(C')| \geq |V(D)| + 3 \geq \frac{3n}{4} - 2 + 3 = \frac{3n}{4} + 1 > \frac{3n}{4}$. As C is a longest cycle of G, $|V(C)| \geq |V(C')| \geq \frac{3n}{4}$.

We then assume that for any $u, v \in V(G)$ with $u \not\sim v$, it holds that $|N(u) \cup N(v)| > \frac{n}{4}$. Note that $\delta(G) \leq \frac{n}{4} - 1$ by our earlier assumption. Let $u \in V(G)$ with $d_G(u) = \delta(G)$. Define $G_1 = G - (N_G(u) \cup \{u\})$. We know that G_1 is $(P_2 \cup 2P_1)$-free, as there are no edges between G_1 and u and the original graph is $(P_2 \cup 3P_1)$-free. If G_1 is 1-tough, then it has a Hamiltonian cycle by Lemma 3 which has at least $\frac{3n}{4}$ vertices, thus $|V(C)| \geq \frac{3n}{4}$. Thus we may assume G_1 is not 1-tough. Let W be a tough set of G_1, i.e. W is a cut set of G_1 such that $\frac{|W|}{c(G_1 - W)} = \tau(G_1)$.

We claim below that $c(G_1 - W) = 2$. We first note that $G_1 - W$ has at least one nontrivial component. Otherwise, all components of $G_1 - W$ are trivial, i.e. $c(G_1 - W) = |V(G_1)| - |W|$. Since $\tau(G_1) < 1$, we have $|W| < c(G_1 - W)$. Then $c(G_1 - W) > \frac{1}{2}|V(G_1)| \geq \frac{1}{2}(n - \frac{n}{4}) = \frac{3n}{8}$. Let $S = N(u) \cup \{u\} \cup W$. Then $c(G - S) = c(G_1 - W)$ and $\frac{|S|}{c(G-S)} = \frac{|S|}{c(G_1-W)} \leq \frac{5n/8}{3n/8} < 3$, which is a contradiction as G is 3-tough. Thus $G_1 - W$ has a nontrivial component.

Next, assume that $G_1 - W$ has more than 2 components. Then by the argument above, at least one of the components is nontrivial. Taking an edge from a nontrivial component and a vertex from each of the two others gives a copy of $P_2 \cup 2P_1$, which contradicts the $(P_2 \cup 2P_1)$-freeness of G_1. Thus, we must have $c(G_1 - W) = 2$.

Let D_1, D_2 be the two components of $G_1 - W$. Then as G_1 is not 1-tough, we get $|W| = 1$. Note that $|V(D_i)| \geq 2$. Otherwise, say $|V(D_2)| = 1$ and let $V(D_2) = \{v\}$, then $|N(u) \cup N(v)| \leq \frac{n}{4} - 1 + 1 = \frac{n}{4}$ which contradicts the assumption that $|N(u) \cup N(v)| > \frac{n}{4}$ for any nonadjacent $u, v \in V(G)$. Since both D_1 and D_2 are nontrivial and G_1 is $(P_2 \cup 2P_1)$-free, each D_i is a complete graph.

Then by Lemma 4, we can find in G two disjoint paths P_1 from some $x_1 \in V(D_1)$ to some $x_2 \in V(D_2)$, and P_2 from some $y_1 \in V(D_1)$ to some $y_2 \in V(D_2)$ where $x_1 \neq y_1, x_2 \neq y_2$. As each D_i is complete, we can find a Hamiltonian path Q_i in D_i from x_i

to y_i. Then the cycle

$$C' = x_1 P_1 x_2 Q_2 y_2 P_2 y_1 Q_1 x_1$$

satisfies $|V(C')| \geq |V(D_1)| + |V(D_2)| + 2 \geq \frac{3n}{4}$. As C is a longest cycle, we have $|V(C)| \geq |V(C')| \geq \frac{3n}{4}$. $\square$

Assume that C is not Hamiltonian, as otherwise we are done. Thus $G - V(C)$ has components. Orient C in the clockwise direction and denote the orientation by $\overrightarrow{C}$.

Claim 2. *Let H be any component of $G - V(C)$. Then we have the following statements:*

(a) $|N_C(H)| \geq 2\tau(G) \geq 6$.

(b) for any two $x, y \in N_C(H)$, $xy \notin E(C)$.

(c) for any two $x, y \in N_C(H)$, $x^+y^+ \notin E(G)$.

(d) H is a trivial component.

Proof. Let H be a component of $G - V(C)$, and $x, y \in N_C(H)$. Note that since G is 3-tough, $2\tau(G) \geq 6$.

For part (a), assume $|N_C(H)| < 2\tau(G)$. Then $\frac{|N_C(H)|}{c(G - N_C(H))} < \frac{2\tau(G)}{2} = \tau(G)$, contradicting the toughness of G. Thus we have $|N_C(H)| \geq 2\tau(G) \geq 6$.

For part (b), $|N_C(H)| \geq 6$ by part (a). If there exist distinct $x, y \in N_C(H)$ such that $xy \in E(C)$, then let $h_1 \in N_H(x), h_2 \in N_H(y)$. Assume without loss of generality that $y = x^+$. As H is connected, there exists some (h_1, h_2)-path P in H. Then the cycle $C' = xh_1Ph_2y\overrightarrow{C}x$ is a cycle longer than C, contradicting the maximality of C.

For part (c), assume for the sake of contradiction that $x^+y^+ \in E(G)$. Let $h_1 \in N_H(x), h_2 \in N_H(y)$. Assume without loss of generality that x appears before y on the cycle. Again, as H is connected, there exists some (h_1, h_2)-path P in H. Then the cycle $C' = xh_1Ph_2y\overleftarrow{C}x^+y^+\overrightarrow{C}x$ is a cycle longer than C, contradicting the maximality of C.

For part (d), note that $\{x^+ \mid x \in N_C(H)\}$ is an independent set of G by Claim 2 (c). We assume that H is nontrivial, then taking an edge from H and three vertices from the independent set $\{x^+ \mid x \in N_C(H)\}$ gives an induced copy of $(P_2 \cup 3P_1)$. $\square$

Claim 3. *$c(G - V(C)) \leq 3$.*

Proof. By Claim 2 (d), each component of $G - V(C)$ is trivial. For the sake of contradiction, assume $G - V(C)$ has at least 4 components, and let x, y, z, w be 4 of them. Then the set $S = V(C) \setminus N_C(\{x, y, z, w\})$ is independent. Otherwise, taking an edge from the set S and three of $\{x, y, z, w\}$ gives an induced copy of $P_2 \cup 3P_1$. Then there must exist 3 consecutive vertices $u_1, u_2, u_3 \in N_C(\{x, y, z, w\})$ such that u_1, u_2, u_3 appear in this order along $\overrightarrow{C}$. Otherwise $|S| \geq \frac{n}{4}$, therefore the independent set $S \cup \{x, y, z, w\}$ has size at least $|S| + 4 > \frac{n}{4}$, contradicting $\tau(G) \geq 3$. Note that none of $\{x, y, z, w\}$ can be adjacent to two consecutive u_i, as otherwise we may easily extend the cycle C. Note also that each u_i $(1 \leq i \leq 3)$ is adjacent to at least two of $\{x, y, z, w\}$, as otherwise the set $\{u_i, x, y, z, w\}$ gives an induced copy of $P_2 \cup 3P_1$. Then, we may assume that $N(u_1) \cap \{x, y, z, w\} = N(u_3) \cap \{x, y, z, w\} = \{x, y\}$ and $N(u_2) \cap \{x, y, z, w\} = \{z, w\}$. Then by Claim 2 ($c$), we have $u_2 \not\sim u_3^+$. Then, taking the edge $u_2 z$ and the vertices x, y, u_3^+ gives an induced copy of $P_2 \cup 3P_1$, a contradiction. Therefore $c(G - V(C)) \leq 3$. $\square$

Claim 4. *Let H be any component of $G - V(C)$, and $w, z \in N_C(H)$ be any two distinct vertices. Let $v \in V(C)$ be a vertex such that the edge vv^+ is on $w^+\overrightarrow{C}z^+$. Then $v \not\sim z^+$ or $v^+ \not\sim w^+$.*

Proof. By Claim 2 (d), H is trivial. Let $h \in N_H(w) \cap N_H(z)$. Assume for the sake of contradiction that $v \sim z^+$ and $v^+ \sim w^+$. Then

$$C' = whz\overleftarrow{C}v^+w^+\overrightarrow{C}vz^+\overrightarrow{C}w$$

is a longer cycle than C, contradicting the maximality of C. $\square$

By Claim 2 (d), we let $x \in V(G) \setminus V(C)$ be a component of $G - V(C)$. By Claim 2 (a), $|N_C(x)| \geq 2\tau(G) \geq 6$. We let $x_1, x_2, \cdots, x_6 \in N_C(x)$ be all distinct vertices and assume they appear in the order $x_1, \cdots, x_6$ along $\overrightarrow{C}$. Note that $\{x_1^+, \cdots, x_4^+\}$ is an independent set in G by Claim 2 (c). Assume without loss of generality that $|V(x_4\overrightarrow{C}x_1)| \geq (|V(C)| - 2)/2$. Label the vertices on this segment by $x_4, x_4^+, y_1, y_2, \cdots, y_t, x_1$.

Let $X = \{x, x_1^+, x_2^+, x_3^+, x_4^+\}$. We claim that for each edge vv^+ in $y_1\overrightarrow{C}y_{t-1}$, one of v and v^+ is adjacent to at least three vertices in X, and the other is adjacent to none of X. Suppose not. Considering that G is $(P_2 \cup 3P_1)$-free, we have to consider the case where both v and v^+ are adjacent to at least three vertices of X. Then, since C is a longest cycle, we have $N_X(v) = \{x_2^+, x_3^+, x_4^+\}$ and $N_X(v^+) = \{x, x_1^+, x_2^+\}$. As $v^{++} \neq x_1$, the edge vx_2^+ and the vertices x, x_1^+, v^{++} gives an induced copy of $P_2 \cup 3P_1$, for otherwise we would have an edge contradicting Claim 2 (c) or Claim 4. This proves the claim.

Let Y be the set of even-indexed vertices on the segment $y_1\overrightarrow{C}y_{t-1}$. Then the above claim together with the fact $y_1 \sim x_4^+$ implies that every vertex of Y is not adjacent to any of X.

Note that Y is an independent set of G, as otherwise taking an edge with end-vertices in Y and 3 vertices from the set $\{x_1^+, \cdots, x_4^+\}$ gives an induced copy of $P_2 \cup 3P_1$. Thus, $Y \cup \{x_1^+, \cdots, x_4^+\}$ is also an independent set in G.

By Claim 2 (d) and Claim 3, we have $|V(C)| \geq n - 3$. Thus $|Y| \geq \lfloor \frac{1}{2}(\frac{|V(C)|-2}{2}) - 1 \rfloor \geq \lfloor \frac{n-9}{4} \rfloor$. Therefore $|Y \cup \{x_1^+, \cdots, x_4^+\}| > \frac{n}{4}$, contradicting $\tau(G) \geq 3$.

This completes the proof. $\square$

CHAPTER III: PROOF OF THEOREMS 2, 3, 4, 5, AND 6

III.1 Preliminaries

One of the main proof ingredients of Theorems 2 to 4 is to apply Tutte's 2-factor Theorem. We start with some notation. Let S and T be disjoint subsets of vertices of a graph G, and D be a component of $G-(S\cup T)$. The component D is said to be an *odd component* (resp. *even component*) of $G-(S\cup T)$ if $e_G(D,T)\equiv 1 \pmod 2$ (resp. $e_G(D,T)\equiv 0 \pmod 2$). Let $h(S,T)$ be the number of all odd components of $G-(S\cup T)$. Define

$$\delta(S,T)=2|S|-2|T|+\sum_{y\in T} d_{G-S}(y)-h(S,T).$$

It is easy to see that $\delta(S,T)\equiv 0 \pmod 2$ for every S, $T\subseteq V(G)$ with $S\cap T=\emptyset$. We use the following criterion for the existence of a 2-factor, which is a restricted form of Tutte's f-factor Theorem.

Lemma 8 (Tutte [21])**.** *A graph G has a 2-factor if and only if $\delta(S,T)\geq 0$ for every S, $T\subseteq V(G)$ with $S\cap T=\emptyset$.*

An ordered pair (S,T), consisting of disjoint subsets of vertices S and T in a graph G, is called a *barrier* if $\delta(S,T)\leq -2$. By Lemma 8, if G does not have a 2-factor, then G has a barrier. In [13], a *biased barrier* of G is defined as a barrier (S,T) of G such that among all the barriers of G,

(1) $|S|$ is maximum; and

(2) subject to (1), $|T|$ is minimum.

The following properties of a biased barrier were derived in [13].

Lemma 9. *Let G be a graph without a 2-factor, and let (S,T) be a biased barrier of G. Then each of the following holds.*

(1) The set T is independent in G.

(2) If D is an even component with respect to (S,T), then $e_G(T,D)=0$.

(3) If D is an odd component with respect to (S,T), then for any $y\in T$, $e_G(y,D)\le 1$.

(4) If D is an odd component with respect to (S,T), then for any $x\in V(D)$, $e_G(x,T)\le 1$.

Let G be a graph without a 2-factor and (S,T) be a barrier of G. For an integer $k\ge 0$, let $\mathcal{C}_{2k+1}$ denote the set of odd components D of $G-(S\cup T)$ such that $e_G(D,T)=2k+1$. The following result was proved as a claim in [13] but we include a short proof here for self-completeness.

Lemma 10. *Let G be a graph without a 2-factor, and let (S,T) be a biased barrier of G. Then $|T|\ge |S|+\sum_{k\ge 1}k|\mathcal{C}_{2k+1}|+1$.*

Proof. Let $U=V(G)\setminus S$. Since (S,T) is a barrier,

$$\begin{aligned}\delta(S,T) &= 2|S|-2|T|+\sum_{y\in T}d_{G-S}(y)-h(S,T)\\ &= 2|S|-2|T|+\sum_{y\in T}d_{G-S}(y)-\sum_{k\ge 0}|\mathcal{C}_{2k+1}|\le -2.\end{aligned}$$

By Lemma 9 (1) and (2),

$$\sum_{y\in T}d_{G-S}(y)=\sum_{y\in T}e_G(y,U)=e_G(T,U)=\sum_{k\ge 0}(2k+1)|\mathcal{C}_{2k+1}|.$$

Therefore, we have

$$-2\ge 2|S|-2|T|+\sum_{k\ge 0}(2k+1)|\mathcal{C}_{2k+1}|-\sum_{k\ge 0}|\mathcal{C}_{2k+1}|,$$

which yields $|T|\ge |S|+\sum_{k\ge 1}k|\mathcal{C}_{2k+1}|+1$. $\square$

We use the following lemmas in our proof.

Lemma 11. *Let $t \geq 1$, G be a t-tough graph on at least three vertices containing no 2-factor, and (S,T) be a barrier of G. Then the following statements hold.*

(1) If $\mathcal{C}_1 \neq \emptyset$, then $|S|+1 \geq 2t$. Consequently, $S=\emptyset$ implies $\mathcal{C}_1=\emptyset$, and $|S|=1$ implies $\mathcal{C}_1=\emptyset$ when $t>1$.

(2) $\bigcup_{k\geq 1}\mathcal{C}_{2k+1} \neq \emptyset$.

Proof. Since G is 1-tough and thus is 2-connected, $d_G(y) \geq 2$ for every $y \in T$. This together with Lemma 9 (1)-(3) implies that $|S| + \sum_{k\geq 0}|\mathcal{C}_{2k+1}| \geq 2$.

For the first part of (1), suppose to the contrary that $|S|+1 < 2t$. Let $D \in \mathcal{C}_1$ and $y \in V(T)$ be adjacent in G to some vertex $v \in V(D)$. As $e_G(D,T)=e_G(D,y)=1$, $|S| + \sum_{k\geq 0}|\mathcal{C}_{2k+1}| \geq 2$ and $|T| \geq |S|+1$ by Lemma 10, we have $c(G-(S\cup\{y\})) \geq 2$ regardless of whether or not $S=\emptyset$. But $c(G-(S\cup\{y\})) \geq 2$ implies $\tau(G) < 2t/2 = t$, contradicting G being t-tough. The second part of (1) is a consequence of the first part.

For (2), suppose to the contrary that $\bigcup_{k\geq 1}\mathcal{C}_{2k+1} = \emptyset$. By Lemma 11 (1), $|S|+|\mathcal{C}_1| \geq 2$ implies $|S| \geq 1$. Consequently, $|T| \geq 2$ by Lemma 10. As every component of $G-(S\cup T)$ in $\mathcal{C}_1$ is connected to exactly one vertex of T, S is a cutset of G with $c(G-S) \geq |T|$. However, $|T| \geq |S| + \sum_{k\geq 1} k|\mathcal{C}_{2k+1}| + 1 = |S|+1$, implying $\tau(G)<1$, a contradiction. $\square$

A path P connecting two vertices u and v is called a (u,v)*-path*, and we write uPv or vPu in specifying the two endvertices of P. Let uPv and xQy be two disjoint paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx. Let G be a graph without a 2-factor, and let (S,T) be a barrier of G. For $y \in T$, define

$$h(y) = |\{D : D \text{ is an odd component of } G-(S\cup T), e_G(D,T) \geq 3, e_G(y,D) \geq 1\}|.$$

Lemma 12. *Let G be a graph without a 2-factor, and let (S,T) be a biased barrier of G. Then the following statements hold.*

(1) If $|\bigcup_{k\geq 1}\mathcal{C}_{2k+1}| \geq 1$, then G contains an induced $P_4 \cup aP_1$, where $a=|T|-2$.

(2) If there exists $y_0 \in T$ with $h(y_0) \geq 2$, then for some integer $b \geq 7$, G contains an induced $P_b \cup aP_1$, where $a = |T| - 3$. Furthermore, an induced $P_b \cup aP_1$ can be taken such that the vertices in aP_1 are from T and the path P_b has the form $y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$, where $y_0, y_1, y_2 \in T$ and $x_1^* P_1 x_1$ and $x_2^* P_2 x_2$ are respectively contained in two distinct components from $\bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ such that $e_G(x, T) = 0$ for every internal vertex x from P_1 and P_2.

Proof. Lemma 9 (1), (3) and (4) will be applied frequently in the arguments sometimes without mentioning it.

Let $D \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$. The existence of D implies $|T| \geq 3$ and $|V(D)| \geq 3$ by Lemma 9 (3) and (4). We claim that for a fixed vertex $x_1 \in V(D)$ such that $e_G(x_1, T) = 1$, there exists distinct $x_2 \in V(D)$ and an induced (x_1, x_2)-path P in D with the following two properties: (a) $e_G(x_2, T) = 1$, and (b) $e_G(x, T) = 0$ for every $x \in V(P) \setminus \{x_1, x_2\}$. Note that the vertex x_1 exists by Lemma 9 (4). Let $y_1 \in T$ be the vertex such that $e_G(x_1, T) = e_G(x_1, y_1) = 1$ and $W = N_G(T \setminus \{y_1\}) \cap V(D)$. By Lemma 9 (4), $x_1 \notin W$. Now in D, we find a shortest path P connecting x_1 and some vertex from W, say x_2. Then x_2 and P satisfy properties (a) and (b), respectively. Let $y_2 \in T$ such that $e_G(x_2, T) = e_G(x_2, y_2) = 1$. The vertex y_2 uniquely exists by the choice x_2 and Lemma 9 (4). By Lemma 9 (1) and (4), and the choice of P, $y_1 x_1 P x_2 y_2$ and $T \setminus \{y_1, y_2\}$ together contains an induced $P_4 \cup aP_1$. This proves (1).

We now prove (2). By Lemma 9 (3), the existence of y_0 implies $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| \geq 2$, which in turn gives $|T| \geq 3$ by Lemma 9 (3) again. We let $D_1, D_2 \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ be distinct such that $e_G(y_0, D_1) = 1$ and $e_G(y_0, D_2) = 1$. Let $x_i \in D_i$ such that $e_G(y_0, D_i) = e_G(y_0, x_i) = 1$. By the argument in the first paragraph above, we can find $x_i^* \in V(D_i) \setminus \{x_i\}$ and an (x_i, x_i^*)-path P_i in D_i for each $i \in \{1, 2\}$. By the choice of P_i and Lemma 9 (4), there are unique $y_1, y_2 \in T \setminus \{y_0\}$ such that $x_i^* y_i \in E(G)$. If $y_1 \neq y_2$, by the choice of P_1 and P_2 and Lemma 9 (1) and (4), we know that $y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$ and $T \setminus \{y_0, y_1, y_2\}$ together contain an induced $P_b \cup aP_1$ for some integer $b \geq 7$. Thus we assume $y_1 = y_2$. Then the vertex y_1 can also play the role of y_0. Let $W = N_G(T \setminus \{y_0, y_1\}) \cap V(D_2)$. By Lemma 9 (4), $x_2, x_2^* \notin W$. Now in D_2, we find a shortest path P_2^* connecting some vertex of $\{x_2, x_2^*\}$ and some vertex from W, say z. If P_2^* is an (x_2, z)-path, then $y_1x_1^*P_1x_1y_0x_2P_2^*z$ and $T \setminus \{y_0, y_1, y_2\}$ together contain an induced $P_b \cup aP_1$. If P_2^* is an (x_2^*, z)-path, then $y_0x_1P_1x_1^*y_1x_2^*P_2^*z$ and $T \setminus \{y_0, y_1, y_2\}$ together contain an induced $P_b \cup aP_1$. The second part of (2) is clear by the construction above. $\square$

Let G be a non-complete graph. A cutset S of $V(G)$ is a *tough-set* of G if

$$\frac{|S|}{c(G-S)} = \tau(G).$$

Lemma 13. *If G is a connected graph and S is a tough-set of G, then for every $x \in S$, x is adjacent in G to vertices from at least two components of $G - S$.*

Proof. Assume to the contrary that there exists $x \in S$ such that x is adjacent in G to vertices from at most one component of $G - S$. Then $c(G - (S \setminus \{x\})) = c(G - S) \geq 2$ and

$$\frac{|S \setminus \{x\}|}{c(G-(S\setminus\{x\}))} < \frac{|S|}{c(G-S)} = \tau(G),$$

giving a contradiction. $\square$

III.2 Proof of Theorems 2, 3, and 4

Let R be any linear forest on at most 7 vertices. If G is R-free, then G is also R^*-free for any supergraph R^* of R. To prove Theorems 2 to 4, we will show that the corresponding statements hold for a supergraph R^* of R, which simplifies the cases of possibilities of R. Let us first list the supergraphs that we will use.

(1) $P_4 \cup 3P_1$ is a supergraph of the following graphs: $P_4 \cup 2P_1, P_3 \cup 3P_1$, and $P_2 \cup 4P_1$;

(2) $6P_1$ is a supergraph of $5P_1$;

(3) $P_3 \cup 2P_2$ is a supergraph of $3P_2$;

(4) $P_7 \cup 2P_1$ is a supergraph of the following graphs:

(a) $P_5, P_3 \cup P_2, 2P_2 \cup P_1$;

(b) $P_6, P_5 \cup P_1, P_4 \cup P_2, 2P_3, P_3 \cup P_2 \cup P_1, 2P_2 \cup 2P_1$;

(c) $P_7, P_6 \cup P_1, P_5 \cup 2P_1, P_4 \cup P_2 \cup P_1, 2P_3 \cup P_1, P_3 \cup P_2 \cup 2P_1, 2P_2 \cup 3P_1$.

Those supergraphs above together with the graphs R listed below cover all the 33 R graphs described in Theorems 2 to 4. Theorems 2 to 4 are then consequences of the theorem below.

Theorem 7. *Let $t > 0$ be a real number, R be a linear forest, and G be a t-tough R-free graph on at least 3 vertices. Then G has a 2-factor provided that*

(1) $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ and $t = 1$ unless

(a) $R = P_2 \cup 3P_1$, and $G \cong H_0$ or G contains H_1, H_2, H_3 or H_4 as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ for each $i \in [1, 3]$, where H_i, S and T are defined in Figure 1.

(b) $R = P_3 \cup 2P_1$ and G contains H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$.

(2) $R \in \{P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1\}$ and $t > 1$ unless

(a) when $R \neq P_4 \cup 3P_1$, G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$, where H_5, S and T are defined in Figure 2.

(b) $R = P_2 \cup 5P_1$ and G contains one of $H_6, \ldots, H_{11}$ as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S]) \cup E(G[V(G) \setminus (T \cup S)])$, where H_i, S and T are defined in Figure 3 for each $i \in [6, 11]$.

(3) $R = 7P_1$ and $t > \frac{7}{6}$ unless G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$.

(4) $R \in \{P_7 \cup 2P_1, P_5 \cup P_2, P_4 \cup P_3, P_3 \cup 2P_2, 3P_2 \cup P_1\}$ *and* $t = 3/2$.

Proof. Assume by contradiction that G does not have a 2-factor. By Lemma 8, G has a barrier. We choose (S,T) to be a biased barrier. Thus (S,T) and G satisfy all the properties listed in Lemma 9. These properties will be used frequently even without further mentioning sometimes. By Lemma 10,

$$|T| \geq |S| + \sum_{k\geq 1} k|\mathcal{C}_{2k+1}| + 1. \tag{III.1}$$

Since $t \geq 1$, by Lemma 11(2), we know that

$$\bigcup_{k\geq 1} \mathcal{C}_{2k+1} \neq \emptyset. \tag{III.2}$$

This implies $|T| \geq 3$ and so G contains an induced $P_4 \cup P_1$ by Lemma 12 (1). Thus we assume $R \neq P_4 \cup P_1$ in the rest of the proof.

Claim 5. $R \notin \{P_3 \cup 2P_1, P_2 \cup 3P_1\}$ *unless G falls under one of the exceptional cases as in (a) and (b) of Theorem 7(1).*

Proof. Assume instead that $R \in \{P_3 \cup 2P_1, P_2 \cup 3P_1\}$. Thus $t = 1$. We may assume that G does not fall under any of the exceptional cases as in (a) and (b) of Theorem 7 (1). It must be the case that $|T| = 3$, as otherwise G contains an induced $P_4 \cup 2P_1$ by Lemma 12(1), and so contains an induced R. By Equation (III.1), we have $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| + |S| \leq 2$. By Lemma 11(1), we have that $\mathcal{C}_0 = \emptyset$ if $S = \emptyset$. Since G is 1-tough and so $\delta(G) \geq 2$, Lemma 9(1)-(3) implies that $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| + |S| = 2$. By (III.2), we have the two cases below.

Case 1: $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| = 2$ and $S = \emptyset$.

Let $D_1, D_2 \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ be the two odd components of $G - (S \cup T)$. Since $|T| = 3$, Lemma 9(3) implies that $e_G(D_i, T) = 3$ for each $i \in [1, 2]$. Let $y \in T$ and $x \in V(D_1)$ such that $xy \in E(G)$. We let x_1 be a neighbor of x from D_1. Then yxx_1 is an induced P_3 by

Lemma 9(3). Let $y_1 \in T \setminus \{y\}$ such that $y_1x_1 \notin E(G)$, which is possible as $|T| = 3$ and $e_G(x_1, T) \leq 1$ by Lemma 9(4). We now let $x_2 \in V(D_2)$ such that $e_G(x_2, \{y, y_1\}) = 0$, which is again possible as $|N_G(T) \cap V(D_2)| = 3$ and each vertex of D_2 is adjacent in G to at most one vertex of T. However, yxx_1, y_1 and x_2 together form an induced copy of $P_3 \cup 2P_1$. Therefore, we assume $R = P_2 \cup 3P_1$.

We first claim that $|V(D_i)| = 3$ for each $i \in [1, 2]$. Otherwise, say $|V(D_2)| \geq 4$. Let $y \in T$ and $x \in V(D_1)$ such that $xy \in E(G)$. Take $x_1 \in V(D_2)$ such that $e_G(x_1, T) = 0$, which exists as $|N_G(T) \cap V(D_2)| = 3$. Then xy, x_1 and $T \setminus \{y\}$ together form an induced copy of $P_2 \cup 3P_1$, giving a contradiction. We next claim that $D_i = K_3$ for each $i \in [1, 2]$. Otherwise, say $D_1 \neq K_3$. As D_1 is connected, it follows that $D_1 = P_3$. If also $D_2 \neq K_3$ and so $D_2 = P_3$, then deleting the two vertices of degree 2 from both D_1 and D_2 gives three components (note that each vertex of T is adjacent in G to one vertex of D_1 and one vertex of D_2), showing that $\tau(G) \leq 2/3 < 1$. Thus $D_2 = K_3$. We let $x_1, x_2 \in V(D_1)$ be nonadjacent, $y_1, y_2 \in T$ such that $e_G(x_i, y_i) = 1$ for each $i \in [1, 2]$, and $z_1, z_2 \in V(D_2)$ such that $e_G(y_i, z_i) = 1$ for each $i \in [1, 2]$. Let $y \in T \setminus \{y_1, y_2\}$. Then z_1z_2, y, x_1 and x_2 together form an induced copy of $P_2 \cup 3P_1$, giving a contradiction.

Thus $|V(D_i)| = 3$ and $D_i = K_3$ for each $i \in [1, 2]$. However, this implies that $G \cong H_0$.

Case 2: $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| = 1$ and $|S| = 1$.

Let $D \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ be the odd component of $G - (S \cup T)$. Assume first that $R = P_3 \cup 2P_1$. Then we have $|V(D)| = 3$. Otherwise, $|V(D)| \geq 4$. Let $x \in V(D)$ such that $e_G(x, T) = 0$ and P be a shortest path of D from x to a vertex, say $x_1 \in V(D) \cap N_G(T)$. Let $y \in T$ such that $e_G(x_1, y) = 1$. Then xPx_1y and $T \setminus \{y\}$ form an induced copy of R, a contradiction. Since G does not contain H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$, it follows that $D \neq K_3$. As D is connected, it follows that $D = P_3$. Now deleting the vertex in S together with the degree 2 vertex of D produces three components, showing that $\tau(G) \leq 2/3 < 1$.

Therefore, we assume now that $R = P_2 \cup 3P_1$. Since G does not contain H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$, the argument for the case $R = P_3 \cup 2P_1$ above implies that $|V(D)| \geq 4$. We claim that $|V(D)| = 4$. If $|V(D)| \geq 5$, we let $x_1, x_2 \in V(D) \setminus N_G(T)$ be any two distinct vertices. If $x_1x_2 \in E(G)$, then x_1x_2 together with T form an induced copy of R, a contradiction. Thus $V(D) \setminus N_G(T)$ is an independent set in G. However, $c(G - (S \cup (N_G(T) \cap V(D)))) = |T| + |V(D) \setminus N_G(T)| \geq 5$, implying that $\tau(G) \leq 4/5 < 1$.

Thus $|V(D)| = 4$. Let $x \in V(D)$ such that $e_G(x, T) = 0$. Since G does not contain H_i as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ for each $i \in [2, 4]$, it follows that either $d_D(x) \leq 2$ or $d_D(x) = 3$ and $D = K_{1,3}$. If $d_D(x) = 3$, then as $D = K_{1,3}$, we have $c(G - (S \cup \{x\})) = 3$, implying $\tau(G) \leq 2/3 < 1$. Thus $d_D(x) \leq 2$. Let $V(D) = \{x, x_1, x_2, x_3\}$ and assume $xx_1 \notin E(D)$. Then $c(G - (S \cup \{x_2, x_3\})) = 4$, implying $\tau(G) \leq 3/4 < 1$. The proof of Case 2 is complete. $\square$

Thus by Claim 5 and the fact that $R \neq P_4 \cup P_1$, we can assume $R \notin \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ from this point on. Therefore we have $t > 1$. This implies that G is 3-connected and so $\delta(G) \geq 3$. Thus $|S| + |\bigcup_{k\geq 0} \mathcal{C}_{2k+1}| \geq 3$ by Lemma 9(1)-(4).

Claim 6. $|T| \geq 5$.

Proof. Equation (III.2) implies $|T| \geq 3$. Assume to the contrary that $|T| \leq 4$. We consider the following two cases.

Case 1: $|T| = 3$.

Since $|S| + |\bigcup_{k\geq 0} \mathcal{C}_{2k+1}| \geq 3$, we already have a contradiction to Equation (III.1) if $\mathcal{C}_1 = \emptyset$. Thus $\mathcal{C}_1 \neq \emptyset$, which gives $|S| \geq 2$ by Lemma 11(1). However, we again get a contradiction to Equation (III.1) as $\bigcup_{k\geq 1} \mathcal{C}_{2k+1} \neq \emptyset$ by Equation (III.2).

Case 2: $|T| = 4$.

By Lemma 9 (3), we know that $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$. First assume $|S| \leq 1$. Then $\mathcal{C}_1 = \emptyset$ by Lemma 11 (1). By Lemma 9, there are at least $3|T| = 12$ edges going from T to vertices in S and components in $\bigcup_{k\geq 1} \mathcal{C}_{2k+1}$. As $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$, it follows that

$|\mathcal{C}_3| \geq 4$ if $|S| = 0$ and $|\mathcal{C}_3| \geq 3$ if $|S| = 1$, contradicting Equation (III.1).

Next, assume $|S| \geq 2$. By Equations (III.1) and (III.2), we have $|S| = 2$. Let D be the single component in $\mathcal{C}_3$. Define W_D to be a set of 2 vertices in D which are all adjacent in G to some vertex from T. Then $S \cup W_D$ is a cutset in G such that $|S \cup W_D| = 4$ and $c(G - (S \cup W_D)) \geq |T| = 4$, contradicting $\tau(G) \geq t > 1$. □

By Claim 6 and Lemma 12 (1), we see that G contains an induced $R = P_4 \cup 3P_1$. Thus we may assume $R \notin \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1, P_4 \cup 3P_1\}$ from this point on.

Claim 7. *$R \notin \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$ unless G falls under the exceptional cases as in (a) and (b) of Theorem 7(2).*

Proof. We may assume that G does not fall under the exceptional cases as in (a) and (b) of Theorem 7(2). Thus we show that $R \notin \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$.

Assume to the contrary that $R \in \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$. By Lemma 12(1), G contains an induced $P_4 \cup aP_1$, where $a = |T| - 2$. If $a \geq 5$, then each of $P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1$, and $7P_1$ is an induced subgraph of $P_4 \cup aP_1$, a contradiction. Thus $a \leq 4$ and so $|T| \leq 6$. As $|T| \leq 6$, we have that $\bigcup_{k>2} \mathcal{C}_{2k+1} = \emptyset$ by Lemma 9 (3). Since G is more than 1-tough and so is 3-connected, we have $\delta(G) \geq 3$. By Claim 6, $|T| \geq 5$. Thus, we have two cases.

Case 1: $|T| = 5$.

As $|T| = 5$, we have $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 3$. We consider two cases regarding whether or not $|\mathcal{C}_3 \cup \mathcal{C}_5| \geq 2$.

Case 1.1: $|\mathcal{C}_3 \cup \mathcal{C}_5| = 1$.

Let $D \in \mathcal{C}_{2k+1} \subseteq \mathcal{C}_3 \cup \mathcal{C}_5$. By Equation (III.1), $5 \geq |S| + k + 1$, so $|S| \leq 4 - k$. If $k = 1$, let W_D be a set of $2k$ vertices (which exist by Lemma 9 (4)) from D which are adjacent in G to vertices from T. Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 2k}{5} \leq \frac{4 + k}{5} = \frac{5}{5} = 1,$$

contradicting $t > 1$. Thus we assume $k = 2$. We consider two subcases.

Case 1.1.1: $|V(D)| \geq 6$.

For $R = P_3 \cup 4P_1$, let $x \in V(D)$ such that $e_G(x, T) = 0$. Let P be a shortest path in D from x to a vertex, say x^* from $N_G(T) \cap V(D)$. Let $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then xPx^*y^* and $T \setminus \{y^*\}$ contain $P_3 \cup 4P_1$ as an induced subgraph. We consider next that $R = 6P_1$. Then T and the vertex of D that is not adjacent in G to any vertex from T for an induced $6P_1$, giving a contradiction. For $R = 7P_1$, let W_D be the set of $2k + 1$ vertices (which exist by Lemma 9(4)) from D which are adjacent in G to vertices from T. Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 2k + 1}{|T| + 1} \leq \frac{4 + k + 1}{6} = \frac{7}{6},$$

giving a contradiction to $t > 7/6$.

Lastly, we consider $R = P_2 \cup 5P_1$. For any $x \in V(D)$ such that $e_G(x, T) = 0$, it must be the case that x is adjacent in G to every vertex from $N_G(T) \cap V(D)$. Otherwise, let $x^* \in N_G(T) \cap V(D)$ such that $xx^* \notin E(G)$. Let $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then x^*y^* and $(T \setminus \{y^*\}) \cup \{x\}$ contain $P_2 \cup 5P_1$ as an induced subgraph. Furthermore, if $|V(D)| - |N_G(T) \cap V(D)| \geq 2$, then $V(D) \setminus (N_G(T) \cap V(D))$ is an independent set in G. Otherwise, an edge with both endvertices from $V(D) \setminus (N_G(T) \cap V(D))$ together with T induces $P_2 \cup 5P_1$. Thus if $|V(D)| \geq 7$, let W_D be the set of $2k + 1$ vertices (which exist by Lemma 9(4)) from D which are adjacent in G to vertices from T. Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 5}{|T| + 2} \leq \frac{7}{7},$$

giving a contradiction to $t > 1$. Thus $|V(D)| = 6$. Let $x \in V(D)$ be the vertex such that $e_G(x, T) = 0$. Then it must be the case that $D - x$ has at most two components. Otherwise, we have $t \leq \frac{|S \cup \{x\}|}{3} = 1$.

Assume first that $c(D-x)=2$. Let D_1 and D_2 be the two components of $D-x$, and assume further that $|V(D_1)| \leq |V(D_2)|$. Then as $|V(D-x)|=5$, we have two possibilities: either $|V(D_1)|=1$ and $|V(D_2)|=4$ or $|V(D_1)|=2$ and $|V(D_2)|=3$. Since $\delta(G) \geq 3$, if $|V(D_1)|=1$, then the vertex from D_1 must be adjacent in G to at least one vertex from S. When $|V(D_2)|=4$ and $D_2 \neq K_4$, then D_2 has a cutset W of size 2 such that $c(D_2-W)=2$. Then $S \cup W \cup \{x\}$ is a cutset of G such that $c(G-(S \cup W \cup \{x\}))=5$, showing that $t \leq 1$. Thus $D_2=K_4$. However, this shows that G contains H_6 as a spanning subgraph. When $|V(D_2)|=3$ and $D_2 \neq K_3$, then D_2 has a cutvertex x^*. Then $S \cup \{x, x^*\}$ is a cutset of G such that $c(G-(S \cup \{x, x^*\}))=4$, showing that $t \leq \frac{4}{4}=1$. Thus $D_2=K_3$; however, this shows that G contains H_7 as a spanning subgraph.

Assume then that $c(D-x)=1$. Let $D^*=D-x$. If $\delta(D^*) \geq 3$, then D^* is Hamiltonian and so G contains H_{10} as a spanning subgraph. Thus we assume $\delta(D^*) \leq 2$.

Assume first that D^* has a cutvertex x^*. Then $c(D^*-x)=2$: as if $c(D^*-x) \geq 3$, then $c(G-(S \cup \{x, x^*\})) \geq 4$, implying $t \leq 1$. Let D_1^* and D_2^* be the two components of D^*-x^*, and assume further that $|V(D_1^*)| \leq |V(D_2^*)|$. Then as $|V(D^*-x^*)|=4$, we have two possibilities: either $|V(D_1^*)|=1$ and $|V(D_2^*)|=3$ or $|V(D_1^*)|=2$ and $|V(D_2^*)|=2$. Since $\delta(G) \geq 3$, if $|V(D_1^*)|=1$, then the vertex from D_1^* must be adjacent in G to at least one vertex from S. When $|V(D_2^*)|=3$ and $D_2^* \neq K_3$, then D_2^* has a cutvertex x^{**}. Then $S \cup \{x, x^*, x^{**}\}$ is a cutset of G such that $c(G-(S \cup \{x, x^*, x^{**}\}))=5$, showing that $t \leq 1$. Thus $D_2^*=K_3$. The vertex x^* is a cutvertex of D^* and so is adjacent in D^* to a vertex of D_1^* and a vertex of D_2^*. However, this shows that G contains H_8 as a spanning subgraph. When $|V(D_2^*)|=2$, as G does not contain H_8 or H_9 as a spanning subgraph, x^* is adjacent in G to exactly one vertex, say x_1^*, of D_1^* and to exactly one vertex, say x_2^*, of D_2^*. Then $S \cup \{x, x_1^*, x_2^*\}$ is a cutset of G whose removal produces 5 components, showing that $\tau(G) \leq 1$.

Assume then that D^* is 2-connected. As $\delta(D^*) \leq 2$, D^* has a minimum cutset W of size 2. If $c(D^*-W)=3$, then we have $c(G-(S \cup W \cup \{x\}))=5$, showing that $t \leq 1$. Thus

$c(D^* - W) = 2$. Then by analyzing the connection in D^* between W and the two components of $D^* - W$, we see that D^* contains C_5 as a spanning subgraph, showing that G contains H_{10} as a spanning subgraph.

Case 1.1.2: $|V(D)| = 5$.

Since G does not contain H_5 as a spanning subgraph, we have $D \neq K_5$. As $D \neq K_5$, D has a cutset W_D of size at most 3 such that each component of $D - W_D$ is a single vertex. Then

$$t \leq \frac{|S| + |W_D|}{|T|} \leq \frac{4 - 2 + 3}{5} = 1,$$

a contradiction.

Case 1.2: $|\mathcal{C}_3 \cup \mathcal{C}_5| \geq 2$.

By Equation (III.1), we have

$$4 \geq |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}|.$$

So one of the following holds:

1. $S = \emptyset$ and either $|\mathcal{C}_5| \leq 2$, $|\mathcal{C}_5| \leq 1$ and $|\mathcal{C}_3| \leq 2$, or $|\mathcal{C}_3| \leq 4$. In this case, $\mathcal{C}_1 = \emptyset$ by Lemma 11(1). Thus by Lemma 9(3), we have $e_G(T, V(G) \setminus T) \leq 12 < 3|T| = 15$.

2. $|S| = 1$ and either $|\mathcal{C}_5| = 1$ and $|\mathcal{C}_3| = 1$ or $|\mathcal{C}_3| \leq 3$. In this case, again $\mathcal{C}_1 = \emptyset$ by Lemma 11(1). This implies there are a maximum of 14 edges incident to vertices in T, a contradiction.

3. $|S| = 2$ and $|\mathcal{C}_3| = 2$.

   Let $\mathcal{C}_3 = \{D_1, D_2\}$. Note that $|V(D_i)| \geq 3$ by Lemma 9(4) for each $i \in [1, 2]$. Since $|T| = 5$, there exists $y_0 \in T$ such that $e_G(y_0, D_i) = 1$ for each $i \in [1, 2]$. If $R = P_3 \cup 4P_1$, then T together with the two neighbors of y_0 from $V(D_1) \cup V(D_2)$ induce R. If $R = 6P_1$, then $T \setminus \{y_0\}$ together with the two neighbors of y_0 from

$V(D_1) \cup V(D_2)$ gives an induced $6P_1$. If $R = 7P_1$, let $W_{D_i} \subseteq V(D_i) \setminus N_G(y_0)$ be the two vertices of D_i that are adjacent in G to vertices from T. Then $c(G - (S \cup W_{D_1} \cup W_{D_2} \cup \{y_0\})) = |T| - 1 + 2 = 6$. Thus $t \leq \frac{2+2+2+1}{6} = \frac{7}{6}$, contradicting $t > \frac{7}{6}$. Lastly, assume $R = P_2 \cup 5P_1$. If one of D_i has at least 4 vertices, say $|V(D_2)| \geq 4$, then let $x \in V(D_2)$ such that $e_G(x, T) = 0$, $x^* \in V(D_1)$ and $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then x^*y^* and $(T \setminus \{y^*\}) \cup \{x\}$ induce $P_2 \cup 5P_1$. Thus $|V(D_1)| = |V(D_2)| = 3$. If one of D_i, say $D_2 \neq K_3$, then D_2 has a cutvertex x. Let W be the set of any two vertices of D_1. Then $S \cup W \cup \{x\}$ is a cutset of G such that $c(G - (S \cup W \cup \{x\})) = 5$, showing that $t \leq \frac{5}{5} = 1$. Thus $D_1 = D_2 = K_3$. However, this shows that G contains H_{11} as a spanning subgraph.

Case 2: $|T| = 6$.

In this case, by Lemma 12(1), G has an induced $P_4 \cup 4P_1$, which contains each of $P_3 \cup 4P_1, P_2 \cup 5P_1$ and $6P_1$ as an induced subgraph. So we assume $R = 7P_1$ in this case and thus $t > \frac{7}{6}$.

Recall for $y \in T$, $h(y) = |\{D : \ D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1} \quad \text{and} \quad e_G(y, D) \geq 1\}|$. If there exists $y_0 \in T$ such that $h(y_0) \geq 2$, we let x_1, x_2 be the two neighbors of y_0 from the two corresponding components in $\bigcup_{k \geq 1} \mathcal{C}_{2k+1}$, respectively. Then $T \setminus \{y_0\}$ together with $\{x_1, x_2\}$ induces $7P_1$. Thus $h(y) \leq 1$ for each $y \in T$. This, together with $|T| = 6$, implies that we have either $|\mathcal{C}_3| \in \{1, 2\}$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$ or $|\mathcal{C}_5| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $1 \leq k \neq 2$.

If $|\mathcal{C}_3| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$, then $|S| \leq 4$ by Equation (III.1). Let W be a set of two vertices from the component in $\mathcal{C}_3$ that are adjacent in G to vertices from T. Then $c(G - (S \cup W)) \geq 6$, indicating that $t \leq \frac{4+2}{6} < \frac{7}{6}$. For the other two cases, we have $|S| \leq 3$. If $|\mathcal{C}_3| = 2$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$, let W be a set of four vertices, with two from one component in $\mathcal{C}_3$ and the other two from the other component in $\mathcal{C}_3$, which are adjacent in G to vertices from T. If $|\mathcal{C}_5| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $1 \leq k \leq 2$, let W be a set of four vertices from the component in $\mathcal{C}_5$ that are adjacent in G to vertices from T. Then we have

$c(G-(S\cup W))\geq 6$, indicating that $t\leq \frac{3+4}{6}=\frac{7}{6}$. $\square$

By Claim 7, we now assume that $R\in\{P_7\cup 2P_1, P_5\cup P_2, P_4\cup P_3, P_3\cup 2P_2, 3P_2\cup P_1\}$ and $t=3/2$.

Claim 8. *There exists $y\in T$ with $h(y)>2$.*

Proof. Assume to the contrary that for every $y\in T$, we have $h(y)\leq 1$. Define the following partition of T:

$$\begin{aligned}
T_0 &= \{y\in T : e_G(y,D)=0 \text{ for all } D\in \bigcup_{k\geq 1}\mathcal{C}_{2k+1}\},\\
T_1 &= \{y\in T : e_G(y,D)=1 \text{ for some } D\in \bigcup_{k\geq 1}\mathcal{C}_{2k+1}\}.
\end{aligned}$$

Note that $|T_1|=\sum_{k\geq 1}(2k+1)|\mathcal{C}_{2k+1}|$ by Lemma 9(3) and (4). For each $D\in\mathcal{C}_{2k+1}$ for some $k\geq 1$, we let W_D be a set of $2k$ vertices that each has in G a neighbor from T. As each $D-W_D$ is connected to exactly one vertex from T and each component from $\mathcal{C}_1$ is connected to exactly one vertex from T, it follows that

$$W=S\cup \bigcup_{D\in\bigcup_{k\geq 1}\mathcal{C}_{2k+1}} W_D$$

satisfies $c(G-W)\geq |T|\geq 5$, where $|T|\geq 5$ is by Claim 6.
By the toughness of G, we have

$$\begin{aligned}
|S|+\sum_{k\geq 1}2k|\mathcal{C}_{2k+1}| &= |W|\geq t|T| = t(|T_0|+|T_1|)\\
&= t\left(|T_0|+\sum_{k\geq 1}(2k+1)|\mathcal{C}_{2k+1}|\right). \qquad \text{(III.3)}
\end{aligned}$$

Since $t = 3/2$, the inequality above implies that $|S| \geq 3|T_0|/2 + \sum_{k\geq 1}(k+3/2)|\mathcal{C}_{2k+1}|$. Thus

$$|S| + \sum_{k\geq 1} k|\mathcal{C}_{2k+1}| \;\; \geq \;\; 3|T_0|/2 + \sum_{k\geq 1}(2k+3/2)|\mathcal{C}_{2k+1}| > |T_0| + \sum_{k\geq 1}(2k+1)|\mathcal{C}_{2k+1}| = |T|,$$

contradicting Equation (III.1). □

By Claim 8, there exists $y \in T$ such that $h(y) \geq 2$. Then as $|T| \geq 5$, by Lemma 12(2), G contains an induced $P_7 \cup 2P_1$. Thus we assume that $R \neq P_7 \cup 2P_1$. We assume first that $|\bigcup_{k\geq 1} \mathcal{C}_{2k+1}| \geq 3$ and let D_1, D_2, D_3 be three distinct odd components from $\bigcup_{k\geq 1} \mathcal{C}_{2k+1}$. Let $y_0 \in T$ such that $h(y_0) \geq 2$. We assume, without loss of generality, that $e_G(y_0, D_1) = e_G(y_0, D_2) = 1$. By Lemma 12(2), G contains an induced $P_b \cup aP_1$, where $b \geq 7$ and $a = |T| - 3$, and the graph $P_b \cup aP_1$ can be chosen such that the vertices in aP_1 are from T and the path P_b has the form $y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$, where $y_0, y_1, y_2 \in T$ and $x_1^* P_1 x_1$ and $x_2^* P_2 x_2$ are respectively contained in D_1 and D_2 such that $e_G(x, T) = 0$ for every internal vertex x from P_1 and P_2. If one of y_1 and y_2, say y_1 has a neighbor z_1 from $V(D_3)$, then $z_1 y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$ and $T \setminus \{y_0, y_1, y_2\}$ induce $P_8 \cup 2P_1$, which contains each of $P_5 \cup P_2$, $P_4 \cup P_3$, and $3P_2 \cup P_1$ as an induced subgraph. Let $z_2 \in V(D_3)$ be a neighbor of z_1. Then $z_2 z_1 y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$ contains an induced $P_3 \cup 2P_2$ whether $e_G(z_2, \{y_0, y_2\}) = 0$ or 1. Thus we assume $e_G(y_i, D_3) = 0$ for each $i \in [1, 2]$ and so we can find $y_3 \in T \setminus \{y_0, y_1, y_2\}$ and $z \in V(D_3)$ such that $y_3 z \in E(G)$. Then $y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$ and $z y_3$ contains an induced $P_7 \cup P_2$, which contains each of $P_5 \cup P_2$, $P_3 \cup 2P_2$ and $3P_2 \cup P_1$ as an induced subgraph. We are only left to consider $R = P_4 \cup P_3$. As $e_G(y_i, D_3) = 0$ for each $i \in [1, 2]$, we can find distinct $y_3, y_4 \in T \setminus \{y_0, y_1, y_2\}$ and distinct $z_1, z_2 \in V(D_3)$ such that $y_3 z_1, y_4 z_2 \in E(G)$. We let P be a shortest path in D_3 connecting z_1 and z_2. If $e_G(y_0, V(P)) = 0$, then $y_3 z_1 P z_2 y_4$ and $y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$ contains an induced $P_4 \cup P_3$. Thus $e_G(y_0, V(P)) = 1$. This in particular, implies that $|V(P)| \geq 3$. Then $y_3 z_1 P z_2 y_4$ and $y_1 x_1^* P_1 x_1$ together contain an induced $P_4 \cup P_3$.

Thus $|\cup_{k\geq 1} \mathcal{C}_{2k+1}| = 2$. Let $D_1, D_2 \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ be the two components. Define the following partition of T:

$$\begin{aligned} T_0 &= \{y \in T : e_G(y, D_1) = e_G(y, D_2) = 0\}, \\ T_{11} &= \{y \in T : e_G(y, D_1) = 1 \text{ and } e_G(y, D_2) = 0\}, \\ T_{12} &= \{y \in T : e_G(y, D_1) = 0 \text{ and } e_G(y, D_2) = 1\}, \\ T_2 &= \{y \in T : e_G(y, D_1) = e_G(y, D_2) = 1\}. \end{aligned}$$

We have either $T_2 = \emptyset$ or $T_2 \neq \emptyset$. First suppose $T_2 = \emptyset$. Define the following vertex sets:

$$W_1 = N_G(T_{11}) \cap V(D_1) \quad \text{and} \quad W_2 = N_G(T_{12}) \cap V(D_2).$$

Then $|W_1| = |T_{11}| = 2k_1 + 1$ and $|W_2| = |T_{12}| = 2k_2 + 1$, where we assume $e_G(T, D_1) = 2k_1 + 1$ and $e_G(T, D_2) = 2k_2 + 1$ for some integers k_1 and k_2. Then $W = S \cup W_1 \cup W_2$ is a cutset of G with $c(G - W) \geq |T|$. By toughness, $|W| \geq \frac{3}{2}|T| = |T| + \frac{1}{2}|T|$. Since $|T| = |T_0| + |T_{11}| + |T_{12}|$, this gives us

$$\begin{aligned} |W| &\geq |T| + \frac{1}{2}|T_0| + \frac{1}{2}(|T_{11}| + |T_{12}|) \\ &= |T| + \frac{1}{2}|T_0| + \frac{1}{2}(2k_1 + 1 + 2k_2 + 1) \\ &= |T| + \frac{1}{2}|T_0| + k_1 + k_2 + 1. \end{aligned}$$

Thus $|W| = |S| + |W_1| + |W_2| = |S| + 2k_1 + 2k_2 + 2 \geq |T| + \frac{1}{2}|T_0| + k_1 + k_2 + 1$, which implies $|S| + k_1 + k_2 + 1 \geq |T| + \frac{1}{2}|T_0|$. Hence, by Equation (III.1), we have $|T| \geq |T| + \frac{1}{2}|T_0|$, giving a contradiction.

So we may assume $T_2 \neq \emptyset$. Now define the following vertex sets:

$$W_1 = N_G(T_{11}) \cap V(D_1), \quad W_2 = N_G(T_{12}) \cap V(D_2), \quad \text{and} \quad W_3 = N(T_2) \cap (V(D_1) \cup V(D_2)).$$

We have that $|W_1| = |T_{11}|$, $|W_2| = |T_{12}|$, and $|W_3| = 2|T_2|$. Now let $W = S \cup W_1 \cup W_2 \cup W_3$.

Then W is a cutset of G with $c(G-W) \geq |T_0| + |T_{11}| + |T_{12}| + 1$ since $T_2 \neq \emptyset$. By toughness, $|W| \geq \frac{3}{2}(|T_0| + |T_{11}| + |T_{12}| + 1)$. Since $|W| = |S| + |W_1| + |W_2| + |W_3| = |S| + |T_{11}| + |T_{12}| + 2|T_2|$, we have $|S| + |T_{11}| + |T_{12}| + 2|T_2| \geq \frac{3}{2}|T_0| + \frac{3}{2}|T_{11}| + \frac{3}{2}|T_{12}| + \frac{3}{2}$. This implies

$$|S| \geq \frac{3}{2}|T_0| + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}| + 1.$$

Thus,

$$|S| + k_1 + k_2 \geq \frac{3}{2}|T_0| + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}| + 1 + k_1 + k_2. \tag{III.4}$$

We have that either $T_{11} \cup T_{12} \cup T_0 = \emptyset$ or $T_{11} \cup T_{12} \cup T_0 \neq \emptyset$. First suppose $T_{11} \cup T_{12} \cup T_0 = \emptyset$. Then $|T| = |T_2| = \frac{1}{2}(2k_1 + 1 + 2k_2 + 1) = k_1 + k_2 + 1$. Thus $|S| + k_1 + k_2 \geq |T|$, showing a contradiction to Equation (III.1).

So we may assume $T_{11} \cup T_{12} \cup T_0 \neq \emptyset$. Then

$$\begin{aligned}
|T| &= |T_0| + (2k_1 + 1 + 2k_2 + 1 - |T_2|) \\
&= |T_0| + (2k_1 + 2k_2 + 2) - \frac{1}{2}(2k_1 + 1 + 2k_2 + 1 - |T_{11}| - |T_{12}|) \\
&= |T_0| + \frac{1}{2}(2k_1 + 2k_2 + 2) + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}| \\
&= |T_0| + k_1 + k_2 + 1 + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}|.
\end{aligned}$$

Using the size of T and (III.4), we get $|S| + k_1 + k_2 \geq |T|$, showing a contradiction to Equation (III.1).

The proof of Theorem 7 is now finished. □

III.3 Proof of Theorems 5 and 6

Recall that for a graph G, $\alpha(G)$, the independence number of G, is the size of a largest independent set in G. We now show the exceptional graphs in Figures 1 to 3 satisfy the corresponding conditions below.

Theorem 5. *The following statements hold.*

(1) The graph H_i is $(P_2 \cup 3P_1)$-free, contains no 2-factor, and $\tau(H_i) = 1$ for each $i \in [0,4]$, the graph H_1 is also $(P_3 \cup 2P_1)$-free (see Figure 1).

(2) The graph H_i is $(P_2 \cup 5P_1)$-free and contains no 2-factor for each $i \in [5,11]$, H_5 with $p = 5$ is $(P_3 \cup 4P_1)$-free and $6P_1$-free. Furthermore, $\tau(H_5) = \frac{6}{5}$ when $p = 5$ and $\tau(H_i) = \frac{7}{6}$ for each $i \in [6,11]$ (see Figure 3).

Proof. For each $i \in [0,11]$, H_i does not contain a 2-factor by Theorem 8. Thus to finish proving Theorem 7, we are only left to show the three claims below.

Claim 9. *The graph H_i is $(P_2 \cup 3P_1)$-free, H_1 is $(P_3 \cup 2P_1)$-free, and $\tau(H_i) = 1$ for each $i \in [0,4]$.*

Proof. We first show that H_i is $(P_2 \cup 3P_1)$-free for each $i \in [0,4]$. We only show this for H_0, as the proofs for H_i for $i \in [1,4]$ are similar. In H_0, there are two types of edges xy: $x, y \in V(D_j)$ or $x \in V(D_j)$ and $y \in V(T)$, where $j \in [1,2]$. Without loss of generality first consider the edge $v_1v_2 \in E(D_1)$ and the subgraph $F_1 = H_0 - (N_{H_0}[v_1] \cup N_{H_0}[v_2])$. We see $\alpha(F_1) = 2$. Now, without loss of generality, consider the edge v_1t_1 and the subgraph $F_2 = H_0 - (N_{H_0}[v_1] \cup N_{H_0}[t_1])$. We see $\alpha(F_2) = 2$. In either case, $P_2 \cup 3P_1$ cannot exist as an induced subgraph in H_0. Thus H_0 is $(P_2 \cup 3P_1)$-free.

Then we show that H_1 is $(P_3 \cup 2P_1)$-free. Two types of induced paths abc of length 3 exist: $a \in S, b \in T, c \in V(D)$ or $a \in T, b, c \in V(D)$. Without loss of generality, consider the path xt_1v_1 and the subgraph $F_1 = H_1 - (N_{H_1}[x] \cup N_{H_1}[t_1] \cup N_{H_1}[v_1])$. We see that F_1 is a null graph. Now, without loss of generality, consider the path $t_1v_1v_2$ and the subgraph $F_2 = H_1 - (N_{H_1}[t_1] \cup N_{H_1}[v_1] \cup N_{H_1}[v_2])$. We see $|V(F_2)| = 1$. In either case, $P_3 \cup 2P_1$ cannot exist as an induced subgraph in H_1. Thus H_1 is $(P_3 \cup 2P_1)$-free.

Let $i \in [0,4]$. As $\delta(H_i) = 2$, $\tau(H_i) \leq 1$. It suffices to show $\tau(H_i) \geq 1$. Since H_i is 2-connected, we show that $c(H_i - W) \leq |W|$ for any $W \subseteq V(H_i)$ such that $|W| \geq 2$. If

$|W| = 2$, by considering all the possible formations of W, we have $c(H_i - W) \leq |W|$. Thus we assume $|W| \geq 3$.

Assume by contradiction that there exists $W \subseteq V(H_i)$ with $|W| \geq 3$ and $c(H_i - W) \geq |W| + 1 \geq 4$. The size of a largest independent set of each H_0, H_2, H_3, and H_4 is 4, and of H_1 is 3. Since $c(H_i - W)$ is bounded above by the size of a largest independent set of H_i, we already obtain a contradiction if $i = 1$ or $|W| \geq 4$. So we assume $i \in \{0, 2, 3, 4\}$ and $|W| = 3$.

As $c(H_i - W) \geq 4$, for the graph H_0, we must have $\{v_1, v_2, v_3\} \cap W \neq \emptyset$ and $\{v_4, v_5, v_6\} \cap W \neq \emptyset$. As $|W| = 3$, we have either $W \cap T = \emptyset$ or $|W \cap T| = 1$. In either case, by checking all the possible formations of W, we get $c(H_0 - W) \leq 2$, contradicting the choice of W.

As $c(H_i - W) \geq 4$, for each $i \in [2, 4]$, we must have $x \in W$. Thus $t_j \notin W$ for $j \in [1, 3]$, as otherwise, $c(H_i - (W \setminus \{t_j\})) \geq 4$, contradicting the argument previously that $c(H_i - W^*) \leq 2$ for any $W^* \subseteq V(H_i)$ and $|W^*| \leq 2$. As $|W| = 3$, we then have $|W \cap \{v_1, v_2, v_3, v_4\}| = 2$. However, $c(H_i - W) \leq 3$ for $W = \{x, v_k, v_\ell\}$ for all distinct $k, \ell \in [1, 4]$. We again get a contradiction to the choice of W. $\square$

Claim 10. *The graph H_5 with $p = 5$ is $(P_3 \cup 4P_1)$-free, $(P_2 \cup 5P_1)$-free, and $6P_1$-free with $\tau(H_5) = \frac{6}{5}$.*

Proof. Let $p = 5$ and D be the odd component of $H_5 - (S \cup T)$. Note that $D = K_p = K_5$. We first show that H_5 is $(P_3 \cup 4P_1)$-free. There are three types of induced paths xyz of length 3 in $H_5 : x \in S, y \in T, z \in V(D)$ or $x \in T, y, z \in V(D)$ or $x, z \in T, y \in S$. Without loss of generality, consider the path $x_1 t_1 y_1$ and the subgraph $F_1 = H_5 - (N_{H_5}[x_1] \cup N_{H_5}[t_1] \cup N_{H_5}[y_1])$. We see that F_1 is a null graph. Now consider the path $t_1 y_1 y_2$ and the subgraph $F_2 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[y_1] \cup N_{H_5}[y_2])$. We see $\alpha(F_2) = 3$. Finally consider the path $t_1 x_1 t_2$ and the subgraph $F_3 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[x_1] \cup N_{H_5}[t_2])$. We see $\alpha(F_3) = 3$. In any case, an induced copy of $P_3 \cup 4P_1$ cannot exist in H_5. Thus H_5 is $(P_3 \cup 4P_1)$-free.

We then show that H_5 is $(P_2 \cup 5P_1)$-free. There are three types of edges xy in $H_5 : x \in S, y \in T$ or $x \in T, y \in V(D)$ or $x, y \in V(D)$. Without loss of generality, consider the edge x_1t_1 and the subgraph $F_1 = H_5 - (N_{H_5}[x_1] \cup N_{H_5}[t_1])$. We see $|V(F_1)| = 4$. Now consider the edge t_1y_1 and the subgraph $F_2 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[y_1])$. We see $|V(F_2)| = 4$. Finally, consider the edge y_1y_2 and the subgraph $F_3 = H_5 - (N_GH_5[y_1] \cup N_{H_5}[y_2])$. We see $\alpha(F_3) = 3$. In any case, no induced copy of $P_2 \cup 5P_1$ can exist in H_5. Thus H_5 is $(P_2 \cup 5P_1)$-free.

We lastly show that H_5 is $6P_1$-free. There are three types of vertices x in $H_5 : x \in S, x \in T$, or $x \in V(D)$. Without loss of generality, consider the vertex x_1 and the subgraph $F_1 = H_5 - N_{H_5}[x_1]$. We see $\alpha(F_1) = 1$. Now consider the vertex t_1 and the subgraph $F_2 = H_5 - N_{H_5}[t_1]$. We see $\alpha(F_2) = 4$. Finally, consider the vertex y_1 and the subgraph $F_3 = H_5 - N_{H_5}[y_1]$. We see $\alpha(F_3) = 4$. In any case, no induced copy of $6P_1$ can exist in H_5. Thus H_5 is $6P_1$-free.

We now show that $\tau(H_5) = \frac{6}{5}$. Let W be a toughset of H_5. Then $S \subseteq W$. Otherwise, by the structure of H_5, we have $c(H_5 - W) \leq 3$ and $|W| \geq 5$. As $S \subseteq W$ and the only neighbor of each vertex of T in $H_5 - S$ is contained in a clique of H_5, we have $T \cap W = \emptyset$. Since $c(H_5 - W) \geq 2$, it follows that $W \cap V(D) \neq \emptyset$. Then $c(H_5 - W) = |W \cap V(D)|$ if $|W \cap V(D)| \leq 3$ or $|W \cap V(D)| = 5$, and $c(H_5 - W) = |W \cap V(D)| + 1$ if $|W \cap V(D)| = 4$. The smallest ratio of $\frac{|W|}{c(H_5 - W)}$ is $\frac{6}{5}$, which happens when $|W \cap V(D)| = 4$. $\square$

Claim 11. *The graph H_i is $(P_2 \cup 5P_1)$-free with $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 11]$.*

Proof. We show first that each H_i is $(P_2 \cup 5P_1)$-free. We do this only for the graph H_6, as the proofs for the rest graphs are similar. For any edge $ab \in E(H_6)$, we see $\alpha(H_6 - (N_{H_6}[a] \cup N_{H_6}[b])) \leq 4$. Thus no induced copy of $(P_2 \cup 5P_1)$ can exist in H_6. Thus H_6 is $(P_2 \cup 5P_1)$-free.

We next show that $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 10]$. We have $c(H_i - (S \cup \{v_1, \ldots, v_5\})) = 6$, implying $\tau(H_i) \leq \frac{7}{6}$. Suppose $\tau(H_i) < \frac{7}{6}$. Let W be a toughset of H_i. As each H_i is 3-connected, we have $|W| \geq 3$. Thus $c(H_i - W) \geq 3$. We have that either $S \subseteq W$ or

$S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_i - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one which contains $S \setminus W$. Since $c(H_i - W) \geq 3$, by the structure of H_i, it follows that we have either $T \subseteq W$ or $\{v_1, \ldots, v_5\} \subseteq W$. In either case, we have $c(H_i - W) \leq 3$, implying $\frac{|W|}{c(H_i - W)} \geq \frac{5}{3} > \frac{7}{6}$, a contradiction. So $S \subseteq W$. By Lemma 13, $t_j \notin W$ for all $j \in [1,5]$. Thus each $t_j \in V(H_i - W)$. Now either $v_0 \in W$ or $v_0 \notin W$. Suppose $v_0 \in W$, then we cannot have all $v_j \in W$ without violating Lemma 13. In this case, the minimum ratio $\frac{|W|}{c(H_i - W)}$ occurs when $|W \cap \{v_1, v_2, v_3, v_4, v_5\}| = 3$. This implies $\frac{|W|}{c(H_i - W)} \geq \frac{6}{5} > \frac{7}{6}$, a contradiction. Thus $v_0 \notin W$ and we must have $v_0 \in V(H_i - W)$. This implies $\{v_1 \ldots v_5\} \subseteq W$ and $\frac{|W|}{c(H_i - W)} = \frac{7}{6}$, a contradiction. Thus $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 10]$.

Lastly we show $\tau(H_{11}) = \frac{7}{6}$. We have $c(H_{11} - (S \cup \{v_1, v_2, t_3, v_4, v_5\})) = 6$, implying $\tau(H_{11}) \leq \frac{7}{6}$. Suppose $\tau(H_{11}) < \frac{7}{6}$. Let W be a tough set of H_{11}. As H_{11} is 3-connected, we have $|W| \geq 3$. Thus $c(H_{11} - W) \geq 3$. We have that either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_{11} - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one which contains $S \setminus W$. Since $c(H_{11} - W) \geq 3$, by the structure of H_{11}, it follows that $|W| \geq 5$ and $c(H_{11} - W) \leq 4$. This implies $\frac{|W|}{c(H_{11} - W)} \geq \frac{5}{4} > \frac{7}{6}$, a contradiction. So $S \subseteq W$. By Lemma 13, $t_i \notin W$ for $i \in \{1, 2, 4, 5\}$. Thus $t_i \in V(H_{11} - W)$ for $i \in \{1, 2, 4, 5\}$ and we must have $W \cap \{v_1, v_2, v_3, v_4, v_5, v_6, t_3\} \neq \emptyset$. If $t_3 \notin W$, then $\frac{|W|}{c(H_{11-W})} \geq \frac{6}{5} > \frac{7}{6}$, a contradiction. Thus $t_3 \in W$. Then v_3 and v_4 are respectively in two distinct components of $H_{11} - W$ by Lemma 13. Thus $W \cap \{v_1, v_2, v_5, v_6\} \neq \emptyset$ as $c(H_{11} - W) \geq 3$. Furthermore, we have $c(H_{11} - W) = |W \cap \{v_1, v_2, v_5, v_6\}| + 2$. The smallest ratio of $\frac{|W|}{c(H_{11} - W)}$ is $\frac{7}{6}$, which happens when $\{v_1, v_2, v_5, v_6\} \subseteq W$. Again we get a contradiction to the assumption that $\tau(H_{11}) < \frac{7}{6}$. Thus $\tau(H_{11}) = \frac{7}{6}$. □

□

We now verify the toughness of the graphs H_5 with $p \geq 6$ and H_{12} (see Figures 2 and 4).

Theorem 6. *The following statements hold.*

(1) $\tau(H_5) = \frac{7}{6}$ *when* $p \geq 6$*;*

(2) $\tau(H_{12}) = 1$.

Proof. Let $p \geq 6$ and D be the odd component of $H_5 - (S \cup T)$. Note that $D = K_p$. Since $c(H_5 - (S \cup \{y_1, \ldots, y_5\})) = 6$, we have $\tau(H_5) \leq \frac{7}{6}$. We show $\tau(H_5) \geq \frac{7}{6}$. Let W be a toughset of H_5. Then either $S \subseteq W$ or $S \nsubseteq W$. Suppose the latter. Then we have $S \cap V(H_5 - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one containing $S \setminus W$. Since $c(H_5 - W) \geq 2$, by the structure of H_5, it follows that we have either $T \subseteq W$ or $\{y_1, \ldots, y_5\} \subseteq W$. In either case, we have $c(H_5 - W) \leq 3$, implying $\frac{|W|}{c(H_5 - W)} \geq \frac{5}{3} > \frac{7}{6}$. Now suppose $S \subseteq W$. By Lemma 13, $t_i \notin W$ for all i. Thus each $t_i \in V(H_5 - W)$. Furthermore, $c(H_5 - W) = |W \cap V(D)| + 1$. Since W is a cutset of G, we have $|W \cap V(D)| \geq 2$. The smallest ratio of $\frac{|W|}{c(H_5 - W)}$ is $\frac{7}{6}$, which happens when $|W \cap V(D)| = 5$.

For the graph H_{12}, we have $c(H_{12} - (S \cup \{y_1, y_2, y_3\})) = 4$, implying $\tau(H_{12}) \leq \frac{4}{4} = 1$. We show $\tau(H_{12}) \geq 1$. Let W be a toughset of H_{12}. Then either $S \subseteq W$ or $S \nsubseteq W$. Suppose the latter. Then we have $S \cap V(H_{12} - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one containing $S \setminus W$. Since $c(H_{12} - W) \geq 2$, by the structure of H_{12}, it follows that we have either $T \subseteq W$ or $\{y_1, y_2, y_3\} \subseteq W$. In either case, we have $c(H_{12} - W) \leq 2$, implying $\frac{|W|}{c(H_{12} - W)} \geq \frac{3}{2} > 1$. Now suppose $S \subseteq W$. By Lemma 13, $t_i \notin W$ for all i. Thus each $t_i \in V(H_{12} - W)$. This implies $|\{y_1, y_2, y_3\} \cap W| = 2$ or 3. In either case we see $\frac{|W|}{c(H_{12} - W)} = 1$. $\square$

CHAPTER IV: FUTURE WORK

We are interested in the following questions:

Question 1: In this thesis, we have shown that every 3-tough $(P_2 \cup 3P_1)$-free graph on at least three vertices is Hamiltonian. Can we confirm Chvátal's Toughness Conjecture for the following classes of graphs:

- $(P_2 \cup kP_1)$-free graphs for any integer $k > 3$
- $(2P_2 \cup 2P_1)$-free graphs
- $3P_2$-free graphs
- P_5-free graphs

Question 2: Is every 1-tough $(P_4 \cup P_1)$-free graph with at least three vertices Hamiltonian? Does the general conjecture of Chvátal hold for $(P_4 \cup P_1)$-free graphs? [1]

Question 3: In this thesis we found for linear forests R on 5, 6, or 7 vertices, the sharpness bound t such that every t-tough R-free graph on at least three vertices has a 2-factor. Can we find, for linear forests R on 8 or 9 vertices, the sharpness bound t such that every t-tough R-free graph on at least three vertices has a 2-factor?

Question 4: Can we generalize our result for k-factors where $k \geq 2$?

Remark 2. *It was found in [18] that for any $\epsilon > 0$ there exists a $(2-\epsilon)$-tough $2P_5$-free graph without 2-factor. In general, it is well known that every 2-tough graph with at least three vertices has a 2-factor. Since $2P_5$ is a linear forest on 10 vertices, the only interesting cases left to consider are linear forests on 8 or 9 vertices.*

REFERENCES

[1] D. Bauer, H. J. Broersma, and E. Schmeichel. Toughness in graphs—a survey. *Graphs Combin.*, 22(1):1–35, 2006.

[2] D. Bauer, H. J. Broersma, J. van den Heuvel, and H. J. Veldman. Long cycles in graphs with prescribed toughness and minimum degree. *Discrete Math.*, 141(1-3):1–10, 1995.

[3] D. Bauer, H.J. Broersma, and H.J. Veldman. Not every 2-tough graph is hamiltonian. *Discrete Applied Mathematics*, 99(1):317–321, 2000.

[4] D. Bauer, G.Y. Katona, D. Kratsch, and H.J. Veldman. Chordality and 2-factors in tough graphs. *Discrete Applied Mathematics*, 99:323–329, 2000.

[5] J. A. Bondy and U. S. R. Murty. *Graph theory with applications*. American Elsevier Publishing Co., Inc., New York, 1976.

[6] H. J. Broersma, V. Patel, and A. Pyatkin. On toughness and Hamiltonicity of $2K_2$-free graphs. *J. Graph Theory*, 75(3):244–255, 2014.

[7] V. Chvátal. Tough graphs and Hamiltonian circuits. *Discrete Math.*, 5:87–228, 1973.

[8] V. Chvátal and P. Erdős. A note on hamiltonian circuits. *Discrete Mathematics*, 2(2):111–113, 1972.

[9] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, s3-2(11):69–81, 1952.

[10] H. Enomoto, B. Jackson, P. Katerinis, and A. Saito. Toughness and the existence of k-factors. III. *Discrete Mathematics*, 189(1):277–282, 1998.

[11] R. Gould. Advances on the hamiltonian problem - a survey. *Graphs and Combinatorics*, 19:7–52, 03 2003.

[12] R. Häggkvist and G.G. Nicoghossian. A remark on hamiltonian cycles. *Journal of Combinatorial Theory, Series B*, 30(1):118–120, 1981.

[13] J. Kanno and S. Shan. Vizing's 2-factor conjecture involving toughness and maximum degree conditions. *Electron. J. Comb.*, 26(2):P2.17, 2019.

[14] D. Kratsch, J. Lehel, and H. Müller. Toughness, hamiltonicity and split graphs. *Discrete Mathematics*, 150(1–3):231–245, 1996.

[15] B. Li, H. J. Broersma, and S. Zhang. Forbidden subgraphs for Hamiltonicity of 1-tough graphs. *Discuss. Math. Graph Theory*, 36(4):915–929, 2016.

[16] O. Ore. Note on Hamiltonian circuits. *The American Mathematical Monthly*, 67(1):55–55, 1960.

[17] K. Ota and M. Sanka. Toughness, 2-factors and hamiltonian cycles in $2k_2$-free graphs. *arXiv:2103.06760*, 2021.

[18] M. Sanka. Forbidden subgraphs and 2-factors in 3/2-tough graphs, 2021.

[19] S. Shan. Hamiltonian cycles in 3-tough $2K_2$-free graphs. *J. Graph Theory*, 94(3):349–363, 2020.

[20] S. Shan. Hamiltonian cycles in tough $(P_2 \cup P_3)$-free graphs. *Electronic Journal of Combinatorics.*, 28(1), 2021.

[21] W.T. Tutte. A short proof of the factor theorem for finite graphs. *Canadian J. Math.*, 6:347–352, 1954.