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Extending Edge-Colorings of Complete Uniform Hypergraphs

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EXTENDING EDGE-COLORINGS OF COMPLETE UNIFORM HYPERGRAPHS

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43 Pages

A hypergraph $V(\mathcal{G})$ is an ordered pair $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ where $V(\mathcal{G})$ is a set of vertices of $\mathcal{G}$ and $E(\mathcal{G})$ is a collection of edge multisets of $\mathcal{G}$. If the size of every edge in the hypergraph is equal, then we call it a uniform hypergraph. A *complete h-uniform hypergraph*, written K_n^h, is a uniform hypergraph with edge sizes equal to h and has n vertices where the edges set is the collection of all h-elements subset of its vertex set (so the total number of the edges is $\binom{n}{h}$)). A hypergraph is called *regular* if the degree of all vertices is the same. An r-factorization of a hypergraph is a coloring of the edges of a hypergraph such that the number of times each element appears in each color class is exactly r. A partial r-factorization is a coloring in which the degree of each vertex in each color class is at most r.

The main problem under consideration in this thesis is motivated by Baranyai's famous theorem and Cameron's question from 1976. Given a partial r-factorization of K_m^h, we are interested in finding the necessary and sufficient conditions under which we can extend this partial r-factorization to an r-factorization of K_n^h. The case $h = 3$ of this problem was partially solved by Bahmanian and Rodger in 2012, and the cases $h = 4, 5$ were partially solved by Bahmanian in 2018. Recently, Bahmanian and Johnsen showed that as long as $n \geqslant (h-1)(2m-1)$, the obvious necessary conditions are also sufficient. In this thesis, we improve this bound for all $h \in \{6, 7, \ldots, 89\}$. Our proof is computer-assisted.

KEYWORDS: factorization, embedding, hypergraph, edge-coloring, baranyai's theorem

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FIGURES

CHAPTER I: INTRODUCTION

A graph, denoted by $\mathcal{G}$, is an ordered pair $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$. An edge $h \in E(\mathcal{G})$ is a subset of $V(\mathcal{G})$ such that $|h| = 2$. If $|h| > 2$, then we call it a *hypergraph*. If $h \in E(\mathcal{G})$ contains $v \in V(\mathcal{G})$, then we say that v is *incident* to h. The number of edges incident with a vertex $v \in V(\mathcal{G})$ is called the *degree of the vertex* v, denoted by $\deg(v)$. Whereas the number of times an edge h appears in the edges set H is called the *multiplicity of the edge* h, denoted by $\text{mult}(h)$. If two or more edges share a common vertex or vertices, then we say the edges are *adjacent* to each other. A hypergraph is called h*-uniform*, if $|e| = h, \forall e \in E(\mathcal{G})$. A d*-regular hypergraph* is a hypergraph such that $\deg(v) = d, \forall v \in V(\mathcal{G})$.

A *complete* h*-uniform hypergraph*, denoted as K_n^h, is a uniform hypergraph with the edge size equal to h and has n number of vertices where the edges set are the set of all h-elements subset of its vertex set with the total number of edges equal to $\binom{n}{h}$. For simplicity of notation, we will write abc to denote the set $\{a, b, c\}$ throughout this thesis. Figure 1 is an example of a complete 3-uniform regular hypergraph with five vertices denoted as K_5^3, with edges set;

$$123, 124, 125, 134, 135,$$

$$145, 234, 235, 245, 345.$$

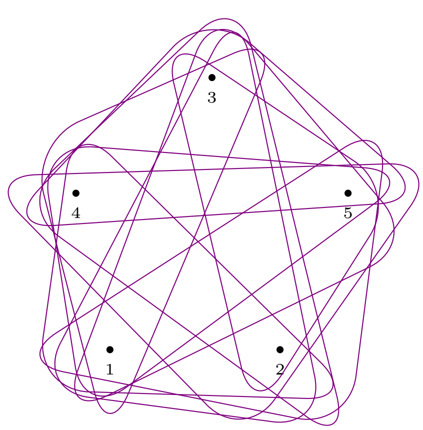

Figure 1: K_5^3

A k-edge coloring of $\mathcal{G}$ is a mapping $f : E(\mathcal{G}) \to [k]$, where $[k] := \{1, ..., k\}$, $k \in \mathbb{N}$, is a set of colors. In other words, a k-edge coloring of $\mathcal{G}$ is a function that assigns a color to every edge of $\mathcal{G}$. A proper k-edge-coloring is a coloring where no adjacent edges have the same color. An r-factorization of $\mathcal{G}$ is a k-edge coloring of $\mathcal{G}$ such that the number of times each element appears in each color class is exactly r. Figure 2 is an example of 3-factorization of K_5^3. A partial r-factorization of $\mathcal{G}$ is a k-edge coloring of $\mathcal{G}$ so that the degree of each vertex in each color class is at most r.

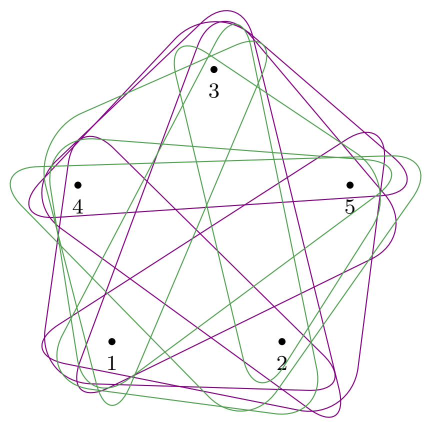

Figure 2: 3-factorization of K_5^3

The edge set are listed below.

$$124, 125, 135, 234, 345,$$

$$123, 134, 145, 235, 245$$

.

I.1 Example

Given $\mathcal{P}$:= a partial 5-factorization of K_4^3 (see Figure 3), we would like to extend it to $\mathcal{Q}$, 5-factorization of K_6^3 (see Figure 8).

123, **234**, **124**, **134**

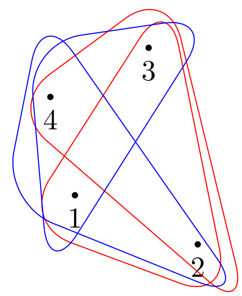

Figure 3: $\mathcal{P}$

First, we need to add 2 more vertices to $\mathcal{P}$. Let $V(\mathcal{Q}) := V(\mathcal{P}) \cup \{v_5, v_6\}$. However, let us amalgamate these two vertices into one vertex u. The degree sum of the vertices will be equal to $rn = 5(6) = 30$ and since each edge contain 3 vertices thus we should have $h|rn \to 3|30$. Since, it is a 5-factorization $\deg_i(v) = 5$, and $k = \frac{\binom{n-1}{h-1}}{r} = \frac{\binom{5}{2}}{5} = 2$. So there are 2 color classes, and since we have $\binom{6}{3} = 20$ edges, thus each color class will contain 10 edges. We will use red and blue colors in these examples.

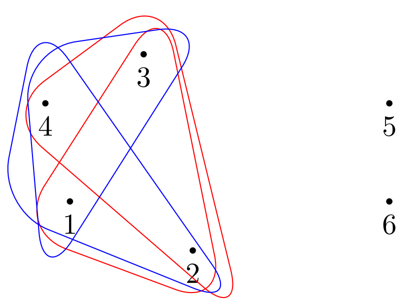

Figure 4: $K_4^3 \cup \{5, 6\}$

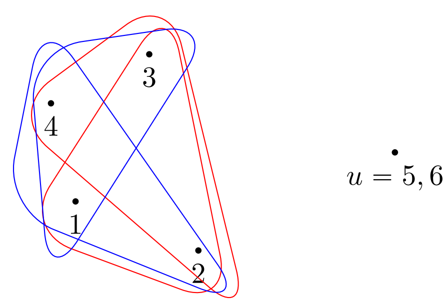

Figure 5: $K_4^3 \cup u$

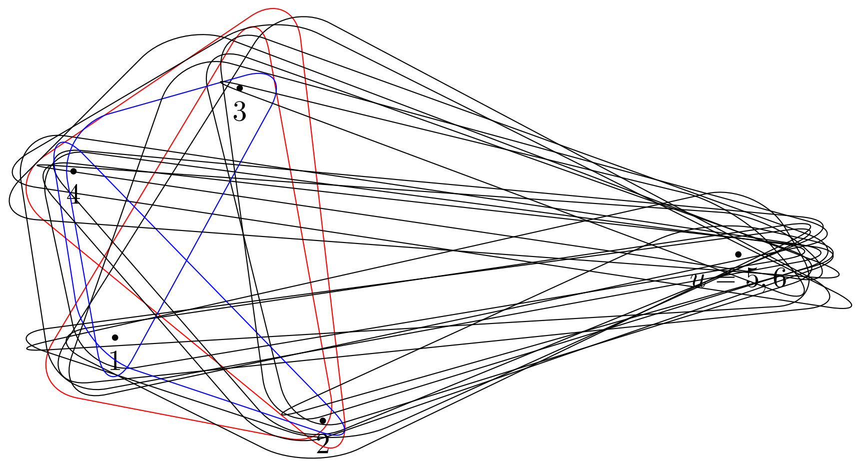

Figure 6: 5-factorization of K_6^3

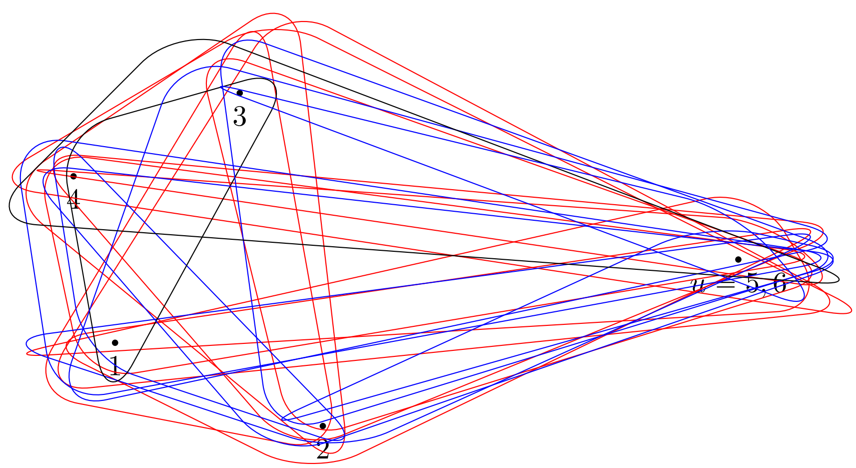

Figure 7: 5-factorization of K_6^3

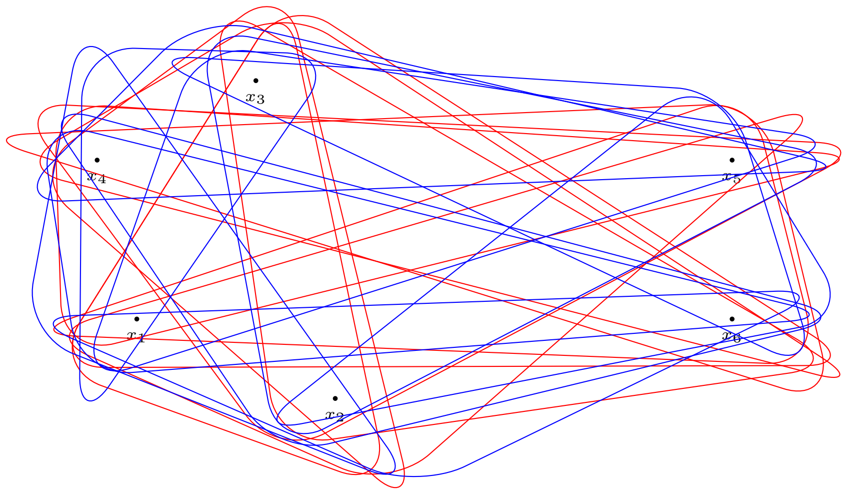

Figure 8: $\mathcal{Q}$

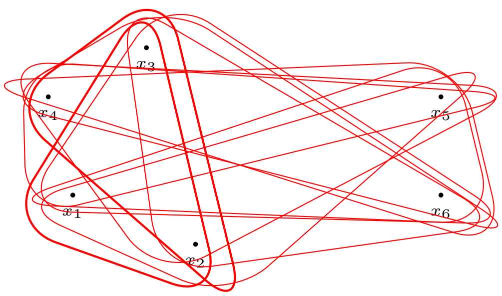

Figure 9: 1^{st} color class of K_6^3

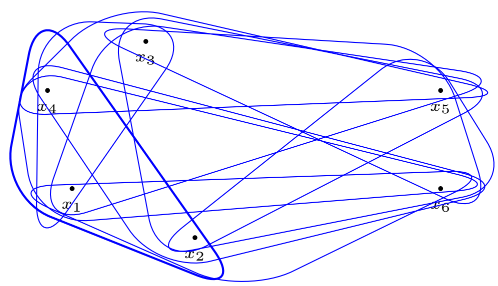

Figure 10: 2^{nd} color class of K_6^3

123, **234**, 346, 456, 125, 136, 145, 156, 245, 236

124, **134**, 126, 135, 146, 235, 246, 256, 345, 356

In the above example, there are some conditions that we need to satisfy first before we can do the extension. For example, if we add four more vertices instead of two vertices, the degree sum of the vertices would be $rn = 5(8) = 40$. Since every edge will contribute three vertices to the degree sum, we should have that $h|rn$; however, here, $3 \nmid 40$. Thus we will not be able to do the extension. Baranyai proved this necessary condition for a hypergraph to r-factorable in 1977.

Theorem I.1.1. *K_n^h is r-factorizable if and only if*

$$h|rn \qquad \text{and} \qquad \frac{rn}{h}\Big|\binom{n}{h}$$

In this thesis, we intend to generalize the above result for $h \in \{6, 7\}$. In chapters 3 and 4, we will answer the following two questions:

1. What are the conditions such that a partial r-factorization of λK_m^6 can be extended to an r-factorization of λK_n^6?

2. What are the conditions such that a partial r-factorization of λK_m^7 can be extended to an r-factorization of λK_n^7?

We also found the bound on n for $h \in \{6, 7, ..., 89\}$.

I.2 Notation and Tools

This thesis will use the amalgamation and detachment technique and greedy coloring. The visual illustration of the technique can be seen in example I.1, and for the detailed information regarding the Detachment theorem we refer the readers to [23]. Greedy coloring is a technique of coloring where we assign the first available colors to the edges sequentially so that no adjacent edges have the same colors. However, if the adjacent edges have the same color, we assign the next available color. For more definitions and algorithms regarding greedy coloring on hypergraphs, we refer the reader to [20] and [21].

The following corollary is an immediate consequence of the Detachment theorem from [1] which is important for our result.

Corollary I.2.1. *Let* $k := \binom{n-1}{h-1}/r \in \mathbb{N}$. *A partial r-factorization of* K_m^h *can be extended to an r-factorization of* K_n^h *if and only if the new edges of* $\mathcal{F} := \widetilde{K_m^h}$ *can be colored so that*

$$\forall i \in [k] \qquad deg_{\mathcal{F}(i)}(v) = \begin{cases} r & if \quad v \neq u \\ \\ r(n-m) & if \quad v = u \end{cases}$$

Where $\mathcal{F} := \widetilde{K_m^h}$ is an amalgamation of K_n^h.

For the proof of the corollary I.2.1 we refer the reader to [1]. We will also use the following combinatorial identities:

Lemma I.2.2. $\binom{n-1}{h-1} = \sum_{i=0}^{h-1} \binom{m-1}{h-i-1}\binom{n-m}{i}$

Lemma I.2.3. $m\left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right] = \sum_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i}$

For the proof of these lemmas we refer the reader to [1] and [15].

CHAPTER II: LITERATURE REVIEW

The problem of hypergraph coloring is a generalization of graph coloring, which researchers have studied for almost 150 years. In 1850, Sylvester [11] proved that K_{15}^3 is 1-factorizable. In 1936, Peltesohn [12] proved a more general case that K_{3m}^3 is 1-factorizable. More than 120 years after Sylvester's theorem, Baranyai proved the generalization of the factorization problem in 1976 by finding the condition in which K_n^h is 1-factorizable.

Theorem II.0.1 (Baranyai, 1976)**.** *If $h|n$ then K_n^h is 1-factorizable.*

Baranyai generalized this theorem to an arbitrary r factorization. He proved the necessary and sufficient condition in which K_n^h is r-factorizable (see I.1.1). This generalization led researchers to more general problems about hypergraphs, especially embedding. Cameron in [22] asked when a partial 1-factorization of K_m^h can be extended to a 1-factorization of K_m^h. In an attempt to answer this question, Baranyai and Brouwer in [13] conjectured that it is possible if and only if $h|n, h|m$ and $n \geqslant 2m$. Häggkvist and Hellgren in [17] proved this conjectured.

Theorem II.0.2 (Häggkvist and Hellgren, 1993)**.** *Let $n = qh$ and $m = ph$ where $q \geqslant 2p$ and $m, n, h, p, q \in \mathbb{Z}^+$. Suppose we are given a proper edge-coloring of K_m^h using $\binom{m-1}{h-1}$ colours. Then this coloring can be extended to a proper edge-coloring of K_n^h using $\binom{n-1}{h-1}$ colors.*

In 1995, Rodger and Wantland [9] extended this theorem to an arbitrary r-factorization in the case of $h = 2$. The same problem for different values of h has also been partially solved. Bahmanian and Rodger in [2] proved the necessary and sufficient condition for $n \geqslant 3.414214m$ when $h = 3$ that any partial r-factorization of K_m^3 can be embedded to an r-factorization of K_n^3. Also, Bahmanian in [1] solved for $n \geqslant 4.847323m$ when $h = 4$ and $n \geqslant 6.285214m$ when $h = 5$.

Theorem II.0.3 (Bahmanian, 2018)**.** *For $n \geqslant 4.847323m$, any partial r-factorization of K_m^4 can be embedded to an r-factorization of K_n^4 if and only if $4|rn$ and $r|\binom{n-1}{3}$.*

Theorem II.0.4 (Bahmanian, 2018). *For $n \geqslant 6.285214m$, any partial r-factorization of K_m^5 can be embedded to an r-factorization of K_n^5 if and only if $5|rn$ and $r|\binom{n-1}{4}$.*

These two results above motivate our result in this thesis. In addition, in the papers [1],[2],[4], [5], [6], [9], [15] the authors used amalgamation and detachment method to prove their results, which is a method that we adapt to prove our results in this thesis.

Bahmanian and Newman in [6] proved a more general result of the problem for $h = 3$ where in addition of the conditions given by Häggkvist and Hellgren, if $\gcd(m, n, h) = \gcd(n, h)$, then r-factorization of K_m^h can be extended to an r-factorization of K_n^h. Furthermore, they also showed that r-factorization of K_m^h can be extended to an s-factorization of K_n^h.

Theorem II.0.5 (Bahmanian and Newman,[6] 2018). *Let m, n, h, r, s, where $m, n, h, r, s \in \mathbb{Z}^+$, satisfy the following necessary conditions for some positive integers p, q, c, d.*

$$h|rm, \qquad h|rn, \qquad r\Big|\binom{m-1}{h-1}, \qquad s\Big|\binom{n-1}{h-1}.$$

Assume furthermore the following condition, where $k = gcd(m, n, h)$.

$$gcd(m,n,h) = gcd(n,h), \quad n \geqslant 2m, \quad 1 \leqslant \frac{s}{r} \leqslant \frac{m}{k}\left[1 - \binom{m-k}{h}\Big/\binom{m}{h}\right]$$

Then there exists an s-factorization of K_n^h containing an embedded r-factorization of K_m^h.

Bahmanian and Haghshenas in [5] proved this theorem for $h = 4, n \geqslant 4m$ and $h = 5, n \geqslant 5m$. Recently in 2021, Bahmanian and Johnsen found a general bound for when a partial **r**-factorization of λK_m^h can be extended to an r-factorization of λK_m^h.

Theorem II.0.6 (Bahmanian and Johnsen, 2021). *For $n \geqslant (h-1)(2m-1)$, a partial **r**-factorization of λK_m^h can be extended to an r-factorization of λK_m^h if and only if it satisfies:*

$$h|rn \qquad \text{and} \qquad r|\lambda\binom{n-1}{h-1}.$$

We used this general bound in comparison to the bound that we found using Mathematica and showed that the bound we found is smaller for $h \in \{6, 7, \ldots, 89\}$. The result can be found in the Appendix, where our bound is on the third column, and Bahmanian and Johnsen's bound is on the fifth column for every h value.

The problem of extending hypergraphs is analogous to extending Latin Squares. In 1951, Ryser's [8] found a necessary and sufficient condition for a partial $r \times s$ Latin rectangle to be completed to an $n \times n$ Latin square. Evans [7] proved that any partial $m \times m$ Latin square can always be extended to a $n \times n$ Latin square when $n \geqslant 2m$. Evan's result is similar to the problem of extending 1-factorization of K_m^h to 1-factorization of K_n^h, when $h = 2$, which was conjectured by Baranyai and Brouwer in [13] and proved by Häggkvist and Hellgren in [17].

Theorem II.0.7 (Ryser, [8] Theorem 2, 1951)**.** *Let T be an $r \times s$ Latin rectangle on the integers 1,2,...,n. Let $\mathcal{N}(i)$ denote the number of times that the integer i occurs in T. T may be extended to an $n \times n$ Latin square if and only if;*

$$\mathcal{N}(i) \geqslant r + s - n, \qquad \textit{for each} \;\; i = 1, 2, ..., n.$$

Lindner and Rodger [16] used edge coloring to prove the sufficient condition of Ryser's theorem, where they constructed an analog of a bipartite hypergraph $K_{r,s}$ for an $r \times s$ Latin rectangle.

CHAPTER III: EXTENDING EDGE-COLORING OF A COMPLETE 6-UNIFORM HYPERGRAPH

In this chapter, we find the necessary and sufficient conditions in which a partial r-factorization of λK_m^6 could be extended to an r-factorization of λK_n^6. This proof is adapted, and a further case of [1].

Theorem III.0.1. *For $n \geqslant 7.72503m$, any partial r-factorization of λK_m^6 can be extended to an $r-$factorization of λK_n^6 if and only if $6|rn$ and $r|\lambda\binom{n-1}{5}$.*

Proof of necessity. Suppose that any partial r-factorization of λK_m^6 can be extended to an r-factorization of λK_n^6. This implies that λK_n^6 is r-factorable. Hence, each vertex must appear r times in each color class. Since $h = 6$, each edge will contain 6 vertices. Thus we should have that $h|rn$. Since each vertex appears r times in each color class, it is regular. This mean $\deg_i(v) = r, \forall i \in [k]$ and since the total degree of each vertex is $\lambda\binom{n-1}{h-1}$, hence we should have $r|\lambda\binom{n-1}{h-1}$ so that each vertices are regular. Therefore, we have that $6|rn$ and $r|\lambda\binom{n-1}{5}$. □

Proof of sufficiency. Suppose $6|rn$ and $r|\binom{n-1}{5}$ and a partial r-factorization of $\mathcal{G} := K_m^6$ is given. We want to show that a partial r-factorization of $\mathcal{G}$ can be extended to an r-factorization of K_n^6.

Let $\mathcal{F}$ be an amalgamated graph obtained by adding an amalgamated vertex u, with $|u| = n - m$, so that $V(\mathcal{F}) = V(\mathcal{G}) \cup \{u\}$ and $E(\mathcal{F})$ be the union of $E(\mathcal{G})$ and the new edges between $N \subseteq V(\mathcal{G})$ and the vertex u, with, $|N| = h - j = 6 - j$ and $|u| = j$, with $j \in \{0, ..., 6\}$. Each edge type in $\mathcal{F}$ will be of the form (v^{6-j}, u^j), where each edge will contain $6 - j$ vertices from $V(\mathcal{F})$ and j copies of u, such that;

$$\text{mult}_{\mathcal{F}}(N, u^j) = \lambda\binom{n-m}{j}$$

Note that there are $\binom{m}{6-j}\binom{n-m}{j}$ edges in each type of edge. By corollary I.2.1, we could extend a partial r-factorization of $\mathcal{G}$ to r-factorization of K_n^6 if and only if we could color the new edges of $\mathcal{F}$ such that the degree conditions in the corollary I.2.1 can be satisfied using $k = \frac{\lambda\binom{n-1}{h-1}}{r} = \frac{\lambda\binom{n-1}{5}}{r}$ colors. In other words, we need to show that the degree of each vertex in $\mathcal{G}$ is r, and the degree of u is equal to $r(n-m)$.

Since the original edges from $\mathcal{G}$ has been colored, we only need to color the new added edges such that $\deg_i(v) = r$ and $\deg(u) = r(n-m)$, for each $v \in V(\mathcal{G})$ with $i \in [k]$. We will color these edges greedily and in order based on each edge's type. Notice that when $|N| = 6$, the edges would only contain vertices from $V(\mathcal{G})$ and when $|N| = 0$, the edges would have no vertices from $V(\mathcal{G})$ and only contain 6 copies of u.

We claim that by greedily coloring these edges so that $\deg_i(v) \leqslant r$ for each $v \in V(\mathcal{G})$ where $i \in [k]$, all edges will be colored. Suppose otherwise. Suppose that there exist an edge that are incident with the vertex $v \in N$ that cannot be colored, meaning that there is a vertex $v \in N$ that already has degree r in each color class i, $\deg_i(v) = r$. Hence, the degree sum of the vertices for each color class $i \in [k]$ is $\sum_{v\in N} \deg_i(v) \geqslant r$. As a result,

$$\sum_{i=1}^{k}\sum_{v\in N} \deg_{\mathcal{G}(i)}(v) \geqslant rk = \lambda\binom{n-1}{h-1} = \lambda\binom{n-1}{5} \tag{III.1}$$

However, we could also count the degree sum using the number of edges and its multiplicity. Hence;

$$\sum_{i=1}^{k}\sum_{v\in N} \deg_{\mathcal{G}(i)}(v) \leqslant \lambda(h-j)\left[\sum_{\ell=0}^{j}\binom{n-m}{h-\ell-1}\binom{m-1}{\ell} - 1\right] \tag{III.2}$$

We need to subtract one at the end since we assume that there exists one edge of each type that cannot be colored, and we also remove one from m since we are assuming that there is $v \in N$ that already has a degree equal to r. We need to multiply $\lambda(h-j)$ since there are $(h-j)$ copies of this type of edge and each edge has multiplicity equal to λ.

Combining III.1 and III.2 we have,

$$\lambda(h-j)\left[\sum_{l=0}^{j}\binom{n-m}{h-\ell-1}\binom{m-1}{\ell} - 1\right] \geqslant \lambda\binom{n-1}{5}$$

$$\iff (h-j)\left[\sum_{l=0}^{j}\binom{n-m}{h-\ell-1}\binom{m-1}{\ell}-1\right] \geqslant \binom{n-1}{5} \tag{III.3}$$

To prove that our claim holds, we need to show that this inequality is not true for every type of edges that contain $v \in N$. Hence, we will prove this for each type of edge successively.

Firstly, edges type 1: $N \cup \{u\}, |N| = 5$. Let's denote this type of edges as x_1

$$\begin{aligned}
&5\left[\binom{m-1}{5}+(n-m)\binom{m-1}{4}-1\right] \geqslant \binom{n-1}{5}\\
\iff &\binom{n-1}{5}-5\left[\binom{m-1}{5}+(n-m)\binom{m-1}{4}-1\right] \leqslant 0\\
\iff &\sum_{i=0}^{5}\binom{m-1}{5-i}\binom{n-m}{i}-5\left[\binom{m-1}{5}+(n-m)\binom{m-1}{4}-1\right] \leqslant 0\\
\iff &\binom{m-1}{5}+\binom{m-1}{4}(n-m)+\binom{m-1}{3}\binom{n-m}{2}+\binom{m-1}{2}\binom{n-m}{3}\\
&\quad +(m-1)\binom{n-m}{4}+\binom{n-m}{5}-5\left[\binom{m-1}{5}+(n-m)\binom{m-1}{4}-1\right] \leqslant 0\\
\iff &\binom{m-1}{2}\binom{n-m}{3}+\binom{m-1}{1}\binom{n-m}{4}+\binom{n-m}{5}+\binom{m-1}{3}\binom{n-m}{2}\\
&\quad -4\binom{m-1}{5}-4(n-m)\binom{m-1}{4}+5 \leqslant 0\\
\iff &\binom{m-1}{2}\binom{n-m}{3}+\binom{m-1}{1}\binom{n-m}{4}+\binom{n-m}{5}\\
&\quad +\binom{m-1}{3}\left(\binom{n-m}{2}-4\left(\frac{m-4}{4}\right)\left(\frac{m-5}{5}\right)-4(n-m)\left(\frac{m-4}{4}\right)\right)+5 \leqslant 0\\
\iff &\binom{m-1}{2}\binom{n-m}{3}+\binom{m-1}{1}\binom{n-m}{4}+\binom{n-m}{5}\\
&\quad +\binom{m-1}{3}\Bigg(\frac{1}{2}(m^2-2mn+m+n^2-n)-\frac{1}{5}(m^2-9m+20)\\
&\quad -(-m^2+mn+4m-4n)\Bigg)+5 \leqslant 0
\end{aligned}$$

$$\begin{aligned}
\iff & \binom{m-1}{2}\binom{n-m}{3} + \binom{m-1}{1}\binom{n-m}{4} + \binom{n-m}{5} + \binom{m-1}{3} \\
& \left(\frac{1}{10}\big(5m^2 - 10mn + 5m + 5n^2 - 5n - 2m^2 + 18m - 40 + 10m^2 - 10mn \right. \\
& \left. - 40m + 40n\big)\right) + 5 \leqslant 0 \\
\iff & \binom{m-1}{2}\binom{n-m}{3} + \binom{m-1}{1}\binom{n-m}{4} + \binom{n-m}{5} \\
& + \binom{m-1}{3}\left(\frac{1}{10}\left(5n^2 - 35n - 20mn + 13m^2 - 17m - 40\right)\right) + 5 \leqslant 0
\end{aligned}$$

Let;

$$\begin{aligned}
a &:= \frac{1}{10}(5n^2 - 35n - 20mn + 13m^2 - 17m - 40) \\
10a &= 5n^2 - 35n - 20mn + 13m^2 - 17m - 40
\end{aligned}$$

Since $\binom{m-1}{2}\binom{n-m}{3} + \binom{m-1}{1}\binom{n-m}{4} + \binom{n-m}{5} \geqslant 0$ and $\binom{m-1}{3} \geqslant 0$, we only need to check whether $10a \geqslant 0$ to ensure if the inequality hold. Since $n \geqslant 7$ and $m \geqslant 7$, then;

$$\begin{aligned}
10a &:= 5n^2 - 35n - 20mn + 13m^2 - 17m - 40 \\
&\geqslant 5(7m)^2 - 35(7m) - 20m(7m) + 13m^2 - 17m - 40 \\
&= 245m^2 - 245m - 140m^2 + 13m^2 - 17m - 40 \\
&= 118m^2 - 262m - 40 \\
&\geqslant 118(7)^2 - 262(7) - 40 \\
&= 3908 > 0
\end{aligned}$$

Since $10a$ is positive, the inequality does not hold. We also used Mathematica to verify that when $n \geqslant 7m$ and $m \geqslant 7$, there is no real solution if $10a \leqslant 0$. This result implied that a is positive; consequently, our initial inequality does not hold. Thus, all edges of this type can be colored.

Secondly, edges type 2: $N \cup \{u^2\}, |N| = 4$. Let's denote this type of edges as x_2

$$
\begin{aligned}
&4\left[\binom{m-1}{5} + (n-m)\binom{m-1}{4} + \binom{m-1}{3}\binom{n-m}{2} - 1\right] \geqslant \binom{n-1}{5}\\
\iff& \binom{n-1}{5} - 4\left[\binom{m-1}{5} + (n-m)\binom{m-1}{4} + \binom{m-1}{3}\binom{n-m}{2} - 1\right] \leqslant 0\\
\iff& \sum_{i=0}^{5}\binom{m-1}{4-i}\binom{n-m}{i} - 4\left[\binom{m-1}{5} + (n-m)\binom{m-1}{4}\right.\\
&\quad \left. + \binom{m-1}{3}\binom{n-m}{2} - 1\right] \leqslant 0\\
\iff& \binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{1} + \binom{m-1}{3}\binom{n-m}{2} + \binom{m-1}{2}\binom{n-m}{3}\\
&\quad + \binom{m-1}{1}\binom{n-m}{4} + \binom{n-m}{5} - 4\left[\binom{m-1}{5} + (n-m)\binom{m-1}{4}\right.\\
&\quad \left. + \binom{m-1}{3}\binom{n-m}{2} - 1\right] \leqslant 0\\
\iff& \binom{m-1}{2}\binom{n-m}{3} + (m-1)\binom{n-m}{4} + \binom{n-m}{5} - 3\binom{m-1}{5}\\
&\quad - 3(n-m)\binom{m-1}{4} - 3\binom{m-1}{3}\binom{n-m}{2} + 4 \leqslant 0\\
\iff& (m-1)\binom{n-m}{4} + \binom{n-m}{5} + \binom{m-1}{2}\binom{n-m}{3}\\
&\quad - 3\binom{m-1}{2}\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\\
&\quad - 3(n-m)\binom{m-1}{2}\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\\
&\quad - 3\left(\frac{m-3}{3}\right)\binom{m-1}{2}\binom{n-m}{2} + 4 \leqslant 0\\
\iff& (m-1)\binom{n-m}{4} + \binom{n-m}{5} + \binom{m-1}{2}\left(\binom{n-m}{3}\right.\\
&\quad - 3\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right) - 3(n-m)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\\
&\quad \left. - 3\left(\frac{m-3}{3}\right)\binom{n-m}{2}\right) + 4 \leqslant 0
\end{aligned}
$$

$$
\begin{aligned}
\iff (m-1)\binom{n-m}{4} + \binom{n-m}{5} + \binom{m-1}{2}\Bigg(&\binom{n-m}{2}\left(\frac{n-m-2}{3}\right) \\
&- 3\left(\frac{m-3}{3}\right)\binom{n-m}{2} - 3\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right) \\
&- 3(n-m)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\Bigg) + 4 \leqslant 0 \\
\iff (m-1)\binom{n-m}{4} + \binom{n-m}{5} + \binom{m-1}{2}\Bigg(&\binom{n-m}{2}\left(\left(\frac{n-m-2}{3}\right) - 3\left(\frac{m-3}{3}\right)\right) \\
&- 3\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\left(\left(\frac{m-5}{5}\right) - (n-m)\right)\Bigg) + 4 \leqslant 0 \\
\iff (m-1)\binom{n-m}{4} + \binom{n-m}{5} + \binom{m-1}{2}\Bigg(&\binom{n-m}{2}\left(\frac{n-4m+7}{3}\right) \\
&- 3\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\left(\frac{5n-4m-5}{5}\right)\Bigg) + 4 \leqslant 0
\end{aligned}
$$

Let;

$$
\begin{aligned}
b &:= \binom{n-m}{2}\left(\frac{1}{3}(n-4m+7)\right) - 3\left(\frac{1}{4}(m-4)\right)\left(\frac{1}{3}(m-3)\right)\left(\frac{1}{5}(5n-4m-5)\right) \\
&= \left(\frac{1}{2}(n-m)(n-m-1)\right)\left(\frac{1}{3}(n-4m+7)\right) - \left(\frac{1}{4}((m-4)(m-3))\right)\left(\frac{1}{5}(5n-4m-5)\right) \\
&= \frac{1}{2}\left(n^2 - 2nm - n + m + m^2\right)\left(\frac{1}{3}(n-4m+7)\right) \\
&\quad - \frac{1}{20}\left(5nm^2 - 4m^3 + 23m^2 - 35nm - 13m + 60n + 60\right) \\
&= \frac{1}{6}\left(-4m^3 + 9m^2n + 3m^2 - 6mn^2 - 9mn + 7m + n^3 + 6n^2 - 7n\right) \\
&\quad - \frac{1}{20}\left(5nm^2 - 4m^3 + 23m^2 - 35nm - 13m + 60n + 60\right) \\
&= \frac{1}{60}\left(-28m^3 + 75nm^2 - 39m^2 - 60n^2m + 15nm + 109m + 10n^3 + 60n^2 - 250n + 420\right)
\end{aligned}
$$

Since $(m-1)\binom{n-m}{4} + \binom{n-m}{5} \geqslant 0$, and $\binom{m-1}{2} \geqslant 0$ we only need to check whether $b \geqslant 0$ to ensure if the inequality hold. Since we have that $n \geqslant 7m$ and $m \geqslant 7$, then

$$
\begin{aligned}
b &:= \frac{1}{60}\left(-28m^3 + 75nm^2 - 39m^2 - 60n^2m + 15nm + 109m + 10n^3 + 60n^2 - 250n + 420\right) \\
&\geqslant \frac{1}{60}(-28m^3 + 75(7m)m^2 - 39m^2 - 60(7m)^2m + 15(7m)m + 109m + 10(7m)^3 + 60(7m)^2 \\
&\quad - 250(7m) + 420) \\
&= \frac{1}{60}((-28m^3 + 525m^3 - 39m^2 - 2940m^3 + 105m^2 + 109m + 3430m^3 + 2940m^2 \\
&\quad - 1750m + 420) \\
&= \frac{1}{60}(987m^3 + 3006m^2 - 1641m + 420) \geqslant 0
\end{aligned}
$$

Since $m \geqslant 7$, and $3006m^2 > 1641m$, then $b \geqslant 0$; consequently, our initial inequality does not hold. Thus, all edges of this type can be colored.

Thirdly, edges Type 3: $N \cup \{u^3\}, |N| = 3$. Let's denote this type of edges as x_3

$$
\begin{aligned}
&3\left[\binom{m-1}{5} + (n-m)\binom{m-1}{4} + \binom{m-1}{3}\binom{n-m}{2} + \binom{m-1}{2}\binom{n-m}{3} - 1\right] \\
&\qquad \leqslant \binom{n-1}{5} \\
\Longleftrightarrow\ & \binom{n-1}{5} - 3\bigg[\binom{m-1}{5} + (n-m)\binom{m-1}{4} + \binom{m-1}{3}\binom{n-m}{2} \\
&\qquad + \binom{m-1}{2}\binom{n-m}{3} - 1\bigg] \leqslant 0 \\
\Longleftrightarrow\ & \sum_{i=0}^{5}\binom{m-1}{5-i}\binom{n-m}{i} - 3\bigg[\binom{m-1}{5} + (n-m)\binom{m-1}{4} + \binom{m-1}{3}\binom{n-m}{2} \\
&\qquad + \binom{m-1}{2}\binom{n-m}{3} - 1\bigg] \leqslant 0 \\
\Longleftrightarrow\ & \binom{m-1}{5}\binom{n-m}{0} + \binom{m-1}{4}\binom{n-m}{1} + \binom{m-1}{3}\binom{n-m}{2} + \binom{m-1}{2}\binom{n-m}{3} \\
&\qquad + \binom{m-1}{1}\binom{n-m}{4} + \binom{m-1}{0}\binom{n-m}{5} - 3\bigg[\binom{m-1}{5} + (n-m)\binom{m-1}{4} \\
&\qquad + \binom{m-1}{3}\binom{n-m}{2} + \binom{m-1}{2}\binom{n-m}{3} - 1\bigg] \leqslant 0 \\
\Longleftrightarrow\ & (m-1)\binom{n-m}{4} + \binom{n-m}{5} - 2\binom{m-1}{5} - 2(n-m)\binom{m-1}{4} \\
&\qquad - 2\binom{m-1}{3}\binom{n-m}{2} - 2\binom{m-1}{2}\binom{n-m}{3} + 3 \leqslant 0 \\
\Longleftrightarrow\ & (m-1)\binom{n-m}{4} - 2\binom{m-1}{5} - 2(n-m)\binom{m-1}{4} + \binom{n-m}{5} \\
&\qquad - 2\binom{m-1}{3}\binom{n-m}{2} - 2\binom{m-1}{2}\binom{n-m}{3} + 3 \leqslant 0 \\
\Longleftrightarrow\ & (m-1)\Bigg(\binom{n-m}{4} - 2\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\left(\frac{m-2}{2}\right) \\
&\qquad - 2(n-m)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\left(\frac{m-2}{2}\right)\Bigg) \\
&\qquad + \binom{n-m}{2}\Bigg(\left(\frac{n-m-4}{5}\right)\left(\frac{n-m-3}{4}\right)\left(\frac{n-m-2}{3}\right) - 2\binom{m-1}{3} \\
&\qquad - 2\binom{m-1}{2}\left(\frac{n-m-2}{3}\right)\Bigg) + 3 \leqslant 0
\end{aligned}
$$

$$
\begin{aligned}
\iff (m-1)\left(\binom{n-m}{4} - 2\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\left(\frac{m-2}{2}\right)\left(\left(\frac{m-5}{5}\right) + (n-m)\right)\right) \\
+ \binom{n-m}{2}\left(\left(\frac{n-m-4}{5}\right)\left(\frac{n-m-3}{4}\right)\left(\frac{n-m-2}{3}\right)\right. \\
\left. - 2\binom{m-1}{2}\left(\frac{m-3}{3} + \frac{n-m-2}{3}\right)\right) + 3 \leqslant 0 \\
\iff (m-1)\left(\binom{n-m}{4} - 2\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\left(\frac{m-2}{2}\right)\left(\frac{5n-4m-5}{5}\right)\right) + \binom{n-m}{2} \\
\left(\left(\frac{n-m-4}{5}\right)\left(\frac{n-m-3}{4}\right)\left(\frac{n-m-2}{3}\right) - 2\binom{m-1}{2}\left(\frac{n-5}{3}\right)\right) + 3 \leqslant 0
\end{aligned}
$$

Let;

$$
\begin{aligned}
c &:= \binom{n-m}{4} - 2\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\left(\frac{m-2}{2}\right)\left(\frac{5n-4m-5}{5}\right) \\
d &:= \left(\frac{n-m-4}{5}\right)\left(\frac{n-m-3}{4}\right)\left(\frac{n-m-2}{3}\right) - 2\binom{m-1}{2}\left(\frac{n-5}{3}\right)
\end{aligned}
$$

Since $m-1 \geqslant 0$ and $\binom{n-m}{2} \geqslant 0$, we only need to check c and d, if both are positive then this inequality does not hold.

$$
\begin{aligned}
c &= \frac{1}{4!}(n-m)(n-m-1)(n-m-2)(n-m-3) - \frac{2}{5!}(m-4)(m-3)(m-2)(5n-4m-5) \\
&= \frac{1}{4!}(n^2 - nm - n - nm + m^2 + m)(n^2 - nm - 3n - nm + m^2 + 3m - 2n + 2m + 6) \\
&\quad - \frac{2}{5!}(m^2 - 7m + 12)(5nm - 4m^2 - 5m - 10n + 8m + 10) \\
&= \frac{1}{4!}(n^4 - 4n^3m - 6n63 + 6n^2m^2 + 18n^2m + 11m^2 + 6m + m^4) \\
&\quad - \frac{1}{5!}(-10nm^3 + 8m^4 - 62m^3 + 90nm^2 + 118m^2 - 260nm + 68m + 240n - 240) \\
&= \frac{1}{5!}(5n^4 - 20n^3m - 30n^3 + 30n^2m^2 + 90n^2m + 55n^2 - 30nm^3 - 370nm + 210n - 32m^3 \\
&\quad + 173m^2 + 98m + 13m^4 - 240)
\end{aligned}
$$

And using Mathematica to expand d we have;

$$d = \frac{1}{60}(176 - 326m + 91m^2 - m^3 - 14n + 78mn - 17m^2n - 9n^2 - 3mn^2 + n^3)$$

By using Mathematica, we have verified that the inverse of these two inequalities does not have any real solution when $m \geqslant 7$ and $n \geqslant 7.72503m$, implying that these two equations will be positive when $m \geqslant 7$ and $n \geqslant 7.72502396m$. Thus, the initial inequality does not hold, which is a contradiction. Therefore, our claim holds, and all edges of this type can be colored using greedy coloring.

Fourthly, edges type 4: $N \cup \{u^4\}, |N| = 2$. Let's denote this type of edges as x_4

$$\begin{aligned}
&2\Big[\binom{m-1}{5} + (n-m)\binom{m-1}{4} + \binom{m-1}{3}\binom{n-m}{2} + \binom{m-1}{2}\binom{n-m}{3} \\
&\qquad + \binom{m-1}{1}\binom{n-m}{4} - 1\Big] \geqslant \binom{n-1}{5} \\
\iff &\binom{n-1}{5} - 2\Big[\binom{m-1}{5} + (n-m)\binom{m-1}{4} + \binom{m-1}{3}\binom{n-m}{2} + \binom{m-1}{2}\binom{n-m}{3} \\
&\qquad + \binom{m-1}{1}\binom{n-m}{4} - 1\Big] \leqslant 0 \\
\iff &\binom{n-1}{5} - 2\Big[\binom{m-1}{5} + (n-m)\binom{m-1}{4} + \binom{m-1}{3}\binom{n-m}{2} \\
&\qquad + \binom{m-1}{2}\binom{n-m}{3} + \binom{m-1}{1}\binom{n-m}{4} + \binom{n-m}{5} - \binom{n-m}{5} - 1\Big] \leqslant 0 \\
\iff &\binom{n-1}{5} - 2\Big[\sum_{i=0}^{5}\binom{m-1}{5-i}\binom{n-m}{i} - \binom{n-m}{5} + 1\Big] \leqslant 0 \\
\iff &\binom{n-1}{5} - 2\Big[\binom{n-1}{5} - \binom{n-m}{5} + 1\Big] \leqslant 0 \\
\iff &\binom{n-1}{5} - 2\binom{n-1}{5} + 2\binom{n-m}{5} - 2 \leqslant 0 \\
\iff &2\binom{n-m}{5} - \binom{n-1}{5} - 2 \leqslant 0
\end{aligned}$$

By using Mathematica, we verified that this inequality does not have any real solution when $m \geqslant 7$ and $n \geqslant 7.72503m$. It is a contradiction. Therefore, every edge of this type can be colored.

Fifthly, edges Type 5: $N \cup \{u^5\}, |N| = 1$. Let's denote this type of edges as x_5. This is the last type of edge that contains a vertex $v \in N$. We would like to show that $\deg_i(v) = r, v \in N$ and since this type of edges would be the last type of edges that contained a vertex $v \in N$, each vertex must be contained in $r - \deg_i(v)$ number of edges of each color $i \in [k]$. Thus,

$$
\begin{aligned}
\sum_{i=1}^{k}(r - \deg_i(v)) &= \sum_{i=1}^{k} r - \sum_{i=1}^{k} \deg_i(v) \\
&= rk - \deg(v) \\
&= \lambda\binom{n-1}{5} - \sum_{\ell=1}^{5} \lambda\binom{n-m}{6-\ell-1}\binom{m-1}{\ell} \\
&= \lambda\sum_{\ell=0}^{5}\binom{n-m}{6-\ell-1}\binom{m-1}{\ell} - \lambda\sum_{\ell=1}^{5}\binom{n-m}{6-\ell-1}\binom{m-1}{\ell} \\
&= \lambda\left(\sum_{\ell=0}^{5}\binom{n-m}{6-\ell-1}\binom{m-1}{\ell} - \sum_{\ell=1}^{5}\binom{n-m}{6-\ell-1}\binom{m-1}{\ell}\right) \\
&= \lambda\left(\binom{n-m}{5} + \sum_{\ell=1}^{5}\binom{n-m}{6-\ell-1}\binom{m-1}{\ell} - \sum_{\ell=1}^{5}\binom{n-m}{6-\ell-1}\binom{m-1}{\ell}\right) \\
&= \lambda\binom{n-m}{5}
\end{aligned}
$$

Hence, we can color all edges of this type using all available k colors, and the degree of every vertex from $\mathcal{G}$ is equal to r.

Lastly, the last type of edges that we need to check is the type of edges that only contain the newly added $n - m$ vertices. Let's denote this last type of edge as x_6. We have colored all the other type edges and verified that $\deg_i(v) = r, v \in N$. Next, the degree of every vertex of the newly added vertices has to equal r as well. Up to this point, all these vertices have been contained in every other type of edge.

Recall that:

$$
\begin{aligned}
rm &= 6x_{0_i} + 5x_{1_i} + 4x_{2_i} + 3x_{3_i} + 2x_{4_i} + x_{5_i} \\
rn - 6rm &= 6x_{0_i} + 6x_{1_i} + 6x_{2_i} + 6x_{3_i} + 6x_{4_i} + 6x_{5_i} + 6x_{6_i} - 36x_{0_i} - 30x_{1_i} \\
&\quad - 24x_{2_i} - 18x_{3_i} - 12x_{4_i} - 6x_{5_i} \\
&= -30x_{0_i} - 24x_{1_i} - 18x_{2_i} - 12x_{3_i} - 6x_{4_i} + 6x_{6_i} \\
6x_{6_i} &= rn - 6rm + 30x_{0_i} + 24x_{1_i} + 18x_{2_i} + 12x_{3_i} + 6x_{4_i}
\end{aligned}
$$

Thus for each color class $i \in [k]$, we have this many edges of this type;

$$x_{6_i} := \frac{rn}{6} - rm + 5x_{5_i} + 4x_{4_i} + 3x_{3_i} + 2x_{2_i} + 1x_{1_i}$$

For the notation, in here x_{6_i} means the edge type 6 in color class i, where $i \in [k]$. Recall that we are given the conditions that $6|rn$, $n \geqslant 7m \geqslant 6m$. Hence, x_{6_i} will be a positive integer. We will show that we could color all edges of this type by showing that the number of edges will equal to $\binom{n-m}{6}$.

Edges Type 6: $\{u^6\}$.

$$\begin{aligned}
\sum_{i=1}^{k} x_{6_i} &= \sum_{i=1}^{k} \frac{rn}{6} - \sum_{i=1}^{k} rm - \sum_{i=1}^{k} 5x_{0_i} + \sum_{i=1}^{k} 4x_{1_i} + \sum_{i=1}^{k} 3x_{2_i} + \sum_{i=1}^{k} 2x_{3_i} + \sum_{i=1}^{k} x_{4_i} \\
&= \frac{n}{6}(rk) - m(rk) + 5\lambda\binom{m}{6} + 4\lambda\binom{m}{5}\binom{n-m}{1} + 3\lambda\binom{m}{4}\binom{n-m}{2} - 2\lambda\binom{m}{3}\binom{n-m}{3} \\
&\quad + \lambda\binom{m}{2}\binom{n-m}{4} \\
&= \lambda\frac{n}{6}\binom{n-1}{5} - \lambda m\binom{n-1}{5} + \left(\lambda\sum_{i=0}^{5}(h-i)\binom{m}{6-i}\binom{n-m}{i} - \sum_{i=1}^{5}\lambda\binom{m}{6-i}\binom{n-m}{i}\right) \\
&= \lambda\binom{n-1}{6} - \lambda m\binom{n-1}{5} + \left(\lambda\sum_{i=0}^{5}(h-i)\binom{m}{6-i}\binom{n-m}{i} - \sum_{i=1}^{5}\lambda\binom{m}{6-i}\binom{n-m}{i}\right) \\
&= \lambda\sum_{i=1}^{6}\binom{m}{6-i}\binom{n-m}{i} - \lambda m\binom{n-1}{5} + \left(6\lambda\binom{m}{6} + \lambda\sum_{i=1}^{5}(h-i)\binom{m}{6-i}\binom{n-m}{i}\right. \\
&\quad \left. - \lambda\sum_{i=1}^{5}\binom{m}{6-i}\binom{n-m}{i}\right) \\
&= \lambda\binom{n-m}{6} + \lambda\sum_{i=1}^{5}\binom{m}{6-i}\binom{n-m}{i} - \lambda m\binom{n-1}{5} + 6\lambda\binom{m}{6} + \lambda m\left[\binom{n-1}{6-1} - \binom{m-1}{6-1}\right] \\
&\quad - \lambda\sum_{i=1}^{5}\binom{m}{6-i}\binom{n-m}{i} \\
&= \lambda\binom{n-m}{6} + \lambda\sum_{i=1}^{5}\binom{m}{6-i}\binom{n-m}{i} - \lambda m\binom{n-1}{5} + 6\lambda\left(\frac{m}{6}\right)\binom{m-1}{5} + \lambda m\binom{n-1}{5} \\
&\quad - \lambda m\binom{m-1}{5} - \lambda\sum_{i=1}^{5}\binom{m}{6-i}\binom{n-m}{i} \\
&= \lambda\binom{n-m}{6} + \lambda m\binom{m-1}{5} - \lambda m\binom{m-1}{5} \\
&= \lambda\binom{n-m}{6}
\end{aligned}$$

Last step would be to check the degree of every vertex. We have verified that $\deg_i(v) = r, \forall v \in N$. Next, we need to show that $\deg_i(u) = r(n-m)$ for $i \in [k]$.

Recall that $\sum_{v \in \mathcal{V}} \deg(v) = rm = 6x_0 + 5x_1 + 4x_2 + 3x_3 + 2x_4 + 1x_5$

Hence,

$$\begin{aligned}\deg(u) &= x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \\ &= (6x_0 + 6x_1 + 6x_2 + 6x_3 + 6x_4 + 6x_5 + 6x_6) - (6x_0 + 5x_1 + 4x_2 + 3x_3 + 2x_4 + 1x_5) \\ &= rn - rm = r(n-m)\end{aligned}$$

$\square$

CHAPTER IV: EXTENDING EDGE-COLORING OF A COMPLETE 7-UNIFORM HYPERGRAPH

In this chapter, we will show the necessary and sufficient conditions in which a partial r-factorization of λK_m^7 could be extended to an r-factorization of λK_n^7. This proof is adapted, and a further case of [1].

Theorem IV.0.1. *For $n \geqslant 9.16580m$, any partial r-factorization of λK_m^7 can be extended to an $r-$factorization of λK_n^7 if and only if $7|rn$ and $r|\lambda\binom{n-1}{7}$.*

Proof of necessity. Suppose that any partial r-factorization of λK_m^7 can be extended to an r-factorization of λK_n^7. This implies that λK_n^7 is r-factorable. Hence, each vertex needs to appear r times in each color class. Since $h = 7$, each edge will contain 7 vertices. Thus we should have that $h|rn$. Since each vertex appear r times in each color class, it is regular. This mean $\deg_i(v) = r, \forall i \in [k]$ and since the total degree of each vertex is $\lambda\binom{n-1}{h-1}$, hence we should have $r|\lambda\binom{n-1}{h-1}$ so that each vertices are regular. Therefore, we have that $7|rn$ and $r|\lambda\binom{n-1}{6}$ □

Proof of sufficiency. Suppose $7|rn$ and $r|\binom{n-1}{6}$ and a partial r-factorization of $\mathcal{G} := K_m^7$ is given. We would like to show that a partial r-factorization of $\mathcal{G}$ can be extended to an r-factorization of K_n^7.

Let $\mathcal{F}$ be an amalgamated graph obtained by adding an amalgamated vertex u, with $|u| = n - m$, so that $V(\mathcal{F}) = V(\mathcal{G}) \cup \{u\}$ and $E(\mathcal{F})$ be the union of $E(\mathcal{G})$ and the new edges between $N \subseteq V(\mathcal{G})$ and the vertex u, with, $|N| = h - j = 7 - j$ and $|u| = j$, with $j \in \{0, ..., 7\}$. Each edge type in $\mathcal{F}$ will be of the form (v^j, u^{7-j}), where each edge will contain j vertices from $V(\mathcal{F})$ and $7 - j$ copies of u, such that;

$$\mathrm{mult}_{\mathcal{F}}(N, u^j) = \lambda\binom{n-m}{j}$$

Note that there are $\binom{m}{7-j}\binom{n-m}{j}$ edges in each type of edge. By corollary I.2.1, we could extend a partial r-factorization of $\mathcal{G}$ to r-factorization of K_n^7 if and only if we could color the new edges of $\mathcal{F}$ such that the degree conditions in the corollary I.2.1 can be satisfied using $k = \frac{\lambda\binom{n-1}{h-1}}{r} = \frac{\lambda\binom{n-1}{6}}{r}$ colors. In other words, we need to show that the degree of each vertex in $\mathcal{G}$ is r, and the degree of u is equal to $r(n-m)$.

Since the original edges from $\mathcal{G}$ has been colored, we only need to color the new added edges such that $\deg_i(v) = r$ and $\deg_i(u) = r(n-m)$, for each $v \in V(\mathcal{G})$ with $i \in [k]$. We are going to color these edges greedily and in order, based on the types of each edge. Notice that when $|N| = 7$, the edges would only contain vertices from $V(\mathcal{G})$ and when $|N| = 0$, the edges would have no vertices from $V(\mathcal{G})$ and only contain 7 copies of u

We claim that by greedily coloring these edges so that $\deg_i(v) \leqslant r$ for each $v \in V(\mathcal{G})$ where $i \in [k]$, all edges will be colored. Suppose otherwise. Suppose there exist an edge that are incident with the vertex $v \in N$ that cannot be colored, meaning that there is a vertex $v \in N$ that already has degree r in each color class i, $\deg_i(v) = r$. Hence, the degree sum of the vertices for each color class $i \in [k]$ is $\sum_{v \in N} \deg_i(v) \geqslant r$. As a result,

$$\sum_{i=1}^{k} \sum_{v \in N} \deg_{\mathcal{F}(i)}(v) \geqslant rk = \lambda\binom{n-1}{h-1} = \lambda\binom{n-1}{6} \tag{IV.1}$$

However,

$$\sum_{i=1}^{k} \sum_{v \in N} \deg_{\mathcal{F}(i)}(v) \leqslant \lambda(h-j)\left[\sum_{\ell=0}^{j} \binom{n-m}{\ell}\binom{m-1}{h-\ell-1} - 1\right] \tag{IV.2}$$

We need to subtract one at the end since we assume that there exists one edge of each type that cannot be colored, and we also take away one from m since we are assuming that there is $v \in N$ that already has a degree equal to r. We need to multiply $\lambda(h-j)$ since there are $(h-j)$ copies of this type of edge, and each edge has multiplicity equal to λ.

Combining IV.1 and IV.2 we have,

$$\lambda(h-j)\left[\sum_{\ell=0}^{j} \binom{n-m}{\ell}\binom{m-1}{h-\ell-1} - 1\right] \geqslant \lambda\binom{n-1}{6}$$

$$\iff (h-j)\left[\sum_{\ell=0}^{j}\binom{n-m}{\ell}\binom{m-1}{h-\ell-1}-1\right] \geqslant \binom{n-1}{6} \tag{IV.3}$$

To prove that our claim holds, we need to show that this inequality is not true for every type of edge that contains $v \in N$. Hence, we are going to prove this for each type of edge successively.

Firstly, edges Type 1: $N \cup u, |N| = 6$. Lets' denote this type of edges as y_1

$$\begin{aligned}
&6\left[\binom{m-1}{6} + (n-m)\binom{m-1}{5} - 1\right] \geqslant \binom{n-1}{6}\\
\iff &\binom{n-1}{6} - 6\left[\binom{m-1}{6} + (n-m)\binom{m-1}{5} - 1\right] \leqslant 0\\
\iff &\sum_{i=0}^{6}\binom{m-1}{6-i}\binom{n-m}{i} - 6\left[\binom{m-1}{6} + (n-m)\binom{m-1}{5} - 1\right] \leqslant 0\\
\iff &\binom{m-1}{6} + \binom{m-1}{5}(n-m) + \binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3}\\
&\quad + \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5} + \binom{n-m}{6} - 6\binom{m-1}{6}\\
&\quad - 6(n-m)\binom{m-1}{5} + 6 \leqslant 0\\
\iff &\binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3} + \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5}\\
&\quad + \binom{n-m}{6} - 5\binom{m-1}{6} - 5(n-m)\binom{m-1}{5} + 6 \leqslant 0\\
\iff &\binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3} + \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5}\\
&\quad + \binom{n-m}{6} - 5\binom{m-1}{5}\left(\frac{m-6}{6}\right) - 5(n-m)\binom{m-1}{5} + 6 \leqslant 0\\
\iff &\binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3} + \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5}\\
&\quad + \binom{n-m}{6} - 5\binom{m-1}{5}\left(\frac{m-6}{6} + n - m\right) + 6 \leqslant 0\\
\iff &\binom{m-1}{3}\binom{n-m}{3} + \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5} + \binom{n-m}{6}\\
&\quad + \binom{m-1}{4}\binom{n-m}{2} - 5\left(\frac{m-5}{5}\right)\binom{m-1}{4}\left(\frac{6n-5m-6}{6}\right) + 6 \leqslant 0
\end{aligned}$$

$$\begin{aligned}
\iff &\binom{m-1}{3}\binom{n-m}{3}+\binom{m-1}{2}\binom{n-m}{4}+(m-1)\binom{n-m}{5}+\binom{n-m}{6}\\
&+\binom{m-1}{4}\Big(\binom{n-m}{2}-5\Big(\frac{m-5}{5}\Big)\Big(\frac{6n-5m-6}{6}\Big)\Big)+6\leqslant 0\\
\iff &\binom{m-1}{3}\binom{n-m}{3}+\binom{m-1}{2}\binom{n-m}{4}+(m-1)\binom{n-m}{5}+\binom{n-m}{6}\\
&+\binom{m-1}{4}\Big(\frac{(n-m)(n-m-1)}{2}-\frac{(m-5)(6n-5m-6)}{6}\Big)+6\leqslant 0
\end{aligned}$$

Let;

$$\begin{aligned}
f := &\frac{1}{2}(n-m)(n-m-1)-\frac{1}{6}(m-5)(6n-5m-6)\\
= &\frac{1}{6}\left(3(n-m)(n-m-1)-(m-5)(6n-5m-6)\right)\\
= &\frac{1}{6}(8m^2+3n^2+27n-12mn-16m-30)\\
\geqslant &\frac{1}{6}(8m^2+3(9m)^2+27(9m)-12m(9m)-16m-30)\\
= &\frac{1}{6}(mn^2+3(9m)^2+27(9m)-12m(9m)-16m-30)\\
= &\frac{1}{6}(8m^2+243m^3+243m-108m^2-16m-30)\\
= &\frac{1}{6}(243m^3-100m^2+227m-30)
\end{aligned}$$

Since $m \geqslant 8$, $243m^3 \geqslant 100m^2$ and $227 \geqslant 30$. Thus, $f \geqslant 0$. This implies that the inequality is positive, which is a contradiction. Therefore, all edges of this type can be colored.

Edges Type 2: $N \cup u, |N| = 5$. Denote this type of edges as y_2.

$$
\begin{aligned}
& 5\Big[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} - 1\Big] \geqslant \binom{n-1}{6} \\
\iff & \binom{n-1}{6} - 5\Big[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} \\
& \quad - 1\Big] \leqslant 0 \\
\iff & \sum_{i=0}^{6}\binom{m-1}{6-i}\binom{n-m}{i} - 5\Big[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} \\
& \quad - 1\Big] \leqslant 0 \\
\iff & \binom{m-1}{6} + \binom{m-1}{5}(n-m) + \binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3} \\
& \quad + \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5} + \binom{n-m}{6} - 5\binom{m-1}{6} \\
& \quad - 5(n-m)\binom{m-1}{5} - 5\binom{m-1}{4}\binom{n-m}{2} + 5 \leqslant 0 \\
\iff & \binom{m-1}{3}\binom{n-m}{3} + \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5} + \binom{n-m}{6} \\
& \quad - 4\binom{m-1}{6} - 4(n-m)\binom{m-1}{5} - 4\binom{m-1}{4}\binom{n-m}{2} + 6 \leqslant 0 \\
\iff & \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5} + \binom{n-m}{6} + \binom{m-1}{3}\binom{n-m}{3} \\
& \quad - 4\left(\frac{m-6}{6}\right)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\binom{m-1}{3} \\
& \quad - 4(n-m)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\binom{m-1}{3} \\
& \quad - 4\left(\frac{m-4}{4}\right)\binom{m-1}{3}\binom{n-m}{2} + 6 \leqslant 0 \\
\iff & \binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5} + \binom{n-m}{6} \\
& \quad + \binom{m-1}{3}\left(\binom{n-m}{3} - 4\left(\frac{m-6}{6}\right)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\right. \\
& \quad \left. - 4(n-m)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right) - 4\left(\frac{m-4}{4}\right)\binom{n-m}{2}\right) + 6 \leqslant 0
\end{aligned}
$$

Let;

$$
\begin{aligned}
s := & \binom{n-m}{3} - 4\left(\frac{m-6}{6}\right)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right) \\
& - 4(n-m)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right) - 4\left(\frac{m-4}{4}\right)\binom{n-m}{2} \\
= & \binom{n-m}{3} - 4\left(\frac{m-4}{4}\right)\binom{n-m}{2} - 4\left(\frac{m-6}{6}\right)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right) \\
& - 4(n-m)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right) \\
= & (n-m)(n-m-1)\left(\frac{n-m-2}{6} - \frac{m-4}{8}\right) - \left(\frac{(m-5)(m-4)}{5}\right)\left(\frac{m-6}{6} + (n-m)\right) \\
= & (n^2 - n - 2mn + m + m^2)\left(\frac{4n-7m+4}{24}\right) - \left(\frac{m^2-9m+20}{5}\right)\left(\frac{6n-5m-6}{6}\right) \\
= & \frac{1}{24}\left(4n^3 - 4n - 15mn^2 + 3mn + 18m^2n + 4m - 3m^2 - 7m^3\right) \\
& - \frac{1}{30}\left(-5m^3 + 39m^2 - 46m + 6m^2n - 54mn + 120n - 120\right) \\
= & \frac{1}{120}(20n^3 - 500n - 75mn^2 + 66m^2n + 231mn - 15m^3 - 171m^2 + 204m + 480)
\end{aligned}
$$

Since $\binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5} + \binom{n-m}{6} \geqslant 0$ and $\binom{m-1}{3} \geqslant 0$, we only need to check whether $s \geqslant 0$ or not to ensure if the inequality hold. Using Mathematica, we verified that $s \leqslant 0$ does not have any real solution when $n \geqslant 9m$ and $m \geqslant 8$. Thus imply that s is positive when $n \geqslant 9m$ and $m \geqslant 8$. This brings us to conclude that the original inequality does not hold; thus, we can color all the edges of this type.

Thirdly, edges Type 3: $N \cup u^2, |N| = 4$. Let's denote this type of edges as y_3

$$
\begin{aligned}
&4\left[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3} - 1\right] \\
&\qquad \geqslant \binom{n-1}{6} \\
\iff &\binom{n-1}{6} - 4\Bigg[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} \\
&\qquad + \binom{m-1}{3}\binom{n-m}{3} - 1\Bigg] \leqslant 0 \\
\iff &\sum_{i=0}^{6}\binom{m-1}{6-i}\binom{n-m}{i} - 4\Bigg[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} \\
&\qquad + \binom{m-1}{3}\binom{n-m}{3} - 1\Bigg] \leqslant 0 \\
\iff &\binom{m-1}{6}\binom{n-m}{0} + \binom{m-1}{5}\binom{n-m}{1} + \binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3} \\
&\qquad + \binom{m-1}{2}\binom{n-m}{4} + \binom{m-1}{1}\binom{n-m}{5} + \binom{m-1}{0}\binom{n-m}{6} - 4\binom{m-1}{6} \\
&\qquad - 4(n-m)\binom{m-1}{5} - 4\binom{m-1}{4}\binom{n-m}{2} - 4\binom{m-1}{3}\binom{n-m}{3} + 4 \leqslant 0 \\
\iff &\binom{m-1}{2}\binom{n-m}{4} + (m-1)\binom{n-m}{5} + \binom{n-m}{6} - 3\binom{m-1}{6} \\
&\qquad - 3(n-m)\binom{m-1}{5} - 3\binom{m-1}{4}\binom{n-m}{2} - 3\binom{m-1}{3}\binom{n-m}{3} + 4 \leqslant 0 \\
\iff &(m-1)\binom{n-m}{5} + \binom{n-m}{6} + \binom{m-1}{2}\binom{n-m}{4} \\
&\qquad - 3\left(\frac{m-6}{6}\right)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\binom{m-1}{2} \\
&\qquad - 3(n-m)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\binom{m-1}{2} \\
&\qquad - 3\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\binom{m-1}{2}\binom{n-m}{2} - 3\left(\frac{m-3}{3}\right)\binom{m-1}{2}\binom{n-m}{3} \\
&\qquad + 4 \leqslant 0
\end{aligned}
$$

$$\begin{aligned}\iff (m-1)\binom{n-m}{5} + \binom{n-m}{6} + \binom{m-1}{2}\Bigg(\binom{n-m}{4} \\ -3\left(\frac{m-6}{6}\right)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right) \\ -3(n-m)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right) \\ -3\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\binom{n-m}{2} - 3\left(\frac{m-3}{3}\right)\binom{n-m}{3}\Bigg) + 4 \leqslant 0\end{aligned}$$

Let;

$$\begin{aligned}t :=&\binom{n-m}{4} - 3\left(\frac{m-6}{6}\right)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right) \\ &- 3(n-m)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right) - 3\left(\frac{m-4}{4}\right)\left(\frac{m-3}{3}\right)\binom{n-m}{2} \\ &- 3\left(\frac{m-3}{3}\right)\binom{n-m}{3} \\ =&\binom{n-m}{4} - 3\left(\frac{m-3}{3}\right)\Bigg(\left(\frac{m-6}{6}\right)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right) + (n-m)\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right) \\ &+ \left(\frac{m-4}{4}\right)\binom{n-m}{2} + \binom{n-m}{3}\Bigg) \\ =&\binom{n-m}{4} - (m-3)\Bigg(\left(\frac{m-5}{5}\right)\left(\frac{m-4}{4}\right)\left(\left(\frac{m-6}{6}\right) + (n-m)\right) \\ &+ \binom{n-m}{2}\left(\left(\frac{m-4}{4}\right) + \left(\frac{n-m-2}{3}\right)\right)\Bigg) \\ =&\binom{n-m}{4} - (m-3)\Bigg(\left(\frac{1}{20}(m^2-9m+20)\right)\left(\frac{1}{6}(6n-5m-6)\right) \\ &+ \left(\frac{1}{2}(n-m)(n-m-1)\right)\left(\frac{1}{12}(4n-m-20)\right)\Bigg)\end{aligned}$$

Since $(m-1)\binom{n-m}{5} + \binom{n-m}{6} \geqslant 0$ and $\binom{m-1}{2} \geqslant 0$, we only need to check whether $t \geqslant 0$ or not to ensure if the inequality hold.

However, since the equation in t is getting bigger, it would be efficient to use Mathematica to verify whether this equality is positive or not on the bound $m \geqslant 8$ and $n \geqslant 9m$. Using Mathematica, we verified that $t \leqslant 0$ does not have any real solution when $n \geqslant 9m$

and $m \geqslant 8$. Thus implying that t is positive when $n \geqslant 9m$ and $m \geqslant 8$. This brings us to conclude that the original inequality does not hold; thus, all edges of this type can be colored.

Fourthly, edges Type 4: $N \cup u^3, |N| = 3$. Let's denote this type of edges as y_4.

$$
\begin{aligned}
&3\Bigg[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3} \\
&\qquad + \binom{m-1}{2}\binom{n-m}{4} - 1\Bigg] \geqslant \binom{n-1}{6} \\
\iff &\binom{n-1}{6} - 3\Big[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} \\
&\qquad + \binom{m-1}{3}\binom{n-m}{3} + \binom{m-1}{2}\binom{n-m}{4} - 1\Big] \leqslant 0 \\
\iff &\binom{n-1}{6} - 3\Big[\sum_{i=0}^{6}\binom{m-1}{6-i}\binom{n-m}{i} - (m-1)\binom{n-m}{5} - \binom{n-m}{6} - 1\Big] \leqslant 0 \\
\iff &\binom{n-1}{6} - 3\Big[\binom{n-1}{6} - (m-1)\binom{n-m}{5} - \binom{n-m}{6} - 1\Big] \leqslant 0 \\
\iff &\binom{n-1}{6} - 3\binom{n-1}{6} + 3(m-1)\binom{n-m}{5} + 3\binom{n-m}{6} + 3 \leqslant 0 \\
\iff &-2\binom{n-1}{6} + 3\binom{n-m}{5}\left((m-1) + \left(\frac{n-m-5}{6}\right)\right) + 3 \leqslant 0 \\
\iff &-2\binom{n-1}{6} + \binom{n-m}{5}\left(\frac{n+5m-11}{2}\right) + 3 \leqslant 0
\end{aligned}
$$

Using Mathematica, we verified that this inequality does not have any real solution when $n \geqslant 9m, m \geqslant 8$. Thus implying that this inequality will be positive when $n \geqslant 9m, m \geqslant 8$. Therefore, a contradiction and every edge of this type can all be colored.

Fifthly, edges Type 5: $N \cup u^4, |N| = 2$. Let's denote this type of edges as y_5.

$$
\begin{aligned}
& 2\Big[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} + \binom{m-1}{3}\binom{n-m}{3} \\
& \qquad + \binom{m-1}{2}\binom{n-m}{4} + \binom{m-1}{1}\binom{n-m}{5} - 1\Big] \geqslant \binom{n-1}{6} \\
\iff & \binom{n-1}{6} - 2\Big[\binom{m-1}{6} + (n-m)\binom{m-1}{5} + \binom{m-1}{4}\binom{n-m}{2} \\
& \qquad + \binom{m-1}{3}\binom{n-m}{3} + \binom{m-1}{2}\binom{n-m}{4} + \binom{m-1}{1}\binom{n-m}{5} - 1\Big] \leqslant 0 \\
\iff & \binom{n-1}{6} - 2\Big[\sum_{i=0}^{6}\binom{m-1}{6-i}\binom{n-m}{i} - \binom{n-m}{6} - 1\Big] \leqslant 0 \\
\iff & \binom{n-1}{6} - 2\Big[\binom{n-1}{6} - \binom{n-m}{6} - 1\Big] \leqslant 0 \\
\iff & 2\binom{n-m}{6} - \binom{n-1}{6} + 2 \leqslant 0
\end{aligned}
$$

Using Mathematica, we checked that this inequality does not have any solution when $n \geqslant 9.16580m, m \geqslant 8$. Therefore, the inequality does not hold, which leads to a contradiction between our assumption and our claim holds.

Lastly, edges Type 6: $N \cup u^6, |N| = 1$. Let's denote this set of edges as y_6.

This is the last type of edge that contains a vertex $v \in N$. We would like to show that $\deg_i(v) = r, v \in N$ and since this type of edges would be the last type of edges that contained a vertex $v \in N$, each vertex must be contained in $r - \deg_i(v)$ number of edges of each color $i \in [k]$. Thus,

$$
\begin{aligned}
\sum_{i=1}^{k}(r-\deg_i(v)) &= \sum_{i=1}^{k} r - \sum_{i=1}^{k}\deg_i(v) \\
&= rk - \deg(v) \\
&= \lambda\binom{n-1}{6} - \sum_{\ell=1}^{6}\lambda\binom{n-m}{7-\ell-1}\binom{m-1}{\ell} \\
&= \lambda\sum_{\ell=0}^{6}\binom{n-m}{7-\ell-1}\binom{m-1}{\ell} - \lambda\sum_{\ell=1}^{6}\binom{n-m}{7-\ell-1}\binom{m-1}{\ell} \\
&= \lambda\left(\sum_{\ell=0}^{6}\binom{n-m}{7-\ell-1}\binom{m-1}{\ell} - \sum_{\ell=1}^{6}\binom{n-m}{7-\ell-1}\binom{m-1}{\ell}\right) \\
&= \lambda\left(\binom{n-m}{6} + \sum_{\ell=1}^{6}\binom{n-m}{7-\ell-1}\binom{m-1}{\ell} - \sum_{\ell=1}^{6}\binom{n-m}{7-\ell-1}\binom{m-1}{\ell}\right) \\
&= \lambda\binom{n-m}{6}
\end{aligned}
$$

Hence, we can color all edges of this type using all available k colors, and the degree of every vertex from $\mathcal{G}$ is equal to r.

The last type of edges we need to check is the edges that only contain the newly added $n-m$ vertices. Let's denote this last type of edge as y_7. We have colored all the other type edges and verified that $\deg_i(v) = r, v \in N$. Next, the degree of every vertex of the newly added vertices must equal r. Up to this point, all these vertices have been contained in every other edge type.

Recall that:

$$
\begin{aligned}
rm &= 7y_{0_i} + 6y_{1_i} + 5y_{2_i} + 4y_{3_i} + 3y_{4_i} + 2y_{5_i} + y_{6_i} \\
rn - 7rm &= 7y_{0_i} + 7y_{1_i} + 7y_{2_i} + 7y_{3_i} + 7y_{4_i} + 7y_{5_i} + 7y_{6_i} + 7y_{7_i} - 49y_{0_i} - 42y_{1_i} \\
&\quad - 35y_{2_i} - 28y_{3_i} - 21y_{4_i} - 14y_{5_i} - 7y_{6_i} \\
&= -42y_{0_i} - 35y_{1_i} - 28y_{2_i} - 21y_{3_i} - 14y_{4_i} - 7y_{5_i} + 7y_{7_i} \\
7y_{7_i} &= rn - 7rm + 42y_{0_i} + 35y_{1_i} + 28y_{2_i} + 21y_{3_i} + 14y_{4_i} + 7y_{5_i}
\end{aligned}
$$

Thus for each color class $i \in [k]$, we have this many edges of this type;

$$y_{7_i} := \frac{rn}{7} - rm + 6y_{0_i} + 5y_{1_i} + 4y_{2_i} + 3y_{3_i} + 2y_{4_i} + y_{5_i}$$

For the notation, in here y_{7_i} means the edge type 7 in color class i, where $i \in [k]$. Recall that we are given the conditions that $7|rn$, $n \geqslant 9m \geqslant 7m$. Hence, y_{7_i} will be a positive integer. We will show that we could color all edges of this type by showing that the number of edges will equal to $\binom{n-m}{7}$

Edges Type : $\{u^7\}$.

$$\begin{aligned}
\sum_{i=1}^{k} y_{7_i} &= \sum_{i=1}^{k} \frac{rn}{7} - \sum_{i=1}^{k} rm - \sum_{i=1}^{k} 6y_{0_i} + \sum_{i=1}^{k} 5y_{1_i} + \sum_{i=1}^{k} 4y_{2_i} + \sum_{i=1}^{k} 3y_{3_i} + \sum_{i=1}^{k} 2y_{4_i} + \sum_{i=1}^{k} y_{5_i} \\
&= \frac{n}{7}(rk) - m(rk) + 6\lambda\binom{m}{7} + 5\lambda\binom{m}{6}\binom{n-m}{1} + 4\lambda\binom{m}{5}\binom{n-m}{2} \\
&\quad + 3\lambda\binom{m}{4}\binom{n-m}{3} + 2\lambda\binom{m}{3}\binom{n-m}{3} + \lambda\binom{m}{2}\binom{n-m}{4} \\
&= \lambda\frac{n}{7}\binom{n-1}{6} - \lambda m\binom{n-1}{6} + \left(\lambda\sum_{i=0}^{6}(h-i)\binom{m}{7-i}\binom{n-m}{i} - \sum_{i=1}^{6}\lambda\binom{m}{7-i}\binom{n-m}{i}\right) \\
&= \lambda\binom{n-1}{7} - \lambda m\binom{n-1}{6} + \left(\lambda\sum_{i=0}^{6}(h-i)\binom{m}{7-i}\binom{n-m}{i} - \sum_{i=1}^{6}\lambda\binom{m}{7-i}\binom{n-m}{i}\right) \\
&= \lambda\sum_{i=1}^{7}\binom{m}{7-i}\binom{n-m}{i} - \lambda m\binom{n-1}{6} + \left(7\lambda\binom{m}{7} + \lambda\sum_{i=1}^{6}(h-i)\binom{m}{7-i}\binom{n-m}{i}\right. \\
&\quad \left. - \lambda\sum_{i=1}^{6}\binom{m}{7-i}\binom{n-m}{i}\right) \\
&= \lambda\binom{n-m}{7} + \lambda\sum_{i=1}^{6}\binom{m}{7-i}\binom{n-m}{i} - \lambda m\binom{n-1}{6} + 7\lambda\binom{m}{7} \\
&\quad + \lambda m\left[\binom{n-1}{7-1} - \binom{m-1}{7-1}\right] - \lambda\sum_{i=1}^{6}\binom{m}{7-i}\binom{n-m}{i}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda\binom{n-m}{7} + \lambda\sum_{i=1}^{6}\binom{m}{7-i}\binom{n-m}{i} - \lambda m\binom{n-1}{6} + 7\lambda\left(\frac{m}{7}\right)\binom{m-1}{6} + \lambda m\binom{n-1}{6} \\
&\quad - \lambda m\binom{m-1}{6} - \lambda\sum_{i=1}^{6}\binom{m}{7-i}\binom{n-m}{i} \\
&= \lambda\binom{n-m}{7} + \lambda m\binom{m-1}{6} - \lambda m\binom{m-1}{6} \\
&= \lambda\binom{n-m}{7}
\end{aligned}
$$

Last step would be to check the degree of every vertex. We have verified that $\deg_i(v) = r, v \in N$. Next, we need to show that $\deg_i(u) = r(n-m)$ for $i \in [k]$.

Recall that, $\sum_{v\in\mathcal{V}} \deg_i(v) = rm = 7y_{0_i} + 6y_{1_i} + 5y_{2_i} + 4y_{3_i} + 3y_{4_i} + 2y_{5_i} + 1y_{6_i}$.

Hence,

$$
\begin{aligned}
\deg(u) &= y_1 + 2y_2 + 3y_3 + 4y_4 + 5y_5 + 6y_6 + 7y_7 \\
&= (7y_0 + 7y_1 + 7y_2 + 7y_3 + 7y_4 + 7y_5 + 7y_6 + 7y_7) \\
&\qquad - (7y_0 + 6y_1 + 5y_2 + 4y_3 + 3y_4 + 2y_5 + 1y_6) \\
&= rn - rm = r(n-m)
\end{aligned}
$$

$\square$

By following the pattern that we did in our proof for $h = 6$ and $h = 7$, we can conclude that by checking the bound on n for when $j = 2$ in both III.3 and IV.3 we would get n that is big enough for us to embed our original hypergraph. We used Mathematica to find the rest of the bound on n for $h \in \{8 - 89\}$. The table can be found in the Appendix.

REFERENCES

[1] M.A. Bahmanian. “Extending edge-colorings of complete hypergraphs into regular colorings,” J Graph Theory **83** (2018), 1–14.

[2] M.A. Bahmanian and C. Rodger. “Embedding Factorizations for 3-Uniform Hypergraphs,” Journal of Graph Theory **73(2)** (2013), 216–224. DOI:10.1002/jgt.21669

[3] M.A. Bahmanian. “Factorizations of complete multipartite hypergraphs,” Discrete Mathematics **340(2)** (2017), 46–50.https://doi.org/10.1016/j.disc.2016.08.007

[4] M.A. Bahmanian. “Detachments of Amalgamated 3-Uniform Hypergraphs Factorization Consequences,” Journal of Combinatorial Designs **20** (2012), 46–50. https://onlinelibrary.wiley.com/doi/10.1002/jcd.21310

[5] M.A. Bahmanian and S. Haghshenas,“Extending regular edge-colorings of complete hypergraphs,”Journal of Graph Theory **94(1)** (2020), 59–74. https://doi.org/10.1002/jgt.22506

[6] M.A. Bahmanian and M. Newman, “Extending factorization of complete uniform hypergraphs,” Combinatorica **38(6)** (2018), 1309–1335. DOI: 10.1007/s00493-017-3396-3

[7] T. Evans. “Embedding Incomplete Latin Squares,” The American Mathematical Monthly **67:10** (1960), , 958–961. DOI: 10.1080/00029890.1960.1199203

[8] H. J. Ryser , “A combinatorial theorem with an application to Latin rectangles,” Proc. Amer. Math. Soc **2** (1951), 550–552. https://www.ams.org/journals/proc/1951-002-04/S0002-9939-1951-0042361-0/S0002-9939-1951-0042361-0.pdf

[9] C. A. Rodger, and E. B. Wantland, “Embedding edge-colourings into 2-edge-connected k-factorizations of K_{kn+1},“ J. Graph Theory **19**, no 2 (1995), 169–185. DOI:10.1002/jgt.3190190205

[10] M.A. Bahmanian and M. Newman. “Embedding factorizations for 3-uniform hypergraphs II: r-factorizations into s-factorizations,” The Electronic Journal Of Combinatorics **23(2)** (2016), P2.421. https://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i2p42/pdf

[11] J. J. Sylvester. “Elementary researches in the analysis of combinatorial aggregation,” Phil. Mag **3(24)** (1844), 285-296.DOI: 10.1080/14786444408644856

[12] R. Peltesohn. “Das Turnierproblem für Spiele zu je dreien,” Ph.D. thesis, Universität Berlin, (1936).

[13] Zs. Baranyai and A.E. Brouwer: “Extension Of Colourings Of The Edges Of a Complete (uniform Hyper)graph“, CWI Technical Report ZW 91/77, (1977)

[14] Z. Baranyai. “The Edge-Coloring of Complete Hypergraph I,” Journal of Combinatorial Theory (B) **26(3)** (1979), 276–294. https://doi.org/10.1016/0095-8956(79)90002-9

[15] A. Johnsen. (2021). “Embedding Factorizations“, Master Thesis, Illinois State University , Normal.

[16] C. C. Lindner and C. A. Rodger. (2009). Chapter 9 - Embeddings. *Design Theory.* CRC Press/Taylor amp. Francis Group.185–194

[17] R. Häggkvist and T. Hellgren: “Extensions of edge-colourings in hypergraphs, I“,in: Combinatorics, Paul Erdos is eighty, Vol. 1, Bolyai Soc. Math. Stud., 215–238. János Bolyai Math. Soc., Budapest, 1993.

[18] M. Hall. “An existence theorem for latin squares,“ Bulletin of the American Mathematical Society **51** (1945), 387–388. https://doi.org/10.1090/S0002-9904-1945-08361-X

[19] Z. Shao, Z. Li, B. Wang, S. Wang and X. Zhang. “Interval edge-coloring: A model of curriculum scheduling,“ AKCE International Journal of Graphs and Combinatorics **17(3)** (2020), 725–729. DOI: 10.1016/j.akcej.2019.09.003

[20] Pluhár, A. “Greedy colorings of uniform hypergraphs,“ Random Struct. Alg **35** (2009), 216–221. https://doi.org/10.1002/rsa.20267

[21] Danila D. Cherkashin and Jakub Kozik. “A note on random greedy coloring of uniform hypergraphs,” andom Struct. Alg **47** (2015), 407–413. https://doi.org/10.1002/rsa.20556

[22] P. J. Cameron: “Parallelisms of complete designs,“ Cambridge University Press, Cambridge-New York-Melbourne, (1976), London Mathematical Society Lecture Note Series, No. 23. https://doi.org/10.1017/CBO9780511662102

[23] M.A. Bahmanian. “Detachments of Hypergraphs I: The Berge–Johnson Problem,“ Combinatorics, Probability and Computing **21(4)** (2012), 483–495. DOI: 10.1017/s0963548312000041

APPENDIX A: BOUND FOR $h = 3 - 89$

h	$m \geqslant h+1$	$n \geqslant$	n/hm	$n \geqslant$	$n \geqslant (h-1)(2m-1)$
3	4	3.41422m	1.13807	13.65688	14
4	5	4.84733m	1.21183	24.23665	27
5	6	6.28522m	1.25704	37.71132	44
6	7	7.72503m	1.28751	54.07521	65
7	8	9.16580m	1.30940	73.32640	90
8	9	10.60712m	1.32589	95.46408	119
9	10	12.04878m	1.33876	120.48840	152
10	11	13.49068m	1.34907	148.39748	189
11	12	14.93273m	1.35753	179.19396	230
12	13	16.37490m	1.36458	212.87370	275
13	14	17.81715m	1.37056	249.44192	324
14	15	19.25952m	1.37568	288.89280	377
15	16	20.70195m	1.38013	331.23120	434
16	17	22.14428m	1.38402	376.45276	495
17	18	23.58674m	1.38746	424.56276	560
18	19	25.02922m	1.39052	475.5567	629
19	20	26.47172m	1.39325	529.43500	702
20	21	27.91425m	1.39572	586.19925	779
21	22	29.35679m	1.39795	645.84938	860
22	23	30.79935m	1.39998	708.38505	945
23	24	32.24192m	1.40183	773.80608	1034
24	25	33.68450m	1.40353	842.11250	1127
25	26	35.12709m	1.40509	913.30434	1224

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Table A-1 – continued from previous page

h	$m \geqslant h+1$	n	n/hm	$n \geqslant$	$n \geqslant (h-1)(2m-1)$
26	27	36.56969m	1.40653	987.38163	1325
27	28	38.01230m	1.40787	1064.3444	1430
28	29	39.45491m	1.40911	1144.19239	1539
29	30	40.89753m	1.41026	1226.92590	1652
30	31	42.34015m	1.41134	1312.54465	1769
31	32	43.78278m	1.41235	1401.04896	1890
32	33	45.22541m	1.413295	1492.43853	2015
33	34	46.66805m	1.41419	1586.7137	2144
34	35	47.11070m	1.41503	1683.87415	2277
35	36	49.55334m	1.41581	1783.92024	2414
36	37	50.99598m	1.41656	1886.85126	2555
37	38	52.43862m	1.41727	1992.66756	2700
38	39	53.88128m	1.41793	2101.36992	2849
39	40	55.32394m	1.41857	2212.95760	3002
40	41	56.76659m	1.41917	2327.43019	3159
41	42	58.20925m	1.41974	2444.7885	3320
42	43	59.65191m	1.42029	2565.03213	3485
43	44	61.09457m	1.42081	2688.16108	3654
44	45	62.53724m	1.42131	2814.17580	3827
45	46	63.97990m	1.42178	2943.07540	4004
46	47	65.42257m	1.42223	3074.86079	4185
47	48	66.86523m	1.42267	3209.53104	4370
48	49	68.30790m	1.42309	3347.08710	4559

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Table A-1 – continued from previous page

h	$m \geqslant h+1$	n	n/hm	$n \geqslant$	$n \geqslant (h-1)(2m-1)$
49	50	69.75057m	1.42349	3487.5285	4752
50	51	71.19324m	1.42387	3630.85524	4949
51	52	72.63591m	1.42424	3777.06732	5150
52	53	74.07858m	1.42459	3926.16474	5355
53	54	75.52126m	1.42493	4078.14804	5564
54	55	76.96393m	1.42526	4233.01615	5777
55	56	78.40661m	1.42558	4312.36355	5994
56	57	79.84928m	1.42589	4551.40896	6215
57	58	81.29196m	1.42618	4714.93368	6440
58	59	82.73464m	1.42646	4881.34376	6669
59	60	84.17731m	1.42674	5050.63860	6902
60	61	85.61999m	1.42700	5222.81939	7139
61	62	87.06267m	1.42726	5397.88554	7380
62	63	88.50536m	1.42751	5575.83768	7625
63	64	89.94803m	1.42775	5756.67392	7874
64	65	91.39071m	1.42798	5940.39615	8127
65	66	92.83340m	1.42821	6127.00440	8384
66	67	94.27606m	1.42843	6316.49602	8645
67	68	95.71875m	1.42864	6508.87500	8910
68	69	97.16143m	1.42885	6704.13867	9179
69	70	98.60412m	1.42905	6902.28840	9452
70	71	100.04680m	1.42924	7103.32280	9729
71	72	101.48949m	1.42943	7307.24328	10010

Continued on next page

Table A-1 – continued from previous page

h	$m \geqslant h+1$	n	n/hm	$n \geqslant$	$n \geqslant (h-1)(2m-1)$
72	73	102.93217m	1.42962	7514.04841	10295
73	74	104.37485m	1.42980	7723.73890	10584
74	75	105.81753m	1.42997	7936.31475	10877
75	76	107.26021m	1.43014	8151.77596	11174
76	77	108.70289m	1.43030	8370.12253	11475
77	78	110.14558m	1.43046	8591.135524	11780
78	79	111.58827m	1.43062	8815.47333	12089
79	80	113.03096m	1.43077	9042.4768	12402
80	81	114.47365m	1.43092	9272.36565	12719
81	82	115.91633m	1.43107	9505.13906	13040
82	83	117.35902m	1.43121	9740.79866	13365
83	84	118.80171m	1.43135	9979.34364	13694
84	85	120.24440m	1.43148	10220.774	14027
85	86	121.68709m	1.43161	10465.0897	14364
86	87	123.12978m	1.43174	10712.2909	14705
87	88	124.57245m	1.43187	10962.3774	15050
88	89	126.01514m	1.43199	11215.3492	15399
89	90	127.45782m	1.43211	11471.2065	15752