Mathieu-Zhao Subspaces of Burnside Algebras of Some Finite Groups

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In 2010, W. Zhao introduced the notion of a Mathieu subspace as a common framework for study of the Jacobian conjecture and related topics. As a generalization of ideals, Mathieu subspaces provide a new viewpoint to investigate the structure of associative algebras and rings. In this paper, we classify Mathieu subspaces of the Burnside algebras \( B_k(G) \) and \( B_k(D_{2p}) \) where \( k \) is a field of characteristic \( p > 0 \), \( G = H \times K \) for a \( p \)-group \( H \) and a \( p' \)-group \( K \), and \( D_{2p} \) is the dihedral group of order \( 2p \) (for \( p \) odd).

KEYWORDS: Mathieu subspaces (Mathieu-Zhao subspaces), Burnside rings (Burnside algebras), finite groups, \( p \)-groups, \( p' \)-groups, dihedral groups
MATHIEU-ZHAO SUBSPACES OF BURNSIDE ALGEBRAS
OF SOME FINITE GROUPS

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MATHIEU-ZHAO SUBSPACES OF BURNSIDE ALGEBRAS
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ANDREW B. HATFIELD

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Wenhua Zhao, Chair
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CONTENTS

<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>CONTENTS</td>
<td>i</td>
</tr>
<tr>
<td>CHAPTER I: INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER II: PRELIMINARIES</td>
<td>5</td>
</tr>
<tr>
<td>CHAPTER III: MATHIEU-ZHAO SUBSPACES OF $\mathcal{B}_k(G)$</td>
<td>8</td>
</tr>
<tr>
<td>CHAPTER IV: MATHIEU-ZHAO SUBSPACES OF $\mathcal{B}<em>k(D</em>{2p})$</td>
<td>12</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>15</td>
</tr>
</tbody>
</table>
CHAPTER I: INTRODUCTION

Let \( f_1, \ldots, f_n \) be a set of polynomials over \( \mathbb{C} \) in variables \( x_1, \ldots, x_n \). Define the polynomial map \( F : \mathbb{C}^n \to \mathbb{C}^n \) by

\[
F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)).
\]

Denote by \( J_F \) the Jacobian of \( F \), which is the determinant of the \( n \times n \) matrix

\[
M = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}.
\]

First formulated by O.-H. Keller in 1939, the Jacobian conjecture can be stated as follows.

**Conjecture 1.1. (Keller, [6])** Let \( F : \mathbb{C}^n \to \mathbb{C}^n \) be a polynomial map. If \( J_F \) is a nonzero constant, then \( F \) has an inverse polynomial map \( G \).

Although the statement of the conjecture is quite simple, the conjecture remains today wide open in general with very few special cases proven. For a survey of related results, we refer the reader to [1], [10], and [11].

Although direct proof attempts have been mostly unsuccessful, it has been shown that numerous conjectures imply the Jacobian conjecture. Motivated by Mathieu’s conjecture [8] and the Image conjecture [13], W. Zhao introduced in [12] the following notion with the goal of creating a common framework for the study of the Jacobian conjecture and related conjectures.

**Definition 1.2.** Let \( R \) be a commutative ring and \( \mathcal{A} \) be a commutative \( R \)-algebra. We say that a subspace \( \mathcal{M} \) of \( \mathcal{A} \) is a **Mathieu subspace** of \( \mathcal{A} \) if the following condition holds: for
a, b ∈ A with \(a^m \in M\) for all \(m \geq 1\), we have \(a^mb \in M\) when \(m \gg 0\), i.e. there exists \(N \geq 1\) (depending on \(a, b\)) such that \(a^m b \in M\) for all \(m \geq N\).

Note that Mathieu subspaces are now commonly called Mathieu-Zhao subspaces in the literature.

Several conjectures related to the Jacobian conjecture can be restated in terms of Mathieu-Zhao subspaces (for example, the Mathieu and Image conjecture as in [12]). Mathieu-Zhao subspaces are also a natural generalization of the concept of ideals, and as such, the study of Mathieu-Zhao subspaces of associative algebras and rings has grown into a field of its own. In this paper, we investigate the Mathieu-Zhao subspaces of Burnside algebras of certain classes of finite groups over fields of prime characteristic. We begin by fixing some definitions.

Let \(G\) denote a finite group and \(R\) a commutative, unital ring. Let \(S\) be a finite set and denote by \(A(S)\) the symmetric group of all permutations of \(S\). We say that \(S\) is a \(G\)-set if there exists a group homomorphism \(\tau : G \to A(S)\). We call \(\tau\) a group action of \(G\) on \(S\) and typically write \(gs\) to denote \((\tau(g))(s)\) for all \(s \in S\).

Let \(S, T\) be \(G\)-sets. We say that \(S\) and \(T\) are isomorphic as \(G\)-sets if there exists a bijection \(f : S \to T\) such that \(f\) preserves the group actions of \(G\) on \(S\) and \(T\), i.e. for any \(g \in G\) and \(s \in S\), we have \(f(gs) = gf(s)\). The Cartesian product of \(S\) and \(T\) is also a \(G\)-set under the diagonal action \(g(s, t) = (gs, gt)\) for all \(g \in G, (s, t) \in S \times T\). We define the orbit of \(s\) to be the set \(Gs = \{gs \mid g \in G\}\). We say that \(S\) is transitive if \(S = Gs\) for some \(s \in S\), or equivalently, the action of \(G\) on \(S\) has exactly one orbit. Let \(s \in S, t \in T\). For transitive \(G\)-sets \(S\) and \(T\), \(S\) and \(T\) are isomorphic as \(G\)-sets if and only if \(\text{stab}(s)\) is conjugate to \(\text{stab}(t)\), where \(\text{stab}(s) = \{g \in G \mid gs = s\}\) denotes the stabilizer (or isotropy group) of \(s\) [9].

For any \(G\)-set \(S\), we may uniquely decompose \(S\) into the disjoint union of transitive \(G\)-sets which are precisely the orbits of \(S\) under the action of \(G\). For any subgroup \(H \leq G\), the left coset space \(G/H\) defines a transitive \(G\)-set with action given by left multiplication. For any \(s \in S\), we have \(S \cong G/\text{stab}(s)\) if \(S\) is transitive.
Let $P$ denote the set of conjugacy classes of subgroups of $G$. For each $a \in P$, let $H_a$ denote a representative of the class $a$ and $[G/H_a]$ denote the isomorphism class of $G/H_a$. Let $\mathcal{B}_R(G)$ denote the free $R$-module generated by the set $\{ [G/H_a] \mid a \in P \}$. For any two basis elements $[G/H_a], [G/H_b] \in \mathcal{B}_R(G)$, define

$$[G/H_a] \cdot [G/H_b] = \sum [G/K_i]$$

where the sum is taken over all $G$-orbits in $G/H_a \times G/H_b$ and $K_i$ is the stabilizer of the $i$th $G$-orbit. Extending the product by linearity makes $\mathcal{B}_R(G)$ a commutative ring with identity $[G/G]$, and we call $\mathcal{B}_R(G)$ the Burnside ring of $G$ over $R$. If $R$ is a field, we call $\mathcal{B}_R(G)$ the Burnside algebra of $G$ over $R$. The Burnside ring is named after W. Burnside, who introduced the notion in [2].

We say that a finite group $G$ is a $p$-group for a prime $p$ if $|G| = p^k$ for some $k$, and say that $G$ is a $p'$-group if $p \nmid |G|$. Let $A$ be an associative algebra and $\langle e \rangle$ denote the principal ideal of $A$ generated by $e \in A$. In this thesis, we prove the following main theorems.

**Theorem 1.3.** Let $k$ be a field of characteristic $p$ and $G = H \times K$ where $H$ is a $p$-group and $K$ is a $p'$-group. Let $V$ be a subspace of $\mathcal{B}_k(G)$. Then $V$ is a Mathieu-Zhao subspace of $\mathcal{B}_k(G) \cong \mathcal{B}_k(H) \otimes_k \mathcal{B}_k(K)$ if and only if $V$ contains no nonzero idempotents or $\mathcal{B}_k(H) \otimes \langle j \rangle \subseteq V$ for each nonzero idempotent $j$ of $\mathcal{B}_k(K)$ such that $1 \otimes j \in V$, where $\langle j \rangle$ is the principal ideal of $\mathcal{B}_k(K)$ generated by $j$.

**Theorem 1.4.** Let $p$ be an odd prime, $k$ be a field of characteristic $p$, and $A = \mathcal{B}_k(D_{2p})$. Then $A \cong e_1 A \times e_2 A \times e_3 A$ for some nonzero idempotents $e_i$. A subspace $V$ of $A$ is a Mathieu-Zhao subspace of $A$ if and only if $V$ contains no nonzero idempotents or $\bigoplus_{j \in J} e_j A \subseteq V$ for each nonzero idempotent $\sum_{j \in J} e_j$ contained in $V$, where $J \subseteq \{1, 2, 3\}$.

The rest of the paper is organized as follows: in Chapter II, we discuss results necessary for the proof of Theorems 1.3 and 1.4. In Chapter III, we prove Theorem 1.3. In Chapter
IV, we prove Theorem 1.4.
CHAPTER II: PRELIMINARIES

The following theorem due to G. Karpilovsky allows the splitting of Burnside rings over the cross product of groups.

**Theorem 2.1** (Karpilovsky, [5]). Let $G$ and $H$ be groups with representatives of all conjugacy classes given by $G_1, \cdots, G_n$ and $H_1, \cdots, H_m$ respectively. Then the map $\phi : \mathcal{B}_Z(G) \otimes \mathcal{B}_Z(H) \to \mathcal{B}_Z(G \times H)$ given by $\phi([G/G_i] \otimes [H/H_j]) = [(G \times H)/(G_i \times H_j)]$ is an injective ring homomorphism. Furthermore, if $G$ and $H$ are of relatively prime order, then $\phi$ is a ring isomorphism.

Let $p$ be a prime and let $\mathbb{Z}_p$ denote the field of integers modulo $p$. The following theorem due to E. Jacobson classifies local Burnside rings of the form $\mathcal{B}_{\mathbb{Z}_p}(G)$ where $G$ is a finite group.

**Theorem 2.2** (Jacobson, [4]). Let $G$ be a finite group. $G$ is a $p$-group if and only if $\mathcal{B}_{\mathbb{Z}_p}(G)$ is local.

The following theorem is an analogue of the well-known Maschke’s theorem for group algebras. For a unital ring $R$, we say that $e \in R$ is *idempotent* if $e^2 = e$ and we call $e$ *central* if $eR(1-e) = (1-e)Re = 0$. We say that idempotents $e$ and $f$ are *orthogonal* if $ef = fe = 0$, and we say that a central idempotent $e$ is *centrally primitive* if $e \neq 0$ and $e$ cannot be written as the sum of two nonzero orthogonal central idempotents in $R$. Furthermore, we say a set $E$ of orthogonal centrally primitive idempotents is *complete* if $\sum_{e \in E} e = 1$. We note for the Burnside algebras $\mathcal{B}_k(G)$ that all idempotents are central as $\mathcal{B}_k(G)$ is commutative.

**Theorem 2.3** (Solomon, [9]). Let $G$ be a finite group and let $k$ be a field of characteristic 0 or coprime to $|G|$. Then the Burnside algebra $\mathcal{B}_k(G)$ is semisimple and isomorphic to $\bigoplus_{e \in E} ke$ for a complete set of orthogonal centrally primitive idempotents $E$.

The following theorem is a standard result for Burnside rings describing the product of $G$-sets $[G/H]$ and $[G/K]$ such that $H, K$ are normal subgroups of $G$. 
Lemma 2.4. Let $G$ be a finite group and let $k$ be a field of characteristic $p$. Let $H, K$ be normal subgroups of $G$. Then the multiplication of transitive $G$-sets $[G/H], [G/K]$ in $B_k(G)$ is given by

$$[G/H] \cdot [G/K] = \frac{|H \cap K||G|}{|H||K|} [G/H \cap K].$$

Proof. Let $(aH, bK) \in G/H \times G/K$. As $H, K$ are normal, the stabilizer $\text{stab}(aH, bK)$ is given by

$$\text{stab}(aH, bK) = aHa^{-1} \cap bKb^{-1} = H \cap K.$$ 

Counting the number of elements on both sides gives $[G/H] \cdot [G/K] = \frac{|H \cap K||G|}{|H||K|} [G/H \cap K]$. $\square$

The following theorem due to W. Zhao allows for simple classification of the Mathieu-Zhao subspaces of some algebras given their idempotents. Let $k$ be a field and $A$ an associative algebra over $k$. We say $V \subseteq A$ is algebraic over $k$ if every element of $V$ is the root of a monic polynomial with coefficients in $k$. Denote by $\sqrt{V}$ the radical of $V$, i.e., the set of all $a \in A$ such that $a^m \in V$ for sufficiently large $m$.

Theorem 2.5 (Zhao, [14]). Let $k$ be a field and $A$ an associative algebra over $k$. Let $V$ be a $k$-subspace such that $\sqrt{V}$ is algebraic over $k$. Then $V$ is a Mathieu-Zhao subspace of $A$ if and only if for each idempotent $e \in V$, we have the principal ideal $(e) \subseteq V$.

Lemma 2.6. Let $G$ be a finite group and $k$ be a field. Then $B_k(G)$ is algebraic.

Proof. Let $b \in B_k(G)$. Then the set $\{1, b, b^2, \cdots\}$ must be linearly dependent, so there exists a nonconstant polynomial $q$ such that $q(b) = 0$. Let $\alpha \in k$ be the leading coefficient of $q$. Then $\alpha^{-1}p(b) = 0$, hence $b$ is algebraic. Therefore, $B_k(G)$ is algebraic. $\square$

Let $p$ be an odd prime and $D_{2p}$ denote the dihedral group of order $2p$. Write $D_{2p}$ as $\langle r, s \rangle$, where $r$ has order $p$ and $s$ has order 2. In $D_{2p}$, conjugacy classes of some subgroups
are nontrivial. The following theorem due to K. Conrad allows us to classify all subgroups of $D_{2p}$ into one of 4 conjugacy classes.

**Theorem 2.7** (Conrad, [3]). Let $n$ be odd and $m \mid 2n$. If $m$ is odd, then all $m$ subgroups of $D_{2n}$ with index $m$ are conjugate. If $m$ is even, then the only subgroup of $D_{2n}$ with index $m$ is $\langle r^m/2 \rangle$. In particular, all subgroups of $D_{2n}$ with the same index are conjugate to each other.

The following theorem is a well-known result relating the idempotents of a ring and its decomposition (e.g., [7]).

**Theorem 2.8.** Let $R$ be a (not necessarily commutative) ring. Then $R$ can be expressed as a finite direct product of indecomposable rings if and only if $1 \in R$ can be written as a sum of orthogonal centrally primitive idempotents. If such a decomposition exists, each factor of the decomposition of $R$ contains no nontrivial central idempotents.
CHAPTER III: MATHIEU-ZHAO SUBSPACES OF $\mathcal{B}_k(G)$

Let $G = H \times K$ where $H$ is a $p$-group and $K$ is a $p'$-group. Then by Theorem 2.1, we have $\mathcal{B}_k(G) \cong \mathcal{B}_k(H) \otimes_k \mathcal{B}_k(K)$. To find the Mathieu-Zhao subspaces of $\mathcal{B}_k(G)$, we first investigate the idempotents of each of $\mathcal{B}_k(H), \mathcal{B}_k(K)$.

**Theorem 3.1.** Let $H$ be a $p$-group and $k$ be a field of characteristic $p$. Then $\mathcal{B}_k(H)$ is local.

**Proof.** By Theorem 2.2, $\mathcal{B}_{Z_p}(H)$ is local. As $\mathcal{B}_k(H) = k \otimes_{Z_p} \mathcal{B}_{Z_p}(H)$, we see that $\mathcal{B}_k(H)$ must also be local. \qed

Recall that $K$ is a $p'$-group. Let $l = \dim_k \mathcal{B}_k(K)$. By Theorem 2.3, $\mathcal{B}_k(K) \cong \bigoplus_{i=1}^l k$, and $\mathcal{B}_k(K)$ has a complete set of primitive idempotents $\{e_1, \ldots, e_l\}$.

In some cases, it is simple to list the primitive idempotents of $\mathcal{B}_k(K)$. Let $C_n$ denote the cyclic group with $n$ elements.

**Example 3.2.** Let $K = C_{q^s}$ with $q$ prime and let $f_i = q^{i-s}[K/C_{q^s}]$. Then a complete set of primitive idempotents of $\mathcal{B}_k(K)$ is given by $F = \{f_0, f_1 - f_0, \ldots, f_s - f_{s-1}\}$.

**Proof.** For $i \leq j$, we have

$$f_i \cdot f_j = q^{i-s}[K/C_{q^i}] \cdot q^{j-s}[K/C_{q^j}] = q^{i+j-2s}q^{s-j}[K/C_{q^s}] = f_i$$

by Lemma 2.4. Then $f_i^2 = f_i$, and for $i \geq 1$,

$$(f_i - f_{i-1})^2 = f_i^2 - 2f_{i-1} + f_{i-1}^2 = f_i - f_{i-1}.$$ 

For $1 \leq i \leq s$, we have

$$f_0(f_i - f_{i-1}) = f_0 - f_0 = 0$$
and for $1 \leq i < j \leq s$,

$$(f_i - f_{i-1})(f_j - f_{j-1}) = f_i - f_{i-1} - f_i + f_{i-1} = 0,$$

thus $F$ is a set of orthogonal idempotents. As $\dim \mathcal{B}_k(K) = s + 1 = |F|$, each $f \in F$ must be primitive. As $\sum_{f \in F} f = f_s = 1$, $F$ is a complete set of primitive idempotents as desired. \[\square\]

We now investigate the idempotents of $\mathcal{B}_k(H) \otimes_k \mathcal{B}_k(K)$.

**Lemma 3.3.** Let $\{e_1, \cdots, e_l\}$ be a complete set of orthogonal primitive idempotents of $\mathcal{B}_k(K)$. Then the set $E = \{1 \otimes e_1, \cdots, 1 \otimes e_l\}$ is a complete set of orthogonal primitive idempotents in $\mathcal{B}_k(G) \cong \mathcal{B}_k(H) \otimes_k \mathcal{B}_k(K)$.

**Proof.** We have

$$\mathcal{B}_k(G) \cong \mathcal{B}_k(H) \otimes_k \mathcal{B}_k(K)$$

$$\cong \mathcal{B}_k(H) \otimes_k \left( \bigoplus_{i=1}^l k \right)$$

$$\cong \bigoplus_{i=1}^l (\mathcal{B}_k(H) \otimes_k k)$$

$$\cong \bigoplus_{i=1}^l \mathcal{B}_k(H)$$

as $\mathcal{B}_k(H) \otimes_k k \cong \mathcal{B}_k(H)$. Note that every $1 \otimes e_i \in E$ satisfies $(1 \otimes e_i)^2 = 1 \otimes e_i^2 = 1 \otimes e_i$, so each $1 \otimes e_i$ is idempotent. For $i \neq j$, $(1 \otimes e_i)(1 \otimes e_j) = 1 \otimes e_i e_j = 0$, so the elements of $E$ are pairwise orthogonal. Let $f = (f_i)_{i=1}^l$ be an idempotent of $\mathcal{B}_k(G) \cong \bigoplus_{i=1}^l \mathcal{B}_k(H)$. Then

$$f = f \cdot \left( \sum_{i=1}^l 1 \otimes e_i \right) = \sum_{i=1}^l f(1 \otimes e_i).$$

As $\mathcal{B}_k(H)$ is local by Theorem 3.1, any nonzero idempotent $f$ that is not the identity must be in the unique maximal ideal. Similarly, $1 - f$ must also be in the same maximal ideal,
which implies 1 is in this maximal ideal. This is a contradiction, thus each \( f_i \) is either 0 or 1. Then either \( f = 0 \) or

\[
f = \sum_{j \in J} (1 \otimes e_j) = 1 \otimes \sum_{j \in J} e_j
\]

for some \( J \subseteq \{1, \ldots, l\} \). As each nonzero idempotent \( f \) has such a decomposition, we see that \( E \) is a primitive set of idempotents. We have \( \sum_{i=1}^{l} 1 \otimes e_i = 1 \), so \( E \) is a complete set. \( \square \)

With the idempotents of \( \mathcal{B}_k(G) \) clear, Theorem 1.3 becomes a consequence of Theorem 2.5.

**Proof of Theorem 1.3.** \((\Rightarrow)\) Let \( V \) be a Mathieu-Zhao subspace of \( \mathcal{B}_k(H) \otimes \mathcal{B}_k(K) \). If \( V \) contains no nonzero idempotents, then the proof is complete. If \( V \) contains a nonzero idempotent, it must be of the form \( 1 \otimes j \) by Lemma 3.3. As \( V \) is a Mathieu-Zhao subspace, \( \langle 1 \otimes j \rangle = \mathcal{B}_k(H) \otimes \langle j \rangle \) must be contained in \( V \) by Theorem 2.5.

\((\Leftarrow)\). Let \( V \) be a subspace of \( \mathcal{B}_k(H) \otimes \mathcal{B}_k(K) \). Then by Lemma 2.6, \( \sqrt{V} \) is algebraic. If \( V \) contains no nonzero idempotents, then \( V \) is a Mathieu-Zhao subspace by Theorem 2.5. If \( V \) contains a nonzero idempotent \( f \), then by Lemma 3.3, \( f = 1 \otimes j \) for some idempotent \( j \) of \( \mathcal{B}_k(K) \). By assumption, \( \langle 1 \otimes j \rangle \subseteq V \), so \( V \) satisfies Theorem 2.5 and is therefore a Mathieu-Zhao subspace. \( \square \)

**Corollary 3.4.** Let \( V \) be a subspace of \( \mathcal{B}_k(H) \) not containing 1. Then for any subspace \( W \) of \( \mathcal{B}_k(K) \), \( V \otimes W \) does not contain any nonzero idempotents, hence is a Mathieu-Zhao subspace of \( \mathcal{B}_k(G) \).

**Proof.** Note that \( \sqrt{V \otimes W} \) is algebraic by Lemma 2.6. Let \( \{v_1, \ldots, v_m\} \) be a basis of \( V \) and let \( \{w_1, \ldots, w_n\} \) be a basis of \( W \). If \( V \otimes W \) contains a nonzero idempotent \( f \), then by Lemma 3.3, \( f = 1 \otimes j \) for some idempotent \( j \) in \( \mathcal{B}_k(K) \). Then

\[
1 \otimes j = \sum_{s,t} \alpha_{s,t} v_s \otimes w_t,
\]
for some $\alpha_{s,t} \in k$, so $1 \in \text{span}\{v_1, \cdots, v_m\}$, contradicting the assumption that $1 \not\in V$. So $V \otimes W$ contains no nonzero idempotents, and by Theorem 1.3, $V \otimes W$ is a Mathieu-Zhao subspace of $\mathcal{B}_k(G)$. 

**Corollary 3.5.** Let $W$ be a subspace of $\mathcal{B}_k(K)$ containing no nonzero idempotents. Then for any subspace $V$ of $\mathcal{B}_k(H)$, $V \otimes W$ contains no nonzero idempotents, hence is a Mathieu-Zhao subspace of $\mathcal{B}_k(G)$.

**Proof.** Again, note $\sqrt{V} \otimes W$ is algebraic over $k$ by Lemma 2.6. By Corollary 3.4, we may assume $V$ contains 1. Let $\{v_1, \cdots, v_m\}$ be a basis of $V$ with $v_1 = 1$ and $\{w_1, \cdots, w_n\}$ be a basis of $W$. Let $f$ be a nonzero idempotent contained in $V \otimes W$. Then by Lemma 3.3, $f = 1 \otimes j$ for some idempotent $j$ in $\mathcal{B}_k(K)$. Then

\[
f = 1 \otimes j = \sum_{s,t} \alpha_{s,t} v_s \otimes w_t = \sum_t \alpha_{1,t} 1 \otimes w_t + \sum_{s,t \neq 1} \alpha_{s,t} v_s \otimes w_t,
\]

but as $\{v_s \otimes w_t \mid 1 \leq s \leq m, 1 \leq t \leq n\}$ are linearly independent, we see that the second summand must be 0. Then

\[
1 \otimes j = \sum_t 1 \otimes \alpha_{1,t} w_t = 1 \otimes \left( \sum_t \alpha_{1,t} w_t \right)
\]

and we see that $j$ is a linear combination of basis vectors of $W$ and therefore $j \in W$. But $j$ is an idempotent and $W$ contains no nonzero idempotents, so we must have $j = 0$. Then $1 \otimes j = 0$, and by Theorem 1.3, $V \otimes W$ is a Mathieu-Zhao subspace of $\mathcal{B}_k(G)$. 

**Remark 3.6.** By the Classification Theorem of Finite Abelian Groups, every finite abelian group $G$ is isomorphic to $H \times K$ for some $p$-group $H$ and $p'$-group $K$. Therefore, Theorem 1.3 and Corollaries 3.4 and 3.5 hold for all finite abelian groups.
CHAPTER IV: MATHIEU-ZHAO SUBSPACES OF $\mathcal{B}_k(D_{2p})$

Throughout this chapter, let $p$ be an odd prime, $k$ be a field of characteristic $p$, and $G = D_{2p}$ denote the dihedral group of order $2p$. Write $G$ as $\langle r, s \rangle$, where $r$ has order $p$ and $s$ has order 2. Let $\mathcal{A}$ denote the Burnside algebra $\mathcal{B}_k(G)$. As $G$ is not abelian, conjugacy classes of subgroups are sometimes nontrivial, therefore the structure of $\mathcal{A}$ is slightly more complex than the cyclic case.

Let $C_n$ denote the cyclic subgroup of $G$ with $n$ elements and let $S$ be the subgroup $\{1, s\}$. By Theorem 2.7, a complete set of representatives of conjugacy classes of subgroups of $G$ is given by $\{G, C_p, S, C_1\}$. For each representative subgroup $H$, let $T_H$ denote the class $[G/H]$. Note that $C_1$, $C_p$, and $G$ are all normal subgroups of $G$.

**Lemma 4.1.** Let $G = D_{2p}$. The product of $G$-sets in $\mathcal{A}$ is given by the table below.

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<td>$T_S$</td>
<td>$T_{C_1}$</td>
<td>$T_S - \frac{1}{2}T_{C_1}$</td>
<td>0</td>
</tr>
<tr>
<td>$T_{C_1}$</td>
<td>$T_{C_1}$</td>
<td>$2T_{C_1}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

*Proof.* For the product of $G$-sets corresponding to normal subgroups, use Lemma 2.4.

For the product of $T_S$ and $T_N$ where $N$ is a normal subgroup of $G$, note that the stabilizer of a pair $(aS, bN) \in G/S \times G/N$ is given by

$$\text{stab}(aS, bN) = aS a^{-1} \cap bN b^{-1} = aS a^{-1} \cap N.$$ 

If $N = C_1$ or $C_p$, then $aS a^{-1} \cap N = C_1$ as all conjugates of $S$ are of the form $\{1, r^i s\}$ for some $i$. Then every element of $G/S \times G/N$ has stabilizer $C_1$ and counting the number of elements on both sides gives

$$T_S \cdot [G/N] = \frac{|G||C_1|}{|S||N|}T_{C_1} = \frac{p}{|N|}T_{C_1},$$

12
which is 0 for $N = C_1$ and $T_{C_1}$ for $N = C_p$. If $N = G$, the $G$-set $T_G$ is the identity element of $\mathcal{B}_k(G)$ and the product is trivial.

Finally, consider the product $T_S \cdot T_S$. Again, let $(aS, bS) \in G/S \times G/S$. We may assume $a, b \in C_p$. Then

$$\text{stab}(aS, bS) = aSa^{-1} \cap bSb^{-1},$$

so we see that the stabilizer depends on the choice of $(aS, bS)$. The intersection $aSa^{-1} \cap bSb^{-1}$ is trivial unless $aSa^{-1} = bSb^{-1}$, which is the case if and only if $a \equiv b \mod N(S)$ where $N(S)$ denotes the normalizer of $S$. But $N(S) = S$ in $D_{2p}$, and therefore $(aS, bS)$ has stabilizer conjugate to $S$ if and only if $a \equiv b \mod S$ if and only if $a = b$, since $ab^{-1} \in S$ if and only if $ab^{-1} = 1$. Then there are $p^2$ total elements in $G/S \times G/S$, of which $p$ of them have stabilizer conjugate to $S$ and $p^2 - p$ of them have stabilizer conjugate to $C_1$. Then

$$T_S \cdot T_S = \frac{|S|}{|G|} p T_S + \frac{|C_1|}{|G|} (p^2 - p) T_{C_1}$$

$$= \frac{2p}{2p} T_S + \frac{(p^2 - p)}{2p} T_{C_1}$$

$$= T_S + \frac{p - 1}{2} T_{C_1}$$

$$= T_S - \frac{1}{2} T_{C_1}$$

as $k$ has characteristic $p$. □

**Lemma 4.2.** Let $e_1 = \frac{1}{2} T_{C_p}, e_2 = T_S - \frac{1}{2} T_{C_1}$ and $e_3 = 1 - e_1 - e_2$. Then $E = \{e_1, e_2, e_3\}$ is a complete set of orthogonal primitive idempotents in $A$.

**Proof.** Note that $e_1^2 = (\frac{1}{2} T_{C_p})^2 = \frac{1}{2} T_{C_p}$ and $e_2^2 = (T_S - \frac{1}{2} T_{C_1})^2 = T_S - \frac{1}{2} T_{C_1}$, so $e_1$ and $e_2$ are idempotent. Then

$$e_1 e_2 = \left(\frac{1}{2} T_{C_p}\right) \left(T_S - \frac{1}{2} T_{C_1}\right) = \frac{1}{2} T_{C_1} - \frac{2}{4} T_{C_1} = 0,$$

13
so \(e_1e_3 = e_1(1 - e_1 - e_2) = e_1 - e_1^2 - e_1e_2 = 0\) and \(e_2e_3 = e_2(1 - e_1 - e_2) = e_2 - e_2e_1 - e_2^2 = 0\) and we see the elements of \(E\) are pairwise orthogonal. Then \(e_3\) is also idempotent, as

\[
e_3^2 = (1 - e_1 - e_2)^2 = 1 - e_1 - e_2 - e_1 + e_1e_2 - e_2 - e_2e_1 + e_2 = 1 - e_1 - e_2 = e_3.
\]

By Lemma 4.1, \(e_1A = \text{span}\{T_{C_2}, T_{C_1}\}, e_2A = \text{span}\{e_2\}, e_3A = \text{span}\{e_3\}\). Note that \(e_2A, e_3A\) are simple and therefore \(e_2, e_3\) are primitive. Assume that \(e_1\) is not primitive. Then there exist some orthogonal idempotents \(f, f'\) such that \(e_1 = f + f'\) and \(e_1A = fA \oplus f'A\) with each of \(fA, f'A\) simple. Then

\[
A = fA \oplus f'A \oplus e_2A \oplus e_3A \cong k \oplus k \oplus k \oplus k,
\]

so \(A\) has no nonzero nilpotent element. But \(T_{C_1}^2 = 0\) is nilpotent in \(A\), which is a contradiction. As \(\sum_{e \in E} e = 1\), we see that \(E\) is a complete set of orthogonal primitive idempotents.

**Proof of Theorem 1.4.** By Lemma 4.2, \(E = \{e_1, e_2, e_3\}\) is a complete set of orthogonal primitive idempotents. Then by Theorem 2.8, we have \(A = \bigoplus_{i=1}^3 e_iA\) where the idempotents of each subalgebra \(e_iA\) are exactly 0 and \(e_i\).

(\(\Rightarrow\)) Let \(V\) be a Mathieu-Zhao subspace of \(A\). Assume \(V\) contains some nonzero idempotent \(f\). Then by Lemma 4.2, \(f = \sum_{j \in J} e_j\) where \(J\) is a nonempty subset of \(\{1, 2, 3\}\). As \(V\) is a Mathieu-Zhao subspace, \(\langle f \rangle = \bigoplus_{j \in J} e_jA\) is a subset of \(V\) by Theorem 2.5.

(\(\Leftarrow\)) Let \(V\) be a subspace of \(A\). By Lemma 2.6, \(\sqrt{V}\) is algebraic. If \(V\) contains no nonzero idempotents, then \(V\) is a Mathieu-Zhao subspace of \(A\) by Theorem 2.5. If \(V\) contains a nonzero idempotent \(f\), then by Lemma 4.2, \(f = \sum_{j \in J} e_j\) for some nonempty subset \(J\) of \(\{1, 2, 3\}\). By assumption, \(\bigoplus_{j \in J} e_jA\) is contained in \(V\), therefore \(V\) satisfies Theorem 2.5 and is a Mathieu-Zhao subspace. \(\square\)
REFERENCES


