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Shadows of Colored Complexes and Cycle Decompositions of Equipartite Hypergraphs

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SHADOWS OF COLORED COMPLEXES AND CYCLE DECOMPOSITONS
OF EQUIPARTITE HYPERGRAPHS

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Let $K^h_{n\times m}$ be the complete h-uniform n-partite hypergraph with parts of size m. A cycle of length c in a hypergraph is an alternating sequences of distinct vertices, v_i, and distinct edges e_i of the form $v_1, e_1, v_2, e_2, \ldots, e_c, v_c$ such that $\{v_i, v_{i+1}\} \subseteq e_i$, for $1 \leq i \leq c$ and $v_{c+1} = v_1$. By applying the shadows of colored complexes, we nearly settle the problem of partitioning the edges of $K^h_{n\times m}$ into cycles of length c where c is a multiple of m.

KEYWORDS: Hypergraph, Coloring, Shadows, Equipartite, Multipartite, Decomposition, Kruskal-Katona Theorem

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CHAPTER I: INTRODUCTION

An *n-partite graph* is a graph where vertices are partitioned into n disjoint sets such that vertices within the same set are not adjacent. Similarly, an *equipartite* graph is an n-partite graph where each disjoint set, or part, has the same number of vertices. The complete n-partite equipartite graph is denoted by $K_{n\times m}$, where n is the number of parts, and m represents the number of vertices in each part. A *λ-fold* graph is a multigraph where λ copies of each edge are created. The λ-fold complete graph on n vertices is denoted by λK_n. An *complete h-uniform hypergraph* is a hypergraph where each edge contains h vertices, and is denoted by K_n^h. Thus, $K_{n\times m}^h$ denotes the *complete n-partite, h-uniform equipartite hypergraph with parts of size m*, where $n, m, h \in \mathbb{N} \cup \{0\}$. We denote the size of the edge set of $K_{n\times m}^h$ in two different ways, letting $|E(K_{n\times m}^h)| = \binom{|mn|}{h}_n = m^h\binom{n}{h}$, where $[mn] = \{1, \ldots, mn\}$. A *cycle of length c* in a hypergraph is an alternating sequences of distinct vertices, v_i, and distinct edges e_i of the form $v_1, e_1, v_2, e_2, \ldots, e_c, v_c$ such that $\{v_i, v_{i+1}\} \subseteq e_i$, for $1 \le i \le c$ and $v_{c+1} = v_1$. Finally, to *decompose* a graph means to partition the edge set of that graph into subgraphs. Thus, a *c-cycle decomposition* is the decomposition of a graph into subgraphs that are cycles of length c. In this paper we aim to find a c-cycle decomposition of $K_{n\times m}^h$. In order to show this, we must first consider the complete decomposition of the λ-fold n-partite graph with parts of size m, denoted by $\lambda K_{n\times m}$, into cycles of length c.

In order to decompose $\lambda K_{n\times m}$, we will use several results in conjunction. First, in 2014 Bahmanian and Šajna [2] established the following heorem to settle the complete decomposition of the λ-fold complete equipartite multigraph $\lambda K_{n\times m}$. Note that a *$(c_1, c_2, \ldots, c_k)$-cycle decomposition* is a decomposition of a graph into cycles of varying lengths $c_1, c_2, \ldots, c_k$.

Theorem I.0.1 (Bahmanian, Šajna). *[2, Theorem 1.4] Let λ, m, n, and $c_1, c_2, \ldots, c_k$ be positive integers such that there exists a $(c_1, c_2, \ldots, c_k)$-cycle decomposition of $\lambda m K_n$. Then the complete equipartite multigraph $\lambda K_{n\times m}$ admits a $(c_1 m, c_2 m, \ldots, c_k m)$-cycle decomposition.*

Next, in 2015 Bryant, Horsley, Maenhaut, and Smith [3] established the following theorem in order to find a c-cycle decompostion of λ copies of the complete graph K_n, this is commonly referred to as the λ-fold complete graph on n vertices.

Theorem I.0.2 (Bryant, Horsley, Maenhaut, Smith). *[3, Theorem 1.1] There is a decomposition $\{G_1, G_2, \dots, G_k\}$ of λK_n in which G_i is a c_i-cycle for $i = 1, 2, \dots, k$ if and only if the following conditions hold.*

- $\lambda(n-1)$ is even;
- $2 \leq c_1, c_2, \dots, c_k \leq n$;
- $c_1 + c_2 + \cdots + c_k = \lambda\binom{n}{2}$;
- $\max(c_1, c_2, \dots, c_k) + k - 2 \leq \frac{\lambda}{2}\binom{n}{2}$ when λ is even, and
- $\sum_{c_i=2} c_i \leq (\lambda - 1)\binom{n}{2}$ when λ is odd.

For our purposes we will focus on a corollary of this theorem in which our cycle is of a fixed length c in a λ-fold graph. Though the qualities are very similar, they are in many ways simpler.

Corollary I.0.3. *Let $2 \leq c \leq n$, there is a decomposition of λK_n into c-cycles if and only if:*

- *$\lambda(n-1)$ is even;*
- *$c \mid \lambda\binom{n}{2}$;*

Theorem I.0.1 and Corollary I.0.3 provide the basis of our motivation for this paper. We aim to find the necessary and sufficient conditions for the existence of a decomposition of the complete n-partite h-uniform hypergraph, $K^h_{n\times m}$, into c-cycles.

Beyond the results by Bryant et al. and Šajna, in order to create these c-cycle decompositions, we will be applying the Kruskal-Katona Theorem [6, 5] and a related version by Frankl et al. [4] referred to as the Colored Kruskal-Katona Theorem. Our main result relies on the use of this theorem and follows a similar solution method to Kühn and Osthus [7], and Bahmanian and Haghshenas [1].

In order to apply these theorems, we must clarify some notation. For some subset S of vertices of a graph G, let $N(S)$ denote the neighborhood of S, or set of all vertices adjacent to any vertex in S. Let $G[X, Y]$ denote the bipartite graph G with parts X and Y. Finally, for multipartite hypergraph, a *legal edge* is an edge that contains at most one vertex from each part.

In this paper we prove the following result.

Theorem I.0.4. *Let* $n, m, h, c \in \mathbb{N}$ *with* $2 \leq c \leq mn$, $3 \leq h < n$, $m \mid c$ *and let* $\ell = m^h\binom{n}{h} - c\left\lfloor \frac{m^h\binom{n}{h}}{c} \right\rfloor$. *Then for* $L \subsetneq E(K^h_{n\times m})$ *with* $|L| = \ell$, $K^h_{n\times m} \setminus L$ *can be decomposed into cycles of length* c *in the following cases:*

(A) $h = 3$, $n \geq 85$, and L is a matching.

(B) $4 \leq h \leq n-3$, and $n \geq 23$.

(C) $n-2 \leq h \leq n-1$, and $n \geq 16$.

(D) $h = n$, $n \geq 3$, and $m \geq 3$.

CHAPTER II: THE KRUSKAL-KATONA THEOREM

II.1 The Kruskal-Katona Theorem

The Kruskal-Katona Theorem is a combinatorial theorem which provides a lower bound for the size of the shadow of a set. This theorem was independently discovered by a number of mathematicians. However, it was named for the first two people who discovered it, Joseph Kruskal, whose findings were published in 1963 [6], and Gyula O. H. Katona, whose findings were published in 1968 [5]. To discuss this theorem in greater detail, let $n, h \in \mathbb{Z}$ where $0 \leq h \leq n$, and let $\binom{[n]}{h} = \{s \subseteq [n] : |s| = h\}$, where $[n]$ denotes the set $\{1, \ldots, n\}$. For $i \in [h]$, and for any set S with $S \subseteq \binom{[n]}{h}$, we define the i^{th} lower shadow, denoted by $\partial_i^-(S)$, and the i^{th} upper shadow, denoted by $\partial_i^+(S)$, as follows:

$$\partial_i^-(S) = \left\{t \in \binom{[n]}{h-i} : t \subseteq s \text{ for some } s \in S\right\},$$
$$\partial_i^+(S) = \left\{t \in \binom{[n]}{h+i} : t \subseteq s \text{ for some } s \in S\right\}.$$

Example:

Let S be a set of 3-element subsets of $\{1,2,3,4,5\}$ or $[5]$, where $S = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\}\}$.

- The upper shadow by two of S is $\partial_2^+(S) = \{\{1,2,3,4,5\}\}$
- The upper shadow by one of S is $\partial_1^+(S) = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}\}$
- The lower shadow by one of S is $\partial_1^-(S) = \{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{1,5\},\{2,5\},\{3,4\},\{3,5\}\}$
- The lower shadow by two of S is $\partial_2^-(S) = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$

Though finding shadows in small cases is simple, as the size of S increases and the size of its subsets grows larger, estimating the size of a shadow becomes more difficult. For any positive real number s and any integer h with $1 \leq h \leq s$,

$$\binom{s}{h} := \frac{s(s-1)\ldots(s-h+1)}{h!}.$$

In order to apply the Kruskal-Katona Theorem, we must order subsets, representing our edges, in specific ways. Given two sets A and B in lexicographical order, we say $A < B$ if the smallest element between $A \setminus (A \cap B)$ and $B \setminus (A \cap B)$ is in A. For example, if $A = \{1, 3, 5\}$ and $B = \{2, 3, 4\}$, then $A < B$ in lexicographical order as $1 < 2$, since 3 is in the intersection of A and B, and 1 and 2 are the smallest elements of each set. Similarly, we say that $A > B$ in colexicographical order if the largest element between $A \setminus (A \cap B)$ and $B \setminus (A \cap B)$ is in A. Again, if $A = \{1, 3, 5\}$ and $B = \{2, 3, 4\}$, then $A > B$ in colexicoraphical order as $5 > 4$, since 3 is in the intersection of A and B and 5 and 4 are the largest elements of each set. Table 1 demonstrates the differences in these orders on a larger scale, showing the lexicographical order and colexicographical order of all the 3-element subsets of $\{1, 2, 3, 4, 5\}$.

Table 1: An example of lexicographical and colexicographical ordering

Lexicographical order	Colexicographical order
$\{1,2,3\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,2,4\}$
$\{1,2,5\}$	$\{1,3,4\}$
$\{1,3,4\}$	$\{2,3,4\}$
$\{1,3,5\}$	$\{1,2,5\}$
$\{1,4,5\}$	$\{1,3,5\}$
$\{2,3,4\}$	$\{2,3,5\}$
$\{2,3,5\}$	$\{1,4,5\}$
$\{2,4,5\}$	$\{2,4,5\}$
$\{3,4,5\}$	$\{3,4,5\}$

Notice that in lexicographical order all of the subsets containing 1 are listed first, but in colexicographical order all of the subsets containing 5 are listed last. This is important as the Kruskal-Katona Theorem uses such an ordering. Note that given an ordered set A, the initial segment B is a subset of A and is defined as $\forall\ a \in A$, and $\forall\ b \in B$, $b \leq a$. For example in Table 1, an initial segment could be $S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$ is lexicographical order or

$S = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$ in colexicographical order. With this vocabulary defined, below is the Kruskal-Katona Theorem.

Lemma II.1.1 (Kruskal-Katona Theorem). *[6, 5] Let $S \subseteq E(K_n^h)$ with $|S| = \binom{s}{h}$ where $s \geq h$, $s \in \mathbb{R}$. Then $|\partial_1^-(S)|$ is minimized when S is chosen to be the initial segment of all h-element subsets in colexicographical order.*

The Kruskal-Katona Theorem provides us with the concept of how to find the minimum lower shadow of a set. In our case, this allows us to find the minimum lower shadow for a set of edges of a hypergraph. Though powerful, this only provides us with a description of how to find the minimum shadow. It does not give us an approximate lower bound for the size of the first lower shadow. This limitation led to the creation of the quantitative version of this theorem as seen below.

Lemma II.1.2 (Quantitative Kruskal-Katona Theorem). *Let $S \subseteq E(K_n^h)$ be a collection of h-subsets of $[n]$, and suppose that the h-binomial representation of $|S|$ is*

$$|S| = \binom{a_h}{h} + \binom{a_{h-1}}{h-1} + \cdots + \binom{a_t}{t} = \binom{s}{h}$$

for $s \in \mathbb{R}$ and $a_h > a_{h-1} > \cdots > a_t \geq t \geq 1$. Then $|\partial_1^-(S)| \geq \binom{a_h}{h-1} + \binom{a_{h-1}}{h-2} + \cdots + \binom{a_t}{t-1}$.

Expanding on the approximate lower bound of a shadow brought about a more generalized version of the Kruskal-Katona Theorem for not just the first lower shadow, but instead for the t^{th} lower shadow. This extension of the Kruskal-Katona Theorem was proven by Lovász in 1979 [8].

Lemma II.1.3 (Lovász's Theorem). *[8] Let $S \subseteq E(K_n^h)$ with $|S| = \binom{s}{h}$ where $s \geq h$, $s \in \mathbb{R}$. Then $|\partial_t^-(S)| \geq \binom{s}{h-t}$ for $1 \leq t \leq h$.*

Each of these theorems are powerful tools and can be applied to a variety of topics, including our focus of graph theory. Kühn and Ostus applied them in order to create approximations for the minimum size of the lower and upper shadow of the graphs they decomposed into cycles. We will use a similar strategy to their work. However, we will need a multipartite version of each of these theorems.

II.2 The Colored Kruskal-Katona Theorem

In 1988, Frankl, Füredi, and Kalai established a colored, or multipartite, generalization of the Kruskal-Katona Theorem. Note that an n-colored graph is synonymous with a n-partite graph. The following three theorems were a result of their work. First, is the multipartite version of the general Kruskal-Katona Theorem.

Lemma II.2.1 (Frankl, Füredi, Kalai). *[4, Theorem 1.2] IF $S \subseteq E(K^h_{n\times m})$, then $|\partial_1^-(S)|$ is minimized when S is chosen to be the first m legal edges in lexicographical order.*

As seen previously, this theorem works very similarly to the Kruskal-Katona Theorem in strategically ordering a set and then choosing the first legal edges. The creation of the first upper and lower shadow of a set S, this time not the initial segment, can be seen in the following example.

Example:

Let S be a set of 3-element subsets of 2-colored $\{1,2,3,4,5,6,7,8,9,10\}$ or $[10]$, where numbers equivalent modulo 5 are part of the same part or color.

Let $S = \{\{1,2,3\},\{1,2,10\},\{1,3,5\},\{2,4,6\}\}$.

- The upper shadow by two of S is $\partial_2^+(S) = \{\{1,2,3,4,5\},\{1,2,3,4,10\},\{1,2,3,5,9\},$ $\{1,2,3,9,10\},\{1,2,4,8,10\},\{1,2,8,9,10\},\{1,3,4,5,7\},\{1,3,5,7,9\},\{2,3,4,5,6\},$ $\{2,3,4,6,10\},\{2,4,5,6,8\},\{2,4,6,8,10\}\}$
- The upper shadow by one of S is $\partial_1^+(S) = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,9\},$ $\{1,2,3,10\},\{1,2,4,10\},\{1,2,8,10\},\{1,2,9,10\},\{1,3,4,5\},\{1,3,5,7\},\{1,3,5,9\},$ $\{2,3,4,6\},\{2,4,5,6\},\{2,4,6,8\},\{2,4,6,10\}\}$
- The lower shadow by one of S is $\partial_1^-(S) = \{\{1,2\},\{1,3\},\{1,5\},\{1,10\},\{2,3\},$ $\{2,4\},\{2,6\},\{2,10\},\{3,5\},\{4,6\}\}$
- The lower shadow by two of S is $\partial_2^-(S) = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\{10\}\}$

Note that when listing the first upper shadow, the subsets $\{1,2,3,6\}$, $\{1,2,6,10\}$, $\{1,3,5,6\}$ and $\{1,2,4,6\}$ are not listed as part of the shadow as $6 \cong 1 \mod 5$ and thus they are not legal edges. Other edges that are similarly disqualified are $\{1,2,3,7\}$, $\{1,2,3,8\}$, $\{1,2,5,10\}$, $\{1,2,7,10\}$, $\{1,3,5,8\}$, $\{1,3,5,10\}$, $\{2,4,6,7\}$, and $\{2,4,6,10\}$. As seen in this example, approximating a lower bound for the shadow of a colored set is also quite difficult as n grows. Thus, in the same paper Frankl, Füredi, Kalai, also proved a theorem to find an approximated minimum for the size of the first lower shadow. This approximation theorem for multipartite hypergraphs is listed below.

Theorem II.2.2 (Frankl, Füredi, Kalai). *[4] Let $S \subseteq E(K^h_{n\times m})$ be a collection of h-subsets of $[mn]$, and suppose the h-binomial representation of S is $|S| = m^h\binom{s}{h}$ for $s \in \mathbb{R}$. Then*

$$|\partial_1^-(S)| \geq m^{h-1}\binom{s}{h-1}$$

.

Finally, Frankl, Füredi, and Kalai included a version of the Lóvasz Theorem for multipartite hypergraphs. This approximation will be used directly in proving our main theorem.

Theorem II.2.3 (Frankl, Füredi, Kalai). *[4, Theorem 5.1] Suppose $S \subseteq E(K^h_{n\times m})$ where $|S| = m^h\binom{s}{h}$. Then for $1 \leq t \leq h$ and $s \in \mathbb{R}$*

$$|\partial_t^-(S)| \geq m^{h-t}\binom{s}{h-t}$$

.

II.3 Our Application of the Kruskal-Katona Theorem

Using Theorem II.2.3, we are able to generate a tool for the manipulation of the upper and lower shadows of multipartite graphs using only legal edges.

Lemma II.3.1. *Let $n, m, h \in \mathbb{N}$, $m \geq 1$, $2 \leq h \leq n$, and $\emptyset \neq T \subseteq E(K^h_{n\times m})$.*

(i) If $h \geq 3$ and $t \in \mathbb{R}$ such that $|T| = m^h\binom{t}{h}$, then $|\partial^-_{h-2}(T)| \geq m^2\binom{t}{2}$;

(ii) If $h = 2$ and $|T| := m^2 t$ where $t \leq n-1$, then $|\partial^+_2(T)| \geq m^2 t\binom{n-t-1}{2} + m^2\binom{t}{2}(n-t-1)$;

(iii) If $h = 2$ and $p, q \in \mathbb{N} \cup \{0\}$, such that $p < n, q < n-(p+1)$, and

$|T| = m^2[pn - \binom{p+1}{2} + q]$, then

$|\partial^+_1(T)| \geq m^2[\binom{n-1}{2}+\binom{n-2}{2}+\cdots+\binom{n-p}{2}+q(n-p-2)-\binom{q}{2}] \geq m^2[p\binom{n-p}{2}+q(n-p-2)-\binom{q}{2}]$.

Proof. Using Lemma II.2.3, one can see that $|\partial^-_{h-2}| \geq m^2\binom{t}{h-(h-2)} = m^2\binom{t}{h-h+2} = m^2\binom{t}{2}$. Therefore, (i) holds.

For (ii) we will prove this using construction. Consider $|T|$, there exist t 2-subsets from the values $\{1,\ldots,n\}$. As this is a multipartite graph, each of those 2-subsets can also be replicated using the other m elements from the same set. As this replacement can be done with both elements, the m changes can be done with each of the m other options of the second element and we have $m^2 t$ 2-subsets from $\{1,\ldots,mn\}$ with respect to the coloring. Since $h = 2$, in order to generate the second upper shadow, we must bolster each of the $m^2 t$ 2-subsets of T into 4-subsets of numbers $\{1,\ldots,mn\}$. This is done by matching each 2-subset in T with two additional numbers not used in the edges of T. In order to achieve the minimum upper shadow, there must be as much overlap in these combinations as possible. Thus, as seen in the first term of this inequality, each of the t edges will be matched with all 2-subsets combinations of the elements 1 to n that are not in T. This creates the first part of our upper shadow, but does not include any 4-subset of elements used by another edge in t with one of the numbers 1 to $n-1$. In order to ensure that we are finding the minimum second upper shadow, we must reuse as many numbers as possible. Therefore the second term represents all 3-subsets created using elements from the edges in T. $m^2\binom{t}{2}$ represents this by making all combinations of the t edges in T and an additional number in T which is not used in that particular edges. Then this is multiplied by $n-t-1$ to create all combinations of these 3-subsets with the remaining numbers not in T. As the base cases use the smallest element in each part, and then replicate an isomorphic edge with other elements, no illegal edges are created, but a second upper multipartite shadow is generated minimizing extraneous edges.

Similarly, (iii) generates the first upper shadow when $h = 2$. That means that each 2-subset of T needs one additional element. As above, we must find the most efficient way of creating these 3-subsets to minimize the first upper shadow. Thus, let p represent a portion of the numbers 1 to $n - 1$ and q represent the numbers 1 to $n - 1$ that are not in p and does not equal $p + 1$. Considering the first inequality, the first p terms follow a similar pattern. For example $\binom{n-1}{2}$ represents matching 1 with all of the 2-subsets of the values 2 through n. Then, $\binom{n-2}{2}$ represents matching 2 with all of the 2-subsets of the values 3 through n and not 1. This pattern continues until $\binom{n-p}{2}$ represents p matched with all of the 2-subsets of the values $p + 1$ through n. In order to minimize our first upper shadow, as seen in the second inequality, we instead match each number $\{1, \ldots p\}$ with each 2-subset of values $p + 1$ to n, generated by $\binom{n-p}{2}$. Though this estimation excludes the 2-subsets created between 1 and p matched with a number $p + 1$ to n, it does generate a smaller lower bound. These two approaches generate all of the 3-subsets of the numbers 1 through p, though they do not represent the 3-subsets that do not use numbers in p. Thus, the commonality in these two proofs is the last two terms. $q(n - p - 2)$ represents the 3-subsets of $p + 1$ matched with a number in q and some value not $p + 1$ or the number from q. That is carried out for each value in q, which is why $n - p - 2$ is multiplied by q. Finally, $\binom{q}{2}$ must be subtracted because of the repetition that occurs among the numbers in q and $n - p - 2$ as $\binom{q}{2}$ of these 2-subsets will occur twice with $p + 1$. By creating as many 3-subsets as possible using elements in p and reducing the number generated with elements outside of p, we are able to achieve our lower bound of the size of the first upper shadow while still respecting the multipartite properties of the graph.

□

Remark II.3.2. *Note that in Lemma II.3.1 (iii), since $|T|$ is positive,*

$$\begin{aligned}|T| &= m^2\left(pn - \frac{p(p+1)}{2} + q\right) \geq m^2 p\left(\frac{2n-p-1}{2}\right) \\ &> m^2 p\left(\frac{2n-n-1}{2}\right) = m^2 p\left(\frac{n-1}{2}\right) \geq 0\end{aligned}$$

Additionally, note that when $n \geq 85$ and $p \leq 8$, the bound found in Lemma II.3.1 (iii) can be simplified as follows.

Lemma II.3.3. *If p and q are defined as in Lemma II.3.1 (iii), then* $|\partial_1^+(T)| \geq m^2 p\binom{n-p}{2} + \frac{2m^2qn}{5}$.

Proof. Since $q < n - (p+1)$ and $p \geq 0$, we have $q \geq n-2$. Additionally, since $n \geq 85$, $p \leq 8$, and $q \leq n-2$, we have $10n - 10p - 20 - 5q + 5 \geq 4n$. Alternative this can be expressed as $n - p - 2 - \frac{q-1}{2} \geq \frac{2n}{5}$. When both sides are multiplied by q, this implies $q(n-p-2) - \binom{q}{2} \geq \frac{2qn}{5}$. This gives us,

$$\begin{aligned}
|\partial_1^+(T)| &\geq m^2\left(p\binom{n-p}{2} + q(n-p-2) - \binom{q}{2}\right) \\
&\geq m^2\left(p\binom{n-p}{2} + \frac{2qn}{5}\right) \\
&\geq m^2 p\binom{n-p}{2} + \frac{2m^2qn}{5}.
\end{aligned}$$

□

CHAPTER III: PROOF OF THE MAIN RESULT

III.1 Set-up and Supplemental Theorems

Using the following lemma we are able to change the problem of decomposing $K^h_{n\times m}$ into c-cycles into a problem where we match the hyperedges to known c-cycle decompositions of $K_{n\times m}$ and thusly decompose $K^h_{n\times m}$ when a perfect matching exists. An example of how one such perfect matching is applied is shown in III.2. This proof follows similarly constructions to those seen in [7] and [1].

Lemma III.1.1 (Perfect Matching Lemma). *Given an n-colored, h-uniform equipartite hypergraph with parts of size m, $\mathcal{H}$, and a graph G with $V(\mathcal{H}) = V(G)$, let $B[X,Y]$ be a bipartite graph with $X := E(\mathcal{H}), Y := E(G)$, such that for $x \in X$ and $y \in Y$, xy is an edge in B if $y \subseteq x$. If B has a perfect matching, and G can be decomposed into cycles of length c, then $\mathcal{H}$ can be decomposed into cycles of length c.*

Proof. Note that the vertices in X are h-subsets, and the vertices in Y are 2-subsets. Let M be a perfect matching in B. Let C be a c-cycle in the graph G. We can then write the cycle C as the sequence $v_1, b_1, v_2, b_2, \ldots, v_c, b_c, v_1$ where $i \in [c]$, $v_i \in V$, and $b_i \in Y$. For each $i \in [c]$, let e_i be the vertex in X such that $e_ib_i \in M$. By the definition of B, the sequence $v_1, e_1, v_2, e_2, \ldots, v_c, e_c, v_1$ represents a c-cycle in the hypergraph $\mathcal{H}$. Thus, each edge in the decompositon of G corresponds to a cycle of the same length in $\mathcal{H}$. By assumption, M is a perfect matching and the decomposition of G into cycles covers all of the edges in G exactly once, we obtain the desired decomposition of H into cycles of length c. □

In order to generate the bipartite graph described above, we will need to use the results from Corollary I.0.3 and Theorem I.0.1 in conjunction for the following theorem and one of its consequences.

Theorem III.1.2. *There exists a $\lambda K_{n\times m}$ c-cycle decomposition, if:*

- $2 \leq c \leq mn$;
- $m \mid c$;
- $\lambda(n-1)$ *is even;*
- $c \leq \frac{\lambda}{2}m^2\binom{n}{2}$ *when* λ *is even, and* $c \leq (\lambda-1)m^2\binom{n}{2}$ *when* λ *is odd.*

Proof. Let $\lambda(n-1)$ be even and $d \mid \lambda\binom{n}{2}$, thus λK_n admits a d-cycle decomposition by Corollary I.0.3. Using m-fold λK_n we then know $\lambda m(n-1)$ is even and $d \mid \lambda m\binom{n}{2}$. As we are working with fixed length cycles, the degree of each vertex is even, thus $\lambda m(n-1)$ will be even. Then, using Theorem I.0.1, if $\lambda m K_n$ has a d-cycle decomposition, then $\lambda K_{n\times m}$ has a dm-cycle decomposition. Let $dm = c$, or equivalently, $d = \frac{c}{m}$, this provides us with necessary conditions:

- $m \mid c$
- $\frac{c}{m} \mid \lambda m\binom{n}{2}$, or equivalently, $c \mid \lambda m^2\binom{n}{2}$
- $c \leq \frac{\lambda}{2}m^2\binom{n}{2}$ when λ is even, and $c \leq (\lambda-1)m^2\binom{n}{2}$ when λ is odd.

Thus, if these conditions are met, $\lambda K_{n\times m}$ admits a c-cycle decomposition.

□

If we let $\lambda = 2$ then we have an immediate consequence.

Corollary III.1.3. *Let $m \mid c$, $c \mid m^2n(n-1)$ for $2 \leq c \leq mn$, then $2K_{n\times m}$ has a c-cycle decomposition.*

For the remainder of this section, let n, m, h, c and L satisfy the conditions of Theorem I.0.4. The following parameters will also be used in support of our proof:

$$\alpha = \left\lfloor \frac{m^h\binom{n}{h} - |L|}{m^2n(n-1)} \right\rfloor, \beta = \frac{m^h\binom{n}{h} - |L| - \alpha m^2n(n-1)}{c}$$

β can be further simplified using our assumptions from Theorem I.0.4, $c|m^h\binom{n}{h} - |L|$ and $c|m^2n(n-1)$, thus β is a non-negative integer. Since $\alpha > \frac{m^h\binom{n}{h}-|L|}{m^2n(n-1)} - 1$ then implies $\alpha m^2n(n-1) > m^h\binom{n}{h} - |L| - m^2n(n-1)$ and $\beta c < m^2n(n-1)$ implies $\beta \leq \frac{m^2n(n-1)}{c} - 1$. This implies:

$$\beta c \leq m^2 n(n-1) - c. \tag{III.1}$$

To prove Theorem I.0.4, we must combine Lemma III.1.1 with conditions that satisfy Philip Hall's Marriage Theorem. Before we can do this, we must clarify some notation. Note that for graphs A and B, let the union of A and B, denoted by $A \cup B$, and the disjoint union of A and B, denoted by $A + B$, represent graph with vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B)$. It should be noted that $A \cup B$ can be a multigraph, as the edges are not disjoint, and that for $A + B$, $V(A)$ and $V(B)$ are disjoint sets. Finally, note that the disjoint union of graphs $A_1 \ldots A_w$ is denoted by $\sum_{i=1}^{w} A_i$.

Since $m \mid c$ and $c \mid m^2 n(n-1)$, by Corollary III.1.3 we know that $2K_{n\times m}$ has a c-cycle decomposition. Let $G_1 = \sum_{i=1}^{\beta} C_i$, where $C_1, \ldots C_\beta$ are β arbitrary cycles from a $2K_{n\times m}$ decomposition into c-cycles. Similarly, let $G_2 = \sum_{i=1}^{\alpha}(M_{2i-1} \cup M_{2i})$, where M_i is a copy of $K_{n\times m}$ for any $i \in \{1, \ldots, 2\alpha\}$. By III.1, we know that G_1 is well defined. Additionally, since $M_{2i-1} \cup M_{2i} \cong 2K_{n\times m}$ for $i \in \{1, \ldots, \alpha\}$ by Corollary III.1.3, G_2 can be decomposed into c-cycles. Let $\mathcal{H} = K_{n\times m}^h \setminus L$ and $G := G_1 + G_2$. Note that by the definition of G_2, G is a multigraph which can be decomposed into c-cycles. Additionally, $|E(\mathcal{H})| = |E(G)|$. Therefore, let $B[X, Y]$ be the balanced bipartite graph as described in Lemma III.1.1, where $X = E(\mathcal{H})$ and $Y = E(G)$, such that for $x \in X$ and $y \in Y$, xy is an edges in B if $y \subset x$. By Lemma III.1.1 in order to decompose $\mathcal{H}$ into c-cycles, it suffices to show that B has a perfect matching. In order to achieve these, we will rely on the consequences of the Kruskal-Katona Theorem described in Lemma II.3.1 and Lemma II.3.3 to ensure that B satisfies Philip Hall's condition. Therefore, our goal is to show that $|N(S)| \geq |S|$ for any $S \subseteq X$ where $S \neq \emptyset$.

For ease of notation, we will be using the following parameters in the remainder of our proof:

- Let $s \in \mathbb{R}$, with $h \leq s \leq n$, $m \geq 1$ such that $|S| = m^h\binom{s}{h}$.

- $a = \frac{|S|}{m^h\binom{n}{h}} = \frac{m^h\binom{s}{h}}{m^h\binom{n}{h}} = \frac{\binom{s}{h}}{\binom{n}{h}}$. Then $0 \le a \le 1$.

- $b = \frac{|N(S)\cap E(M_1)|}{m^2\binom{n}{2}}$, where M_1 is a copy of $K_{n\times m}$. Note that $0 \le b \le 1$.

- $g = \frac{m^h\binom{n}{h} - |X| + m^2 n(n-1) - c}{m^h\binom{n}{h}}$. Alternatively, $1 - g = \frac{|X| - m^2 n(n-1) + c}{m^h\binom{n}{h}}$.

Lemma III.1.4. $|N(S)| \ge a^{\frac{2}{h}}(|X| - m^2 n(n-1) + c)$

Proof. Note that each $s \in S$ is an n-colored h-subset of [mn] and $N(S) \cap E(M_1) = \partial^-_{h-2}(S)$. By Lemma II.3.1 (i), $|N(S) \cap E(M_1)| \ge m^2\binom{s}{2}$. Therefore, since $bm^2\binom{n}{2} = |N(S) \cap E(M_1)|$, $bm^2\binom{n}{2} \ge m^2\binom{s}{2}$. This implies:

$$\begin{aligned}
b^h &\ge \left(\frac{m^2\binom{s}{2}}{m^2\binom{n}{2}}\right)^h = \frac{\binom{s}{2}^h}{\binom{n}{2}^h} = \frac{(s(s-1))^h}{(n(n-1))^h} \\
&= \left(\prod_{i=1}^{h} \frac{s}{n}\right)\left(\prod_{i=1}^{h} \frac{s-1}{n-1}\right) \\
&= \left(\left(\frac{s-1}{n-1}\right)\prod_{i=1}^{h-1} \frac{s}{n}\right)\left(\left(\frac{s}{n}\right)\prod_{i=2}^{h} \frac{s-1}{n-1}\right) \\
&\ge \left(\frac{s-h+1}{n-h+1}\prod_{i=1}^{h-1} \frac{s-i+1}{n-i+1}\right)\left(\frac{s}{n}\prod_{i=2}^{h} \frac{s-i+1}{n-i+1}\right) \\
&= \left(\prod_{i=1}^{h} \frac{s-i+1}{n-1+1}\right)\left(\prod_{i=1}^{h} \frac{s-i+1}{n-i+1}\right) \\
&= \left(\frac{s(s-1)\dots(s-h+1)}{n(n-1)\dots(n-h+1)}\right)^2 = \frac{\binom{s}{h}^2}{\binom{n}{h}^2} = \left(\frac{\binom{s}{h}}{\binom{n}{h}}\right)^2 = a^2
\end{aligned}$$

Note: this uses the fact that $\frac{s}{n} \ge \frac{s-1}{n-1} \ge \frac{s-i+1}{n-i+1}$, for $2 \le i \le h$.

Therefore, $b^h \ge a^2$ implies $b \ge a^{\frac{2}{h}}$.

$$|N(S)| \ge 2\alpha |N(S) \cap E(M_1)|$$

Since $|N(S) \cap E(M_1)| = bm^2\binom{n}{2} \geq m^2\binom{s}{2}$ and $b \geq a^{\frac{2}{h}}$, which implies:

$$|N(S)| \geq 2\alpha a^{\frac{2}{h}} m^2 \binom{n}{2}$$
$$= a^{\frac{2}{h}} \alpha m^2 n(n-1)$$

By the definition of β, $\beta c = m^h\binom{n}{h} - |L| - \alpha m^2 n(n-1)$. Therefore, $\alpha m^2(n-1) = m^h\binom{n}{h} - |L| - \beta c$. Since $|X| = m^h\binom{n}{h} - |L|$, we have $\alpha m^2 n(n-1) = |X| - \beta c$. Hence:

$$|N(S)| \geq a^{\frac{2}{h}}(|X| - \beta c) \geq a^{\frac{2}{h}}(|X| - (m^2 n(n-1) - c))$$

by (III.1). Therefore,

$$|N(S)| \geq a^{\frac{2}{h}}(|X| - m^2 n(n-1) + c)$$

□

Lemma III.1.5. *If* $a^{1-\frac{2}{h}} \leq 1 - g$, *then* $|N(S)| \geq |S|$.

Proof. By Lemma III.1.4, we have:

$$|N(S)| \geq a^{\frac{2}{h}}(|X| - m^2 n(n-1) + c)$$

Since $1 - g = \frac{|X| - m^2 n(n-1) + c}{m^h\binom{n}{h}}$ and $a^{1-\frac{2}{h}} \leq 1 - g$, we have:

$$|N(S)| \geq a^{\frac{2}{h}} m^h \binom{n}{h}(1-g) \geq a^{\frac{2}{h}} a^{1-\frac{2}{h}} m^h \binom{n}{h} = am^h\binom{n}{h} = |S|$$

Therefore, $|N(S)| \geq |S|$ □

Lemma III.1.6. *Let* $S' = Y \setminus N(S)$. *If* $|N(S')| \geq |S'|$, *then* $|N(S)| \geq |S|$.

Proof. Let $|N(S')| \geq |S'|$.

By the definition of S', $|S'| = |Y| - |N(S)|$.

Thus, $|N(S)| = |Y| - |S'|$. This gives us

$$|N(S)| = |Y| - |S'| \geq |Y| - |N(S')| = |X| - |N(S')| \geq |S|$$

Therefore, it suffices to show that $|N(S')| \geq |S'|$ to prove $|N(S)| \geq |S|$. $\square$

Now let $S'_1 = S' \cap E(M_1)$.

Lemma III.1.7. *If $S' \neq \emptyset$, then*

$$2\alpha \leq |S'| \leq (2\alpha + 2)|S'_1|.$$

Proof. We have

$$\begin{aligned} |S'| &= \sum_{i=1}^{2\alpha} |S' \cap E(M_i)| + |S' \cap E(G_1)| = 2\alpha|S'_1| + |S' \cap E(G_1)| \\ &\leq 2\alpha|S'_1| + 2|S'_1| = (2\alpha + 2)|S'_1|. \end{aligned}$$

Note that since this is chosen from a decomposition, this inequality also indicates that each element in S' can only appear in at most 2 cycles of G_1. This shows us additionally that $|S'| = \sum_{i=1}^{2\alpha} |S' \cap E(M_i)| + |S' \cap E(G_1)| \geq 2\alpha|S'_1| \geq 2\alpha$. Thus, $|S'| \geq 2\alpha$ $\square$

Before beginning the general proof, we will now provide a small case example on how this bipartite graph is generated and a matching is made. In that case, a perfect matching was generated by hand, but the process of proving the existence of a matching is used instead in this proof to generalize our findings.

III.2 A small case example of this proof:

$$h = 4, n = 5, m = 2, c = 4$$

As seen in Lemma III.1.1, we must create a balanced bipartite graph. Let there exist a bipartite graph $B[X, Y]$ such that X is the set of all 4-edges of the graph $K^4_{5\times3}$ and Y is the set of copies of 2-edges from $2K_{5\times3}$. Our goal is to decompose $K^4_{5\times3}$ into cycles of length 4. First, let X be the following set:

$$
\begin{aligned}
X =\{&\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,9\},\{1,2,3,10\},\{1,2,4,5\},\\
&\{1,2,4,8\},\{1,2,4,10\},\{1,2,5,8\},\{1,2,5,9\},\{1,2,8,9\},\\
&\{1,2,8,10\},\{1,2,9,10\},\{1,3,4,5\},\{1,3,4,7\},\{1,3,4,10\},\\
&\{1,3,5,7\},\{1,3,5,9\},\{1,3,7,9\},\{1,3,7,10\},\{1,3,9,10\},\\
&\{1,4,5,7\},\{1,4,5,8\},\{1,4,7,8\},\{1,4,7,10\},\{1,4,8,10\},\\
&\{1,5,7,8\},\{1,5,7,9\},\{1,5,8,9\},\{1,7,8,9\},\{1,7,8,10\},\\
&\{1,7,9,10\},\{1,8,9,10\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,4,10\},\\
&\{2,3,5,6\},\{2,3,5,9\},\{2,3,6,9\},\{2,3,6,10\},\{2,3,9,10\},\\
&\{2,4,5,6\},\{2,4,5,8\},\{2,4,6,8\},\{2,4,6,10\},\{2,4,8,10\},\\
&\{2,5,6,8\},\{2,5,6,9\},\{2,5,8,9\},\{2,6,8,9\},\{2,6,8,10\},\\
&\{2,6,9,10\},\{2,8,9,10\},\{3,4,5,6\},\{3,4,5,7\},\{3,4,6,7\},\\
&\{3,4,6,10\},\{3,4,7,10\},\{3,5,6,7\},\{3,5,6,9\},\{3,5,7,9\},\\
&\{3,6,7,9\},\{3,6,7,10\},\{3,6,9,10\},\{3,7,9,10\},\{4,5,6,7\},\\
&\{4,5,6,8\},\{4,5,7,8\},\{4,6,7,8\},\{4,6,7,10\},\{4,6,8,10\},\\
&\{4,7,8,10\},\{5,6,7,8\},\{5,6,7,9\},\{5,6,8,9\},\{5,7,8,9\},\\
&\{6,7,8,9\},\{6,7,8,10\},\{6,7,9,10\},\{6,8,9,10\},\{7,8,9,10\}\}
\end{aligned}
$$

Similarly let $Y = E(2K_{n\times m})$ where each vertex is a 2-edge of the graph. As such let Y be the following set:

$$
\begin{aligned}
Y =\{&\{1,2\},\{1,2\},\{1,3\},\{1,3\},\{1,4\},\{1,4\},\{1,5\},\{1,5\},\\
&\{1,7\},\{1,7\},\{1,8\},\{1,8\},\{1,9\},\{1,9\},\{1,10\},\{1,10\},\\
&\{2,3\},\{2,3\},\{2,4\},\{2,4\},\{2,5\},\{2,5\},\{2,6\},\{2,6\},\\
&\{2,8\},\{2,8\},\{2,9\},\{2,9\},\{2,10\},\{2,10\},\{3,4\},\{3,4\},\\
&\{3,5\},\{3,5\},\{3,6\},\{3,6\},\{3,7\},\{3,7\},\{3,9\},\{3,9\},\\
&\{3,10\},\{3,10\},\{4,5\},\{4,5\},\{4,6\},\{4,6\},\{4,7\},\{4,7\},\\
&\{4,8\},\{4,8\},\{4,10\},\{4,10\},\{5,6\},\{5,6\},\{5,7\},\{5,7\},\\
&\{5,8\},\{5,8\},\{5,9\},\{5,9\},\{6,7\},\{6,7\},\{6,8\},\{6,8\},\\
&\{6,9\},\{6,9\},\{6,10\},\{6,10\},\{7,8\},\{7,8\},\{7,9\},\{7,9\},\\
&\{7,10\},\{7,10\},\{8,9\},\{8,9\},\{8,10\},\{8,10\},\{9,10\},\{9,10\}\}
\end{aligned}
$$

Note that since this $K_{5\times3}$ is 2-fold, each edge in $K_{5\times3}$ occurs twice. It can be seen that $|X| = 2^4\binom{5}{4} = 80$ and $|Y| = 2(5)(5-1) = 80$. Therefore with one copy of $2K_{5\times3} \subseteq Y$ we have that $|X| = |Y|$. Since this is true, $B[X,Y]$ is a balanced bipartite graph. Thus, if Y can be decomposed into 4-cycles, we will be able to find a perfect matching in $B[X,Y]$. Below is a complete decomposition of Y into 4-cycles, which means that all 80 edges can be partitioned into 4-cycles. It should be noted that our cycles take the form $v_1, e_1, v_2, e_2, \ldots, e_{c-1}, v_c, e_c, v_1$, where $v_i, v_{i+1} \subseteq e_i$ and $v_{c+1} = 1$.

$$1, \{1, 2\}, 2, \{2, 6\}, 6, \{6, 7\}, 7, \{7, 1\}, 1;$$

$$1, \{1, 2\}, 2, \{2, 6\}, 6, \{6, 7\}, 7, \{7, 1\}, 1;$$

$$2, \{2, 3\}, 3, \{3, 7\}, 7, \{7, 8\}, 8, \{8, 2\}, 2;$$

$$2, \{2, 3\}, 3, \{3, 7\}, 7, \{7, 8\}, 8, \{8, 2\}, 2;$$

$$3, \{3, 4\}, 4, \{4, 8\}, 8, \{8, 9\}, 9, \{9, 3\}, 3;$$

$$3, \{3, 4\}, 4, \{4, 8\}, 8, \{8, 9\}, 9, \{9, 3\}, 3;$$

$$4, \{4, 5\}, 5, \{5, 9\}, 9, \{9, 10\}, 10, \{10, 4\}, 4;$$

$$4, \{4, 5\}, 5, \{5, 9\}, 9, \{9, 10\}, 10, \{10, 4\}, 4;$$

$$5, \{5, 1\}, 1, \{1, 10\}, 10, \{10, 6\}, 6, \{6, 5\}, 5;$$

$$5, \{5, 1\}, 1, \{1, 10\}, 10, \{10, 6\}, 6, \{6, 5\}, 5;$$

$$1, \{1, 3\}, 3, \{3, 6\}, 6, \{6, 8\}, 8, \{8, 1\}, 1;$$

$$1, \{1, 3\}, 3, \{3, 6\}, 6, \{6, 8\}, 8, \{8, 1\}, 1;$$

$$2, \{2, 4\}, 4, \{4, 7\}, 7, \{7, 9\}, 9, \{9, 2\}, 2;$$

$$2, \{2, 4\}, 4, \{4, 7\}, 7, \{7, 9\}, 9, \{9, 2\}, 2;$$

$$3, \{3, 5\}, 5, \{5, 8\}, 8, \{8, 10\}, 10, \{10, 3\}, 3;$$

$$3, \{3, 5\}, 5, \{5, 8\}, 8, \{8, 10\}, 10, \{10, 3\}, 3;$$

$$4, \{4, 1\}, 1, \{1, 9\}, 9, \{9, 6\}, 6, \{6, 4\}, 4;$$

$$4, \{4, 1\}, 1, \{1, 9\}, 9, \{9, 6\}, 6, \{6, 4\}, 4;$$

$$5, \{5, 2\}, 2, \{2, 10\}, 10, \{10, 7\}, 7, \{7, 5\}, 5;$$

$$5, \{5, 2\}, 2, \{2, 10\}, 10, \{10, 7\}, 7, \{7, 5\}, 5.$$

Thus, there exists a complete decomposition of $2K_{5\times 3}$ into 4-cycles. Since $|X| = |Y|$ and there exists a complete decomposition of Y into 4-cycles, there can be a perfect matching in $B[X, Y]$. Note that edges in B are defined as xy for $x \in X$ and $y \in Y$ such that $y \subseteq x$. For example, there is an edge between $\{1,2\}$ and $\{1,2,3,4\}$ as $\{1,2\} \subseteq \{1,2,3,4\}$. Table 2 shows a perfect matching of $B[X, Y]$. In a general case, we would need to find the upper or lower shadow to find this perfect matching. Consider the fact that $\{1,2\}$ is part of the $h-2$ lower shadow of $\{1,2,3,4\}$, and $\{1,2,3,4\}$ is part of the second upper shadow of $\{1,2\}$. Thus, those shadows and the Colored Kruskal-Katona Theorem help us create approximations for the neighborhood of a set S.

From this perfect matching we are able to take our 4-cycles from Y and map a hyperedge from X to overlay each of the simple edges. For example, the cycle

$$1, \{1,2\}, 2, \{2,6\}, 6, \{6,7\}, 7, \{7,1\}, 1$$

can have each edge expanded into

$$1, \{1,3,4,2\}, 2, \{2,3,5,6\}, 6, \{6,8,9,7\}, 7, \{7,3,5,1\}, 1$$

using the matching in Table A-1. Note that the order in which the elements appear in these sets does not matter, as such $\{1,2,3,4\}$ is the same edge as $\{4,1,3,2\}$ as both subsets contain the same elements. The mapping process is repeated for all 20 cycles, and this results in the complete decomposition of $K^4_{5\times2}$ into 4-cycles. This decomposition listed below.

$$1, \{1,3,4,2\}, 2, \{2,3,5,6\}, 6, \{6,8,9,7\}, 7, \{7,3,5,1\}, 1;$$
$$1, \{1,3,5,2\}, 2, \{2,3,4,6\}, 6, \{6,8,10,7\}, 7, \{7,3,9,1\}, 1;$$
$$2, \{2,1,9,3\}, 3, \{3,1,10,7\}, 7, \{7,1,4,8\}, 8, \{8,4,5,2\}, 2;$$
$$2, \{2,4,10,3\}, 3, \{3,4,5,7\}, 7, \{7,9,10,8\}, 8, \{8,1,5,2\}, 2;$$
$$3, \{3,5,6,4\}, 4, \{4,1,2,8\}, 8, \{8,2,10,9\}, 9, \{9,5,7,3\}, 3;$$
$$3, \{3,6,7,4\}, 4, \{4,6,10,8\}, 8, \{8,1,7,9\}, 9, \{9,6,7,3\}, 3;$$
$$4, \{4,7,8,5\}, 5, \{5,1,7,9\}, 9, \{9,1,3,10\}, 10, \{10,1,8,4\}, 4;$$
$$4, \{4,6,8,5\}, 5, \{5,1,8,9\}, 9, \{9,1,3,10\}, 10, \{10,1,7,4\}, 4;$$

5, {5, 2, 4, 1}, 1, {1, 2, 3, 10}, 10, {10, 2, 9, 6}, 6, {6, 7, 8, 5}, 5;

5, {5, 3, 9, 1}, 1, {1, 2, 4, 10}, 10, {10, 2, 9, 6}, 6, {6, 7, 9, 5}, 5;

1, {1, 4, 5, 3}, 3, {3, 2, 10, 6}, 6, {6, 9, 10, 8}, 8, {8, 4, 5, 1}, 1;

1, {1, 4, 7, 3}, 3, {3, 2, 9, 6}, 6, {6, 2, 10, 8}, 8, {8, 2, 10, 1}, 1;

2, {2, 3, 5, 4}, 4, {4, 6, 10, 7}, 7, {7, 3, 10, 9}, 9, {9, 3, 5, 2}, 2;

2, {2, 6, 8, 4}, 4, {4, 6, 10, 7}, 7, {7, 1, 10, 9}, 9, {9, 5, 6, 2}, 2;

3, {3, 6, 7, 5}, 5, {5, 6, 9, 8}, 8, {8, 1, 9, 10}, 10, {10, 4, 6, 3}, 3;

3, {3, 6, 9, 5}, 5, {5, 2, 9, 8}, 8, {8, 1, 7, 10}, 10, {10, 2, 9, 3}, 3;

4, {4, 3, 10, 1}, 1, {1, 2, 5, 9}, 9, {9, 2, 8, 6}, 6, {6, 5, 7, 4}, 4;

4, {4, 5, 7, 1}, 1, {1, 2, 8, 9}, 9, {9, 3, 10, 6}, 6, {6, 7, 8, 4}, 4;

5, {5, 4, 6, 2}, 2, {2, 4, 8, 10}, 10, {10, 3, 4, 7}, 7, {7, 8, 9, 5}, 5;

5, {5, 6, 8, 2}, 2, {2, 4, 6, 10}, 10, {10, 3, 6, 7}, 7, {7, 1, 8, 5}, 5.

III.3 Proof of Theorem I.0.4.

Case A. $h = 3, n \geq 85$, and L is a matching.

Case A.1. $|S| \leq |X| - 3m^2n(n-1) + c$.

Proof. Since $|L| = m^3\binom{n}{3} - |X|$ we have:

$$g = \frac{m^3\binom{n}{3} - |X| + m^2n(n-1) - c}{m^3\binom{n}{3}} = \frac{|L| + m^2n(n-1) - c}{m^3\binom{n}{3}}$$

Since $|L| < c \leq mn$ and $n \geq 85$, we have:

$$g \leq \frac{m^2n(n-1)}{\frac{m^3n(n-1)(n-2)}{6}} = \frac{6}{m(n-2)} \leq 3$$

By Lemma III.1.5 we have $a^{1-\frac{2}{3}} = a^{\frac{1}{3}} \leq 1 - g$. Therefore, it suffices to show that $a \leq (1-g)^3$.

We have:

$$am^3\binom{n}{3} = |S| \leq |X| - 3m^2n(n-1) + c$$
$$= |X| - 2c - 3[m^2n(n-1) - c]$$

Since $|L| < c \leq mn$,

$$|S| \leq |X| - 2|L| - 3[m^2n(n-1) - c] = |X| - 2[m^3\binom{n}{3} - |X|] - 3[m^2n(n-1) - c]$$
$$= 3|X| - 2m^3\binom{n}{3} - 3m^2n(n-1) + 3c = 3(|X| - m^2n(n-1) + c) - 2m^3\binom{n}{3}$$
$$= 3(1-g)m^3\binom{n}{3} - 2m^3\binom{n}{3} = m^3\binom{n}{3}[3(1-g) - 2]$$
$$= m^3\binom{n}{3}(1-3g)$$

Since $g \leq 3$, we have:

$$m^3\binom{n}{3}(1-3g) \leq (1-g)^3m^3\binom{n}{3}$$

.

Since $am^3\binom{n}{3} \leq (1-g)^3m^3\binom{n}{3}$, this implies $a \leq (1-g)^3$. Therefore by Lemma III.1.5, $|N(S)| \geq |S|$. Therefore by Lemma III.1.1, $\mathcal{H}$ can be decomposed into c-cycles.

Case A.2. $|S| > |X| - 3m^2n(n-1) + c$.

As stated in Lemma III.1.6, to show $|N(S)| \geq |S|$ it suffices to show $|N(S')| \geq |S'|$. From the assumed statement of Case A.2. we have,

$$|S| > |X| - 3m^2n(n-1) + c$$
$$|S| - |X| > -3m^2n(n-1) + c$$
$$|X| - |S| < 3m^2n(n-1) - c < 3m^2n(n-1)$$

$$|N(S')| \leq |X| - |S| < 3m^2n(n-1) \tag{III.2}$$

Let $p, q \in \mathbb{N} \cup \{0\}$, such that $p < n, q < n - (p+1)$, and $|S'| = m^2(pn - \binom{p+1}{2} + q)$. For our purposes, it should be noted that $|S'_1|$ indicates a 2-edge of $[mn]$ and $N(S'_1) = \partial_1^+(S'_1) \setminus L$. Using this definition in conjunction with Lemma II.3.1 (iii), we have

$$|N(S'_1)| \geq |\partial_1^+(S'_1)| - |L| \geq m^2\left(p\binom{n-p}{2} + q(n-p-2) - \binom{q}{2}\right) - |L|$$

Since $q < n - (p+1)$, we have $q(n-p-2) - \binom{q}{2} \geq 0$.

If $p = 9$, then since L is a matching, $|N(S')| \geq 9m^2\binom{n-9}{2} - \frac{m^2n}{3} > 3m^2n(n-1)$ when $n \geq 85$. This contradicts (III.2) that $|N(S')| \leq 3m^2n(n-1)$. Alternatively, if $p > 9$, then by Lemma II.3.1 (iii),

$$\begin{aligned}
|N(S')| &\geq m^2\left(\binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{n-p}{2} + q(n-p-2) - \binom{q}{2}\right) - |L| \\
&\geq m^2\left(\binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{n-9}{2} + \frac{n}{3}\right) \\
&\geq m^2 9\binom{n-9}{2} - \frac{m^2n}{3} \\
&> 3m^2n(n-1)
\end{aligned}$$

This is also a contradiction of the condition that $|N(S')| \leq 3m^2n(n-1)$. Thus, $p \leq 8$. Let $L(S'_1) = L \cap N(S'_1)$. Since L is a matching in the assumptions of Case A, $|L(S'_1)| \leq |S'_1|$. Assume that $p = 0$. Then, $|S'_1| = q$ and $|L(S'_1)| \leq q$. By these assumptions and since $n \geq 85$, by Lemma

II.3.3 we have,

$$\begin{aligned}
|N(S')| = |N(S'_1)| &\geq \frac{2m^2qn}{5} \\
&\geq \frac{2m^2qn}{5} - |L(S'_1)| \\
&\geq \frac{2m^2qn}{5} - q = q\left(\frac{2m^2n}{5} - 1\right) \\
&\geq q\left(\frac{2mn}{5} - 1\right)
\end{aligned}$$

$$\geq q\left(\frac{m(n-2)}{3} + 2\right) = |S'_1|\left(\frac{m(n-2)}{3} + 2\right) \tag{III.3}$$

Note that since $h = 3$,

$$\begin{aligned}
\alpha = \left\lfloor \frac{m^3\binom{n}{3} - |L|}{m^2n(n-1)} \right\rfloor &= \left\lfloor \frac{\frac{1}{6}m^3n(n-1)(n-2) - |L|}{m^2n(n-1)} \right\rfloor \\
&\leq \left\lfloor \frac{\frac{1}{6}m^3n(n-1)(n-2)}{m^2n(n-1)} \right\rfloor \\
&\leq \frac{m(n-2)}{6}
\end{aligned}$$

Therefore, $2\alpha \leq \frac{m(n-2)}{3}$ using (III.3) and Lemma III.1.7, we have

$$\begin{aligned}
|N(S')| &\geq |S'_1|\left(\frac{m(n-2)}{3} + 2\right) \\
&\geq |S'_1|(2\alpha + 2) \geq |S'|.
\end{aligned}$$

Thus, we have $p \in [8]$.

By Lemma II.3.3 we have

$$\begin{aligned}|N(S')| = |N(S_1')| &\geq m^2 p\binom{n-p}{2} + \frac{2m^3qn}{5} - |L(S_1')| \\ &\geq \frac{4m^2p}{5}\binom{n}{2} + \frac{2m^3qn}{5} - \frac{m^2n}{3}\end{aligned}$$

Since, $\frac{2m^2qn}{5} = \frac{12m^2qn}{30} \geq \frac{11m^2q}{10}\frac{(n-2)}{3}$ because $n-2<n$ and

$$\begin{aligned}\frac{4m^2p}{5}\binom{n}{2} - \frac{m^2}{3} &\geq \frac{4m^2pn(n-1) - 4m^2pn}{10} = \frac{4m^2pn(n-2)}{10} \\ &= \frac{12m^2pn(n-2)}{30} = \frac{12m^2p}{10}\frac{n(n-2)}{3} \\ &\geq \frac{11m^2p}{10}\frac{n(n-2)}{3},\end{aligned}$$

we have,

$$\begin{aligned}|N(S')| &\geq \frac{4m^2p}{5}\binom{n}{2} + \frac{2m^3qn}{5} - \frac{m^2n}{3} \\ &\geq \frac{11pm^2}{10}\frac{n(n-2)}{3} + \frac{2m^2qn}{5} \\ &\geq \frac{11pm^2}{10}\frac{n(n-2)}{3} + \frac{11m^2q}{10}\frac{n-2}{3} \\ &= \frac{11m^2}{10}\frac{(n-2)}{3}(pn+q).\end{aligned}$$

Moreover, $n \geq 85 \geq 62$ implies $\frac{11}{10}\frac{n-2}{3} \geq \frac{n-2}{3} + 2$. Therefore,

$$|N(S')| \geq m^2\left(\frac{n-2}{3}+2\right)|S_1'| \geq |S_1'|\left(\frac{n-2}{3}+2\right) \geq |S'|.$$

Hence, $|N(S')| \geq |S'|$ and Hall's condition is satisfied.

Case B. $4 \leq h \leq n-3$ and $n \geq 23$.

Case B.1. $|S| \leq |X| - 2m^2n(n-1) + c$.

By Lemma III.1.5 it is enough to show that $a \leq (1-g)^2$. Since $|L| < c \leq mn$,

$$\begin{aligned} am^h\binom{n}{h} &= |S| \leq |X| - 2m^2n(n-1) + c = |X| - c - 2[m^2n(n-1) - c] \\ &\leq |X| - \left[m^h\binom{n}{h} - |X|\right] - 2[m^2n(n-1) - c] = (1-2g)m^h\binom{n}{h} \\ &\leq (1-g)^2 m^h\binom{n}{h}. \end{aligned}$$

Therefore, $a \leq (1-g)^2$ and by Lemma III.1.5, Hall's condition is satisfied.

Case B.2. $|S| > |X| - 2m^2n(n-1) + c$

Case B.2.1 $h = 4$

By Lemma III.1.6 it is enough to show that $|N(S')| \geq |S'|$. We have

$$|N(S_1')| \leq |X| - |S| < 2m^2n(n-1) - c < 2m^2n(n-1). \tag{III.4}$$

Note that each element of S_1' is an n-colored 2-subset of $[mn]$ and $N(S_1') = \partial_2^+(S_1') \setminus L$. Alternatively, this can be expressed as $|\partial_2^+(S_1')| = |N(S_1')| + |L|$. If $|S_1'| = 7$, then by Lemma II.3.1 (ii) we have $|N(S_1')| \geq 7m^2\binom{n-8}{2} + 21m^2(n-8) - |L|$. Since $n \geq 23$ and $|L| < c \leq mn$, we have $7m^2\binom{n-8}{2} + 21m^2(n-8) - |L| \geq 7m^2\left(\frac{n^2-17n+72}{2}\right) + 20m^2n - 168m^2 \geq 2m^2n(n-1)$. However, this contradicts (III.4). Similarly, if $|S_1'| > 7$ then $|N(S_1')| > 7m^2\binom{n-8}{2} + 21m^2(n-8) - |L| \geq 2m^2n(n-1)$, contradicting (III.4) again. Therefore, $|S_1'| \leq 6$.

Then by Lemma II.3.1 (ii) we have,

$$\begin{aligned} |N(S_1')| &\geq m^2|S_1'|\binom{n-|S_1'|-1}{2} + m^2\binom{|S_1'|}{2}(n-|S_1'|-1) - |L| \\ &\geq m^2|S_1'|\binom{n-|S_1'|-1}{2} - |L| \\ &\geq m^2|S_1'|\binom{n-7}{2} - mn \end{aligned}$$

Since $|S_1'| \le 6$ and $|L| < c \le mn$. Thus by Lemma III.1.7,

$$|N(S')| = |N(S_1')| \ge m^2|S'|\left(\frac{\binom{n-7}{2}}{2\alpha+2}\right) - mn$$

.

Since $h = 4$, $\alpha = \left\lfloor \frac{m^4\binom{n}{4} - |L|}{m^2 n(n-1)} \right\rfloor$,

$$\begin{aligned}
\alpha &= \left\lfloor \frac{\frac{1}{24}m^4 n(n-1)(n-2)(n-3) - |L|}{m^2 n(n-1)} \right\rfloor \\
&= \left\lfloor \frac{m^2(n-2)(n-3)}{24} - \frac{|L|}{m^2 n(n-1)} \right\rfloor \\
&\le \frac{m^2(n-2)(n-3)}{24} - \frac{|L|}{m^2 n(n-1)} \le \frac{m^2(n-2)(n-3)}{24}.
\end{aligned}$$

Therefore, $2\alpha \le \frac{m^2(n-2)(n-3)}{12}$. Then for $n \ge 23 > 12$, this gives us

$$\begin{aligned}
|N(S')| &\ge m^2|S'|\left(\frac{\binom{n-7}{2}}{2\alpha+2}\right) - mn \\
&\ge |S'|\frac{m^2(n-7)(n-8)}{2} \cdot \frac{12}{m^2(n-2)(n-3)+24} - mn \\
&= |S'|\frac{6(n-7)(n-8)}{(n-2)(n-3)+24} - mn \\
&\ge 2|S'| - mn.
\end{aligned}$$

Note that the following consideration of α can also be made,

$$\begin{aligned}
\alpha &= \left\lfloor \frac{m^2(n-2)(n-3)}{24} - \frac{|L|}{m^2 n(n-1)} \right\rfloor \\
&> \frac{m^2(n-2)(n-3)}{24} - \frac{|L|}{m^2 n(n-1)} - 1 \\
&\ge \frac{m^2(n-2)(n-3)}{24} - 2.
\end{aligned}$$

Therefore, $2\alpha \geq \frac{m^2(n-2)(n-3)}{12} - 4$ as well. Since $n \geq 23$, we have $2\alpha > mn$, so by Lemma III.1.7 $|S'| > mn$. Therefore, we have,

$$\begin{aligned}|N(S')| &\geq 2|S'| - mn \\ &> 2|S'| - |S'| = |S'|\end{aligned}$$

Thus, $|N(S')| > |S'|$, indicating that Hall's condition is satisfied.

Case B.2.2 $5 \leq h \leq n-3$

It should be noted that for $h \geq 5$ and $|L| < c \leq mn$, for any $y \in Y$ we have

$$\begin{aligned}|N(y)| &\geq m^{h-2}\binom{n-2}{h-2} - |L| \\ &\geq m^{h-2}\binom{n-2}{3}\frac{n-5}{h-2}\cdot\frac{n-6}{h-3}\cdots\frac{n-h+1}{4} - mn\end{aligned}$$

Since $h \leq n-3$, we also know that $n \geq h+3$, Thus we have the following:

$$\begin{aligned}|N(y)| &\geq m^{h-2}\binom{n-2}{3}\frac{n-5}{h-2}\cdot\frac{n-6}{h-3}\cdots\frac{n-h+1}{4} - mn \\ &\geq m^{h-2}\binom{n-2}{3} - mn \geq m^2\binom{n-2}{3} - mn\end{aligned}$$

Then since $n \geq 23$, we have the following:

$$\begin{aligned}|N(y)| &\geq m^2\binom{n-2}{3} - mn \\ &\geq 2m^2n(n-1) \\ &\geq 2m^2n(n-1) - c\end{aligned}$$

Note that this shows that y has more than $2m^2n(n-1) - c$ neighbors. Therefore, since $|S| > |X| - (2m^2n(n-1) - c)$, then every $y \in Y$ has a neighbor in S. Hence, $N(S) = Y$. Thus, $|N(S)| \geq |S|$ and Hall's condition is satisfied.

Case C. $n-2 \leq h \leq n-1$, and $n \geq 16$.

Case C.1. $m = 1$

Since $m = 1$, these results follow exactly from work completed by Bahmanian and Haghshenas [1]. When $h = n-2$ $c \in \{n-1, n\}$, and $n \geq 16$, then there exists a c-cycle decomposition of $K_{n\times m}^{h} \setminus L$. Additionally, when $h = n-1$ and $c = n$, K_{n}^{n-1} is itself an n-cycle so there exists a c-cycle decomposition of $K_{n\times m}^{h} \setminus L$.

Case C.2. $m \geq 2$

Case C.2.1. $|S| \leq |X| - m^2n(n-1) + c$.

By Lemma III.1.5 it is enough to show that $a \leq (1-g)^{\frac{n-2}{n-4}}$. Since $|L| < c$,

$$\begin{aligned} am^h\binom{n}{h} &= |S| \leq |X| - m^2n(n-1) + c \\ &\leq (1-g)m^h\binom{n}{h}. \end{aligned}$$

Since $\frac{n-2}{n-4} > 1$ for $n \geq 16 > 5$,

$$\begin{aligned} am^h\binom{n}{h} &\leq (1-g)m^h\binom{n}{h} \\ &\leq (1-g)^{\frac{n-2}{n-4}}m^h\binom{n}{h}. \end{aligned}$$

Therefore, $a \leq (1-g)^{\frac{n-2}{n-4}}$ and by Lemma III.1.5, Hall's condition is satisfied.

Case C.2.2. $|S| > |X| - m^2n(n-1) + c$

By Lemma III.1.6 it is enough to show that $|N(S')| \geq |S'|$. For any $y \in Y$, we have

$$|N(y)| \geq m^h\binom{n}{h} - |L|$$

Since $|L| < c$, $2 < h \leq n-1$, $n \geq 5$, and $m \geq 2$ we have the following:

$$\begin{aligned} |N(y)| &\geq m^h \binom{n}{h} - c \\ &\geq m^2 n(n-1) - c \end{aligned}$$

Note that this shows that y has more than $m^2n(n-1) - c$ neighbors. Therefore, since $|S| > |X| - (m^2n(n-1) - c)$, then every $y \in Y$ has a neighbor in S. Hence, $N(S) = Y$. Thus, $|N(S)| \geq |S|$ and Hall's condition is satisfied.

Case D. $h = n$, $n \geq 3$, and $m \geq 3$.

Case D.1. $|S| \leq |X| - m^2n(n-1) + c$.

By Lemma III.1.5 it is enough to show that $a \leq (1-g)^{\frac{n}{n-2}}$. Since $|L| < c$,

$$\begin{aligned} am^h \binom{n}{h} = |S| &\leq |X| - m^2n(n-1) + c \\ &\leq (1-g)m^h \binom{n}{h}. \end{aligned}$$

Since $\frac{n}{n-2} > 1$ for $n \geq 3$,

$$\begin{aligned} am^h \binom{n}{h} &\leq (1-g)m^h \binom{n}{h} \\ &\leq (1-g)^{\frac{n}{n-2}} m^h \binom{n}{h}. \end{aligned}$$

Therefore, $a \leq (1-g)^{\frac{n}{n-2}}$ and by Lemma III.1.5, Hall's condition is satisfied.

Case D.2. $|S| > |X| - m^2n(n-1) + c$

By Lemma III.1.6 it is enough to show that $|N(S')| \geq |S'|$. For any $y \in Y$, we have

$$|N(y)| \geq m^h \binom{n}{h} - |L|$$

Since $|L| < c$, $2 < h \leq n$, $n \geq 3$, and $m \geq 3$ we have the following:

$$|N(y)| \geq m^h \binom{n}{h} - c$$
$$\geq m^2 n(n-1) - c$$

Note that this shows that y has more than $m^2 n(n-1) - c$ neighbors. Therefore, since $|S| > |X| - (m^2 n(n-1) - c)$, then every $y \in Y$ has a neighbor in S. Hence, $N(S) = Y$. Thus, $|N(S)| \geq |S|$ and Hall's condition is satisfied.

□

Table 2: A perfect matching of $B[X,Y]$

$x \in X$	$y \in Y$	$x \in X$	$y \in Y$
$\{1,2,3,4\}$	$\{1,2\}$	$\{3,4,6,10\}$	$\{3,10\}$
$\{1,2,3,5\}$	$\{1,2\}$	$\{2,3,9,10\}$	$\{3,10\}$
$\{1,3,4,5\}$	$\{1,3\}$	$\{4,5,7,8\}$	$\{4,5\}$
$\{1,3,4,7\}$	$\{1,3\}$	$\{4,5,7,8\}$	$\{4,5\}$
$\{1,3,4,10\}$	$\{1,4\}$	$\{4,5,6,7\}$	$\{4,6\}$
$\{1,4,5,7\}$	$\{1,4\}$	$\{4,6,7,8\}$	$\{4,6\}$
$\{1,3,5,9\}$	$\{1,5\}$	$\{4,6,7,10\}$	$\{4,7\}$
$\{1,2,4,5\}$	$\{1,5\}$	$\{4,7,8,10\}$	$\{4,7\}$
$\{1,3,5,7\}$	$\{1,7\}$	$\{1,2,4,8\}$	$\{4,8\}$
$\{1,3,7,9\}$	$\{1,7\}$	$\{4,6,8,10\}$	$\{4,8\}$
$\{1,4,5,8\}$	$\{1,8\}$	$\{1,4,8,10\}$	$\{4,10\}$
$\{1,2,8,10\}$	$\{1,8\}$	$\{1,4,7,10\}$	$\{4,10\}$
$\{1,2,5,9\}$	$\{1,9\}$	$\{5,6,7,8\}$	$\{5,6\}$
$\{1,2,8,9\}$	$\{1,9\}$	$\{5,6,7,9\}$	$\{5,6\}$
$\{1,2,3,10\}$	$\{1,10\}$	$\{5,7,8,9\}$	$\{5,7\}$
$\{1,2,4,10\}$	$\{1,10\}$	$\{1,5,7,8\}$	$\{5,7\}$
$\{1,2,3,9\}$	$\{2,3\}$	$\{5,6,8,9\}$	$\{5,8\}$
$\{2,3,4,10\}$	$\{2,3\}$	$\{2,5,8,9\}$	$\{5,8\}$
$\{2,3,4,5\}$	$\{2,4\}$	$\{1,5,7,9\}$	$\{5,9\}$
$\{2,4,6,8\}$	$\{2,4\}$	$\{1,5,8,9\}$	$\{5,9\}$
$\{2,4,5,6\}$	$\{2,5\}$	$\{6,7,8,9\}$	$\{6,7\}$
$\{2,5,6,8\}$	$\{2,5\}$	$\{6,7,8,10\}$	$\{6,7\}$
$\{2,3,5,6\}$	$\{2,6\}$	$\{6,8,9,10\}$	$\{6,8\}$
$\{2,3,4,6\}$	$\{2,6\}$	$\{2,6,8,10\}$	$\{6,8\}$
$\{2,4,5,8\}$	$\{2,8\}$	$\{2,6,8,9\}$	$\{6,9\}$
$\{1,2,5,8\}$	$\{2,8\}$	$\{3,6,9,10\}$	$\{6,9\}$
$\{2,3,5,9\}$	$\{2,9\}$	$\{2,6,9,10\}$	$\{6,10\}$
$\{2,5,6,9\}$	$\{2,9\}$	$\{6,7,9,10\}$	$\{6,10\}$
$\{2,4,8,10\}$	$\{2,10\}$	$\{1,4,7,8\}$	$\{7,8\}$
$\{2,4,6,10\}$	$\{2,10\}$	$\{7,8,9,10\}$	$\{7,8\}$
$\{3,4,5,6\}$	$\{3,4\}$	$\{3,7,9,10\}$	$\{7,9\}$
$\{3,4,6,7\}$	$\{3,4\}$	$\{1,7,9,10\}$	$\{7,9\}$
$\{3,5,6,7\}$	$\{3,5\}$	$\{3,4,7,10\}$	$\{7,10\}$
$\{3,5,6,9\}$	$\{3,5\}$	$\{3,6,7,10\}$	$\{7,10\}$
$\{2,3,6,10\}$	$\{3,6\}$	$\{2,8,9,10\}$	$\{8,9\}$
$\{2,3,6,9\}$	$\{3,6\}$	$\{1,7,8,9\}$	$\{8,9\}$
$\{1,3,7,10\}$	$\{3,7\}$	$\{1,8,9,10\}$	$\{8,10\}$
$\{3,4,5,7\}$	$\{3,7\}$	$\{1,7,8,10\}$	$\{8,10\}$
$\{3,5,7,9\}$	$\{3,9\}$	$\{1,2,9,10\}$	$\{9,10\}$
$\{3,6,7,9\}$	$\{3,9\}$	$\{1,3,9,10\}$	$\{9,10\}$

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