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Students’ Concept Images of Triangle Attributes

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STUDENTS' CONCEPT IMAGES OF TRIANGLE ATTRIBUTES

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Researchers have identified many ways that students think about rectangular area. However, research about students' understanding of triangle area is relatively scarce. Although it may seem simple to apply the triangle area formula, there is more to students' understanding of triangle area. In this dissertation, I obtained data to help answer the following questions:

1. What are high school geometry students' concept images of triangle base, height, and area?
2. How are high school geometry students' concept images of triangle base, height, and area related to one another?
3. How are triangle orientation, gravity, and triangle type related to high school geometry students' concept images of triangle base, height, and area?

I used an interpretive lens called concept image and concept definition, introduced by Vinner and Hershkowitz (1980), as a tool to investigate students' interpretations of the triangle attributes (i.e., base and height) that are required to measure triangle area. My main data sources were a survey of 95 high school geometry students and semi-structured task-based interviews with 7 of those students. My study has potential benefits for researchers in the field of mathematics education both by demonstrating yet another use of the lens of concept image and concept definition, and by unveiling critical information about students' understanding of triangle area. Likewise, my study has potential benefits for practitioners by revealing the complexity of triangle attributes.

KEYWORDS: Area; Base; Concept Definition; Concept Image; Height; Orientation; Triangles

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CONTENTS

TABLES

FIGURES

CHAPTER I: INTRODUCTION

The concept of area is pervasive in the school curriculum and is foundational for the development of many concepts in school mathematics including multiplication of integers and fractions, multiplying and factoring polynomials, and polynomial division (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2018). The Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) includes topics related to area beginning in Grade 3 and continuing throughout high school. The concept of area also has practical applications, such as in brickwork, painting, and floor covering (e.g., Cavanagh, 2008; Hirstein et al., 1978).

Despite the curricular and practical importance of the area concept, several researchers have found that both the concept of area and the ability to determine the measure of area are difficult for students (e.g., Cullen, Eames, et al., 2018; Kamii & Kysh, 2006; Miller, 2013; Outhred & Mitchelmore, 2000; Piaget et al., 1960/1970). Addressing one aspect of difficulty, researchers (e.g., Lehmann, 2022; Ulusoy, 2020) have argued that development of students' understanding of triangle area may support their understanding of polygon area formulas.

Challenges Involving Rectangular Area

Many students, including elementary (e.g., Cullen, Eames, et al., 2018), middle school (e.g., Kamii & Kysh, 2006), secondary (e.g., Kordaki & Balomenou, 2006), and preservice teachers (e.g., Baturo & Nason, 1996), experience a variety of challenges in understanding area concepts. Evidence from research suggests that some students experience challenges in understanding general area concepts (e.g., Kamii & Kysh, 2006; Piaget et al., 1960/1970) and rectangular area, in particular (e.g., Barrett et al., 2017; Cullen, Eames, et al., 2018; Doig et al., 1995; Outhred & Mitchelmore, 2000). Although not an exhaustive list, some challenges faced by students include: (a) elementary and middle school students view the square unit as a rigid object, which is unable to be subdivided (e.g., Kamii & Kysh, 2006); (b) elementary students leave gaps between units when tiling a region (e.g., Cullen, Eames, et al., 2018; Miller, 2013; Outhred & Mitchelmore, 2000), (c) children do not conceptualize the square as a unit of area measure (Barrett et al., 2017; Kamii & Kysh, 2006), and (d) young children (e.g., Yuzawa et al., 2000) through preservice teachers (e.g., Tierney et al., 1990) confuse perimeter and area.

Challenges Involving Triangle Area

Students, at many grade levels, also face challenges related to finding triangle area. Their challenges include: (a) subdividing triangles into both whole and partial square units (e.g., Cavanagh, 2008; Reynolds & Wheatley, 1994), (b) correctly counting both partial and square units in triangles that have been gridded (e.g., Barrett et al., 2017; Reynolds & Wheatley, 1994), (c) using incorrect measures in the triangle area formula or using an incorrect formula altogether (e.g., Baturo & Nason, 1996; Cavanagh, 2008), and (d) drawing altitudes—especially exterior altitudes in obtuse triangles (e.g., Gutiérrez & Jaime, 1999; Vinner & Hershkowitz, 1980). In the next section, I unpack some of the challenges that students face with regards to triangle area.

Increased Complexity of Area for Triangles

Although the concepts of rectangular area are challenging for students (e.g., Cullen, Eames, et al., 2018; Outhred & Mitchelmore, 2000), some of those challenges are compounded with triangular regions (e.g., Baturo & Nason, 1996; Hong & Runnalls, 2020; Osborn, 1976). Some reasons for increased complexity of triangle area as compared to rectangular area include: (a) subdividing triangles into square units requires some partial units (Reynolds & Wheatley, 1994), (b) triangle type may influence students' perception of triangle attributes (Cunningham &

Roberts, 2010; Krajcevski & Sears, 2019), (c) the orientation of figures may influence students' perception of angle (Dağlı & Halat, 2016; Davis et al., 2005; Vinner & Hershkowitz, 1980), and (d) the perception of gravity may influence students' perception of angles and triangles (Vinner & Hershkowitz, 1980; Ward, 2004). In this section, I briefly introduce each of those factors that contribute to increased complexity of understanding triangle area.

Triangles and Square Units

Any triangular region that is subdivided into square units will require some partial square units and may cause challenges for some students as they attempt to generate the subdivisions (e.g., Reynolds & Wheatley, 1994). Students may experience similar challenges when examining a triangle that has been gridded with square units (e.g., Barrett et al., 2017). These difficulties may be partially explained by the finding that some students view square units as ridged objects that are unable to be decomposed (Kamii & Kysh, 2006).

Triangle Type

Some students' perceptions of triangles and triangle altitudes are impacted by triangle type. With triangles that are closer to prototypical form (i.e., isosceles with a horizontal[1] base) children are better able to correctly classify those figures as triangles (e.g., Dağlı & Halat, 2016) and middle school students are better able to correctly draw the triangle's altitude (e.g., Vinner & Hershkowitz, 1980).

[1] People perceive objects based on a variety of perceptual factors including head position and gaze direction (Andersen et al., 1997). Pizzamiglio et al. (2000) described the way that people perceive objects based on a body-centered midline. From this perspective, what appears horizontal may change depending on the position of the body. As such, when I describe a segment as horizontal, I mean the perception of being perpendicular, relative to the midline of the body in the plane that separates the left half of the body from the right.

Orientation of Figures

Regarding orientation, researchers find that figures (e.g., triangles, angles, segments) that contained a horizontal segment tend to lead to more success in the following situations: (a) young children classifying figures as triangles (e.g., Dağlı & Halat, 2016), (b) elementary school-aged children creating copies of given angles from memory (e.g., Davis et al., 2005), (c) college students pivoting a segment to make it perpendicular to a given segment (e.g., Onley & Volkmann, 1958), and (d) middle school students identifying obtuse angles (e.g., Vinner & Hershkowitz, 1980).

Gravity

Researchers (e.g., Vinner & Hershkowitz, 1980; Ward, 2004) have found evidence of a gravitational factor that influences how people perceive geometric figures. That is, when people view a triangle or angle, they imagine how the figure would look if it were being acted upon by gravity. As such, they may perceive how the figure would sit if it were to settle downward onto an imaginary horizontal line. Using this visualization strategy, some students may be able to mentally realign a figure with no horizontal edges into one that does have a horizontal edge, and this visualization process may influence how those students perceive base and height (Vinner & Hershkowitz, 1980).

The gravitational factor may be related to the orientation of the figure. For example, Ward (2004) focused attention on shape identification (e.g., identifying which figures were right triangles), and found evidence of gravitational influence when observing preservice teachers rotating figures when making determinations about those figures. For Vinner and Hershkowitz (1980), this effect was hypothesized based on the results of their study.

Although people seem to have an intuitive sense of horizontal and perpendicular (Davis et al., 2005; Onley & Volkmann, 1958), students may not recognize the importance of the relationships between perpendicular linear measures (i.e., base and height) and area measures (e.g., Barrett et al., 2017; Cavanagh, 2008; Kamii & Kysh, 2006; Miller, 2013; Outhred & Mitchelmore, 2000).

Students' Interpretations of Base and Altitude

Students' interpretations of triangle base, altitude, and height likely impacts their ability to calculate the area of triangles (e.g., Cavanagh, 2008). Because of the fundamental importance of the concepts of base and altitude, researchers have identified various types of interpretations of base (e.g., Herbel-Eisenmann & Otten, 2011; Horzum & Ertekin, 2018; Vinner & Hershkowitz, 1980) and of altitude (e.g., Gutiérrez & Jaime, 1999; Şengün & Yılmaz, 2021) associated with triangles. Some examples include perceptions that: (a) the base of a figure is its bottom-most side (Herbel-Eisenmann & Otten, 2011), (b) the altitude of a triangle is the same as its median (Gutiérrez & Jaime, 1999), and (c) the altitude of a parallelogram must be in the interior of the figure (Şengün & Yılmaz, 2021). In Chapter 2, I elaborate on students' various conceptions of base and height. Now I discuss a theoretical lens, which some researchers have used to investigate students' interpretations of base, height, altitude, and area: concept image and concept definition (Vinner & Hershkowitz, 1980).

Concept Image: A Useful Lens for Investigating Students' Interpretations of Concepts

Researchers (e.g., Gutiérrez & Jaime, 1999; Vinner & Hershkowitz, 1980) have found that the theoretical lens of concept image and concept definition are useful tools for investigating and describing students' interpretations of mathematical concepts, including concepts related to triangles. This lens can help researchers describe the collection of ideas, which students have, about a mathematical concept (i.e., concept image) as compared to the generally accepted mathematical definition for the concept known as concept definition (Vinner & Hershkowitz, 1980). One of the primary benefits of the concept image and concept definition lens is the way that it helps researchers to focus on students' interpretations of individual concepts.

Researchers have examined students' concept images of triangle base (e.g., Horzum & Ertekin, 2018; Vinner & Hershkowitz, 1980), triangle altitude (Gutiérrez & Jaime, 1999), and of height (Gürefe & Gültekin, 2016; Horzum & Ertekin, 2018). In addition, researchers using a variety of theoretical approaches have studied students' understanding of area (e.g., Cullen, Eames, et al., 2018; Doig et al., 1995; Kamii & Kysh, 2006; Kordaki & Balomenou, 2006; Outhred & Mitchelmore, 2000; Piaget et al., 1960/1970; Sarama & Clements, 2009), and a few researchers have examined students' concept images of area (e.g., Tossavainen et al., 2017). For this study, I used the theoretical lens of concept image and concept definition (Vinner & Hershkowitz, 1980) to examine high school students' interpretations of concepts related to triangle area.

Rationale for the Present Investigation

Broadly, I designed my study to supplement the existing mathematics research about students' interpretations of triangle base, height, and area. In this section, I situate my study within the existing research and show the potential for my study to contribute to existing research literature.

It is important for students to learn and understand the concept of area, which is embedded within school curriculum (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2018), and which has practical uses outside of the mathematics class (e.g., Cavanagh, 2008;

Hirstein et al., 1978). Many students face challenges with understanding area including unit concepts (Cullen, Eames, et al., 2018; Doig et al., 1995; Outhred & Mitchelmore, 2000), conservation of area (Piaget et al., 1960/1970), and area formula usage (Cavanagh, 2008; Kamii & Kysh, 2006).

Additionally, researchers argue about the importance of students' understanding of triangle area because of its relationship to polygon area formulas (Lehmann, 2022; Ulusoy, 2020). However, students face challenges when thinking about triangle area including grappling with partial units (Reynolds & Wheatley, 1994), the influence of different types of triangles (Dağlı & Halat, 2016; Vinner & Hershkowitz, 1980), as well as other visual perceptual influences such as triangle orientation (Vinner & Hershkowitz, 1980) and gravity (Vinner & Hershkowitz, 1980; Ward, 2004).

Researchers have examined students' interpretations of the base concept (e.g., Herbel-Eisenmann & Otten, 2011; Horzum & Ertekin, 2018; Vinner & Hershkowitz, 1980), and the concept of height or altitude (e.g., Gutiérrez & Jaime, 1999; Krajcevski & Sears, 2019; Vinner & Hershkowitz, 1980), and the area concept (e.g., Tossavainen et al., 2017). However, more research is needed on how students' conceptions of base, height, and triangle area are related especially at the high school level.

I argue that high school geometry classes are an appropriate setting for this study for two reasons. First, students at this level should already have multiple years of experience with area concepts (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and, therefore, may have concept images that are closely aligned with the concept definitions for triangle base and height. Second, there are very few studies involving concept definition and concept image of triangle base and height at the high school level (e.g., Vinner & Hershkowitz, 1980), and so further study with this age group is warranted.

There is reason to believe, both mathematically and based on research (e.g., Cavanagh, 2008; Gutiérrez & Jaime, 1999; Reynolds & Wheatley, 1994) that such relationships exist and that certain triangle types (e.g., Dağlı & Halat, 2016; Vinner & Hershkowitz, 1980) or contextual factors (e.g., Onley & Volkmann, 1958; Vinner & Hershkowitz, 1980; Ward, 2004) may influence how students coordinate triangle attributes.

Therefore, there is a need for research that explores connections related to triangle area for high school students because the results will have implications for research and practice. The present study may extend existing research, create possibilities for future research, and reveal potential influences on students' interpretations of the area concept. Teachers may also benefit from the present study by deepening their knowledge of students' interpretations of triangle concepts and by gaining new ideas for assessment questions to help them to access students' understanding of triangle area in new ways.

Research Questions

To investigate the gaps in research and to extend existing research, I designed the present study to address the following questions:

1. What are high school geometry students' concept images of triangle base, height, and area?
2. How are high school geometry students' concept images of triangle base, height, and area related to one another?
3. How are triangle orientation, gravity, and triangle type related to high school geometry students' concept images of triangle base, height, and area?

CHAPTER II: LITERATURE REVIEW AND THEORETICAL PERSPECTIVE

In this chapter, first I will discuss how I narrowed my theoretical framework from constructivism (Cobb, 2007; Tzur, 1999; von Glasersfeld, 1980) to the lens of concept definition and concept image (Tall & Vinner, 1981; Vinner, 1975; Vinner & Hershkowitz, 1980). I positioned my theoretical perspective section before the review of literature because the lens of concept image and concept definition guided my literature review process—when searching for research literature involving students' interpretations of base, height, and area, I considered what their concept images may have been.

Theoretical Perspective

Constructivism, as a theory of learning, is "the general psychological contention that learning is an active, constructive process" (Cobb, 2007, p. 10). From this point of view, when students learn new concepts, they build on existing terminology, ideas, and images. Tzur (1999) described constructivism as a learning process "of modifying and reorganizing established structures (schemes) to eliminate perturbations, that is, disturbances due to the functioning of the mental system" (p. 391). Simon (1995) provided an overview of tenets of constructivism: (a) individuals construct knowledge based on experiences they have with the world around them, and (b) individuals cannot know if a concept that they have constructed is consistent with a separate external reality.

Three important ideas within constructivism are assimilation, accommodation, and schemes (von Glasersfeld, 1996). According to Phillips (1981), assimilation involves an organism incorporating something from its environment, such as the body ingesting food. From a psychological perspective, assimilation involves the mind interpreting or accepting an object based on previously known objects. For example, von Glasersfeld (1996) described an infant who recently played with a toy rattle. Then when the infant sees a spoon with a similarly rounded shape, they assimilate that object as a rattle.

In contrast, accommodation has the potential to change an organism so that it can assimilate objects easier in the future (Phillips, 1981). Continuing with von Glasersfeld's (1996) example of the infant and the spoon, the infant may soon realize that the spoon is not a rattle toy. This realization is one form of accommodation. Another form of accommodation would be if the infant discovers that banging the spoon can make a loud noise. As a person engages with the world around them, they encounter new events or concepts that challenge their existing knowledge (i.e., perturbation). Then, through the processes of assimilation and accommodation, they achieve a type of equilibrium whereby ideas that "fit" the new situation "survive," and ideas that do not fit the situation are discarded (Ernest, 2010).

Next, the idea of schemes is closely related to Piaget's concept of assimilation and accommodation (von Glasersfeld, 1980). According to von Glasersfeld (1980), "Piaget's conception of assimilation and accommodation remain incomprehensible unless it is placed within the framework of his theory of knowledge and, specifically, into the context that he calls scheme" (p. 81). Also, according to von Glasersfeld, schemes consist of three parts: (a) an initial trigger or event, (b) an external or internal response, and (c) the result of the event. Therefore, a scheme is a way of describing how a person interacts with the world around them.

In the present study, I was interested in accessing students' schemes related to their understanding of the concepts of triangle base, height, and area. area. Because I viewed learning as an individual process whereby each person may have differences in their learning process and knowledge gained, I expected variation in how students understood and perceived those concepts. Therefore, rather than focusing on the process of how students learned concepts, I focused on what students understood about concepts. Vinner and Hershkowitz (1980) provided a theoretical lens that accomplishes this goal—the lens called concept image and concept definition.

Concept Image and Concept Definition

Vinner (1975) theorized that there exists "an underlying impression or an intuition of which people are unaware. It is an unformulated conception" (p. 341) of a concept. Then, using these ideas, Vinner and colleagues (e.g., Tall & Vinner, 1981; Vinner & Hershkowitz, 1980) developed the theoretical lens of concept image and concept definition to distinguish between how a concept is defined versus how it is understood.

Students' interpretations of individual concepts was my concern in this study. I used the interpretive lens of concept image and concept definition (Tall & Vinner, 1981; Vinner, 1975; Vinner & Hershkowitz, 1980) to describe students' interpretations of mathematical concepts at a particular moment in time. Vinner and Hershkowitz (1980) clarified that theirs was a focus on "simple isolated concepts that have several aspects or components"[2] (p. 182) and that those components are not necessarily learned at the same time, nor in a particular order. They positioned their perspective (i.e., concept image and concept definition) in contrast with some other theories because those theories included different stages or levels.

Concept Definition

The term *concept definition* means a non-circular way of explaining a concept using written or spoken words (Vinner & Hershkowitz, 1980). *Formal* concept definitions of

[2] Vinner and Hershkowitz (1980) did not define their usage of the term *component*. I infer, based on their work, that they meant some part of the geometric structure of the object or some a kind of perception about those objects. For example, components of triangle include vertex, sides, and angles. A component of base may include a perception of "bottomness" and components of height may include the perception of perpendicularity to base, a perception of vertical, and the identification of the vertex opposite a base.

mathematics concepts are definitions that are generally accepted by the broader mathematical community. For example, a formal concept definition of a mathematical function could be defined as “a relation between two sets of *A* and *B* in which each element of *A* is related to precisely one element of *B*” (Tall & Vinner, 1981, p. 153).

In contrast, a *personal* concept definition is how an individual describes a concept based on their understanding of that concept (Tall & Vinner, 1981; Viholainen, 2008; Vinner, 1991). For example, a student might describe a function as “an action which maps *a* in *A* to *f*(*a*) in *B*, or as a *graph*, or a *table of values*” (Tall & Vinner, 1981, p. 153). Viholainen (2008) differentiated formal and personal concept definitions as definitions that belong to an institutionalized way of understanding versus a personal way of understanding.

Concept Image

The term *concept image* means the sum collection of mental pictures and ideas relating to a concept, which is formed over time (Tall & Vinner, 1981). For example, consider a young student’s concept image of the subtraction of whole numbers. When first interacting with subtraction, the student "may observe that subtraction of a number always reduces the answer. For such a child this observation is part of his or her concept image and may cause problems later on should subtraction of negative numbers be encountered" (Tall & Vinner, 1981, p. 152).

As such, the concept image may include the personal concept definition, but it also may include additional ideas related to the concept (Viholainen, 2008). So, in verbalizing the meaning of a concept, a person shares their concept definition and reveals a part of their concept image. Tall and Vinner (1981) clarified that concept images can change over time based on newly learned ideas and experiences.[3] It is also the case that some concept images are formed without the need for any formal definitions (Tall & Vinner, 1981).

Herrera (2016) described the complexity of concept images and cautioned that an individual's concept image is not necessarily accessible all the time. It is for that reason that Tall and Vinner (1981) introduced another term called *evoked* concept image to describe the aspects of the concept image that are revealed at one time or another. Different and logically conflicting concept images may be evoked over time. Tall and Vinner suggested that conflict among evoked images can help a student to better understand the concept definition.

Students' concept images may not completely match the respective accepted concept definition within the field, and thereby cause students' responses to appear different than what a teacher expects (Vinner & Dreyfus, 1989). Also, both Vinner (1983) and Vinner and Hershkowitz (1980) argued that personal concept definitions may become inactive and therefore usually part of a person's concept image, as opposed to their personal concept definition, will be evoked when they think about a concept.

Concept Image and Definition: Appropriate Lens for Present Study

For my study, I argue that the lens of concept image and concept definition is appropriate. I was interested in what students understood about the concepts of base, height, and area. As with all mathematical concepts, triangle base and height have the potential for a variety of interpretations by students. Although the concept of area may be considered more complex, Tossavainen et al. (2017) examined preservice teachers' interpretations of area through the lens

[3] The idea of concept image is reminiscent of a theoretical perspective called Herbartianism, which predates constructivism (Ellerton & Clements, 2005). According to Ellerton and Clements (2005), in Herbartianism, learners construct their own knowledge and ideas that are passed from one person to another may be received differently than intended.

of concept image and concept definition. The lens of concept image and concept definition enabled me to dig deeply into students' interpretations of these concepts.

Review of the Relevant Literature

Driven by my research questions, I reviewed literature related to students' understanding of triangle attributes, area, and angle. I utilized the ERIC, JSTOR, and Google Scholar databases as well as the Illinois State University library database. I searched using the keywords *triangle*, *concept image*, *concept definition*, *base*, *height*, *triangle altitude*, *area, perpendicularity, gravity and drawing, Piaget, conservation of area, horizontal, students' mathematics strategies, perception of angle,* and *distance from a point to a line*. After finding some literature, which appeared relevant to the present study, I searched for newer studies that cited the ones that I previously identified. Also, I searched for and read relevant studies which were cited in the research that I had read. Two broad categories, relevant to my study, emerged from my review of the literature:

a. Students' understanding of area in general (e.g., Cullen, Eames, et al., 2018; Doig et al., 1995; Kamii & Kysh, 2006; Kordaki & Balomenou, 2006; Outhred & Mitchelmore, 2000; Piaget et al., 1960/1970; Sarama & Clements, 2009); and
b. Students' understanding related to triangle area (e.g., Huang & Witz, 2013; Kadarisma et al., 2020; Krajcevski & Sears, 2019).

Being guided by my research questions, I identified distinct subcategories among each of the two broad categories.[4] The first broad category involved mathematical ideas related to area:

- Unit concepts (e.g., Barrett et al., 2017; Miller, 2013; Outhred & Mitchelmore, 2000; Sarama & Clements, 2003)

[4] Note that these themes are not an attempt to describe all existing research about students' understanding of area. Rather, my intention was to select themes that were relevant to the focus of my study.

- Area and conservation (e.g., Cavanagh, 2008; Kamii & Kysh, 2006; Piaget et al., 1960/1970)
- Attributes related to area measurement (e.g., Barrett et al., 2017; Doig et al., 1995; Hirstein et al., 1978; Kamii & Kysh, 2006; Kordaki & Balomenou, 2006; Miller, 2013; Outhred & Mitchelmore, 2000; Tierney et al., 1990)
- Spatial structuring (e.g., Battista et al., 1998; Battista & Clements, 1996; Cullen, Eames, et al., 2018)

The second broad category involved mathematical ideas related to triangle area:

- Base and height (e.g., Gutiérrez & Jaime, 1999; Horzum & Ertekin, 2018)
- Square units within triangles (e.g., Barrett et al., 2017; Kamii & Kysh, 2006, Reynolds & Wheatley, 1994)
- The orientation of figures and the gravitational factor (e.g., Dağlı & Halat, 2016; Davis et al., 2005; Vinner & Hershkowitz, 1980; Ward, 2004)
- The impacts of triangle type (e.g., Cunningham & Roberts, 2010; Krajcevski & Sears, 2019; Vinner & Hershkowitz, 1980)
- The influence of nearby objects (e.g., Berman et al., 1974; Piaget & Inhelder, 1948/1999)

For clarification, the above lists of main categories, and respective subcategories, represent my organization for the rest of this chapter. I considered the above topics to inform my study and to be adjacent or directly related to my research questions.

After my literature review process, I read through all articles that directly correlated with my research questions one final time. My goal was to check for any additional information that I may have missed during one of my previous readings. I found this to be an opportunity to gather some summary information about those articles.

Figure 1 includes citations for all the research literature—both empirical and theoretical—that I reviewed, and that directly related to my research questions. Specifically,

Figure **1** contains the following topics: (a) the influence of the orientation of figures, (b) the influence of triangle type, (c) concept image and concept definitions of geometric objects, (d) student's understanding or identification of base, (e) student's understanding or identification of height, and (f) understanding of area. To create

Figure **1**, I first determined whether a particular piece of literature contained one of the topics of interest. Then, I looked to see if the authors specifically and intentionally focused on one of the topics. Next, if they did not describe one of the topics of interest, I carefully read their reported results to see if the topics of interest were included there.

Figure 1

Research Involving Topics in Present Study

Matrix of Themes Versus Studies

Studies	Pre-School	Elementary	Middle School	Secondary School	Post-Secondary	Theoretical	Orientation of Figures	Gravity	Triangle Type	CI/CD	Base	Height/Altitude	Triangle Area Strategies
Barrett et al. (2017)	x	x									x	x	x
Battista and Clements (1996)		x					x						
Battista et al. (1998)	x	x					x						
Baturo and Nason (1996)					x								x
Berman et al. (1974)	x						x						
Blanco (2001)					x		x		x		x	x	
Bütüner and Filiz (2016)			x				x						
Cavanaugh (2008)			x									x	x
Cullen, Cullen, et al. (2018)		x					x						
Cullen, Eames, et al. (2018)		x											x
Cunningham and Roberts (2010)					x					x		x	
Dağlı and Halat (2016)	x						x		x				
Davis et al. (2005)	x						x						
Doig et al. (1995)		x											x
Gürefe and Gültekin (2016)			x							x		x	
Gutiérrez and Jaime (1999)					x		x		x	x		x	
Herbel-Eisenmann and Otten (2011)			x	x	x						x	x	x
Hirstein et al. (1978)													x
Hong and Runnalls (2020)					x							x	x
Horzum and Ertekin (2018)					x					x	x	x	x
Huang and Witz (2013)		x											x
Kadarisma et al. (2020)			x									x	x
Kordaki and Balomenou (2006)				x							x	x	x
Kospentaris et al. (2011)				x	x							x	x
Krajcevski and Sears (2019)					x					x	x	x	x
Miller (2013)		x	x										x
Onley and Volkmann (1958)					x		x						
Outhred and Mitchelmore (2000)		x											x
Piaget et al. (1960/1970)	x	x											x
Prolux and Pimm (2008)						x					x	x	x
Reynolds and Wheatley (1994)		x					x				x	x	x
Sarama and Clements (2009)	x	x					x						x
Şengün and Yılmaz (2021)			x				x		x		x	x	
Tierney (1990)					x		x						x
Tossavainen et al. (2017)					x					x			x
Ulusoy (2020)					x					x			
Vinner and Hershkowitz (1980)			x	x			x	x	x	x	x	x	
Ward (2004)					x		x	x	x	x			
Present Study				x			x	x	x	x	x	x	x

Note. In the above figure, "CI/CD" stands for concept image and concept definition.

Notice that relatively few studies have been conducted involving secondary school-aged students, and no studies included all six topics. One study, by Vinner and Hershkowitz (1980), addressed five of the six topics.[5] They introduced the theoretical lens of concept image and concept definition, found evidence of the potential influence of orientation and triangle type, and they also found a potential concept image relating to triangle altitude: “Our guess here is that the concept [image] of an altitude to the basis in an isosceles triangle took the place of a general concept of an altitude” (p. 182). For clarification, their study was extremely valuable in the sense that it scratched the surface of some of these topics and offered researchers valuable insights and an opportunity to delve deeper into researching those topics.

Understanding Area in General

Many researchers have studied students’ understanding of area (e.g., Cullen, Eames, et al., 2018; Doig et al., 1995; Kamii & Kysh, 2006; Kordaki & Balomenou, 2006; Outhred & Mitchelmore, 2000; Piaget et al., 1960/1970; Sarama & Clements, 2009; Tossavainen et al., 2017). In this section, I describe the aspects of a conceptual understanding of area that are relevant to my study. I begin with a discussion of an important collection of some area concepts called *unit concepts* (e.g., Battista et al., 1998; Cullen, Eames, et al., 2018; Sarama & Clements, 2009). Then I discuss students’ challenges relating to conservation of area (e.g., Cavanagh, 2008; Hong & Runnalls, 2020; Piaget et al., 1960/1970) and attending to the wrong attributes when trying to find area measures (e.g., Doig et al., 1995; Hirstein et al., 1978; Hong and Runnalls, 2020; Kamii & Kysh, 2006; Kospentaris et al., 2011; Tierney et al., 1990). Lastly, I introduce *spatial structuring*—a type of organized thinking, abstraction, and mental actions on objects (Battista & Clements, 1996; Battista et al., 1998; Sarama & Clements, 2009).

[5] I review aspects of this study in later sections involving gravity, orientation, and triangle type.

Unit Concepts

Researchers (e.g., Barrett et al., 2017; Miller, 2013; Outhred & Mitchelmore, 2000; Sarama & Clements, 2003) have identified several important concepts related to units—unitizing, composing units, iterating units, and coordinating units. Cullen, Eames, et al. (2018) described these concepts collectively as *unit concepts*.

Unitizing. The concept of *unitizing* is "the cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity" (Lamon, 1996). Cullen, Eames, et al. (2018) wrote, "In an area measurement context, unitizing requires the identification of a repeatable shape, piece or object (i.e., the unit) that is part of the whole or region and segments or covers the two-dimensional space well" (p. 535). For example, unitizing is involved in understanding that squares or right triangles are a better choice than circles for covering a region of area because circles would create uncovered gaps whereas squares or right triangles could be tessellated without gaps.

Composing Units. The concept of *composing units* describes a process whereby a person may group individual units to form a new unit (Steffe, 1992). "With experience or instruction, children transition to thinking about individual squares as units and then to thinking about grouping units together to compose a composite unit (e.g., a row or column)" (Cullen, Eames, et al., 2018, p. 536; see also Sarama & Clements, 2009). The ability to compose units represents the beginning of structuring a region (Sarama & Clements, 2009).

Iterating Units. The concept of *iterating units*, in the context of area, can include iterating individual units and iterating groups of units to fill a region (e.g., Barrett et al., 2017; Cullen, Eames, et al., 2018; Outhred & Mitchelmore, 2000). Iterating must include some type of mental action that goes beyond counting individual units in an array or covering an array fully with physical units (Cullen, Eames, et al., 2018; Piaget et al. 1960/1970; Steffe, 1983). Cullen, Eames, et al. (2018) described iterating as an advancement beyond composing, claiming that "iteration involves the repetition of an area unit, either an actual physical object (e.g., a square tile) or a mental image of a unit, which is geometrically translated repeatedly through two-dimensional space to occupy successive locations, always in an adjacent position with one concurrent edge" (pp. 536–537; see also Piaget et al., 1960/1970; Steffe, 1992). The ability to iterate units is fundamental to all types of measurement including length, area, and volume (Piaget et al., 1960/1970; Steffe, 1992) and an essential component of understanding area (Barrett et al., 2017; Cullen, Eames, et al., 2018).

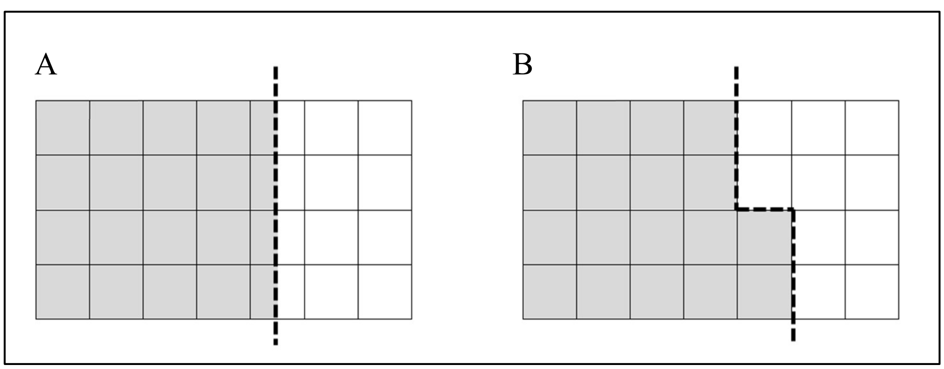
It is well documented that, although offering a large quantity of physical square tiles (i.e., at least enough to tile a rectangle) can dramatically increase young students' ability to correctly find the area of rectangles (Doig et al., 1995), the large number of units may hide students' thinking about the important underlying iterative row and column patterns within the array of square units (Outhred & Mitchelmore, 2000). Therefore, some researchers (e.g., Miller, 2013; Outhred & Mitchelmore, 2000) have limited the quantities of physical tiles given to their study participants to gain a window into students' thinking about iteration.

Coordinating Units. The process of *coordinating units* requires: (a) recognizing a unit as an individual object, but also simultaneously as part of a row and a column, (b) consideration of linear and area units, and (c) iteration of a row or column to cover a region (Cullen, Eames, et al., 2018; Miller, 2013; Steffe, 1992). Steffe (1992) found a variety of types of unit-coordinating thought processes among six students, in a 3-year teaching experiment, starting in Grade 3. Steffe's research demonstrated interconnected relationships between the coordination of units and the concept of multiplication.

Squares as Rigid Objects. Many students consider square units to be rigid objects that cannot be decomposed or partitioned (Hirstein et al, 1978; Kamii & Kysh, 2006). For example, one task, originally from Hirstein et al. (1978) and then repeated by Kamii and Kysh (2006), involved students attempting to create a rectangle with an area of 18 square units by either cutting or drawing a line through a rectangular strip that contained four rows of square units. In other words, a correct solution would require the students to slice halfway through the fifth square to create a rectangle that had four square units in one dimension, and four and a half squares in the other (see Figure 2A). Hirstein et al. (1978) found that many students struggled with this task. Kamii and Kysh (2006) repeated this task with children in Grades 4 through 8, as one among several tasks in a sequence, and found that 94% of their participants either said the task was impossible or created a figure that was not a rectangle (see Figure 2B).

Figure 2

Rendering of Student Solutions of Rectangular Strip Task



Note. Images adapted from "The difficulty of "length× width": Is a square the unit of measurement?" by C. Kamii and J. Kysh, 2006, *Journal of Mathematical Behavior, 25*(2), pp. 105–115 (https://doi.org/10.1016/j.jmathb.2006.02.001). Panel A: Intended correct solution. Panel B: Common incorrect solution.

Understanding Formulae

Researchers have examined connections between students' understanding of unit concepts and area formulae (e.g., Barrett et al., 2017; Hirstein, 1981; Huang & Witz, 2013; Miller, 2013; Nunes et al., 1993; Reynolds & Wheatley, 1994). Other researchers have examined students' understanding of formulae in a way that seemed disjoint from unit concepts (e.g., Baturo & Nason, 1996; Krajcevski & Sears, 2019; Tierney et al., 1990; Ward, 2004). Two broad findings emerged from all of these studies: (a) some students do not recognize that the area of rectangles depends on the measures of two orthogonal sides (e.g., Barrett et al., 2017; Cavanagh, 2008; Kamii & Kysh, 2006; Miller, 2013; Outhred & Mitchelmore, 2000), and (b) some students—including preservice teachers—do not understand why polygonal area formulae make sense (e.g., Baturo & Nason, 1996; Hirstein, 1981; Huang & Witz, 2013; Nunes et al., 1993; Tierney et al., 1990).

Area and Conservation

Lehrer (2003) defined *conservation* as the "recognition of invariance under transformation, undergirded by such mental operations as reversibility, a form of mental undoing" (p. 180). Many researchers have identified challenges faced by students regarding the concept of conservation of area (e.g., Cavanagh, 2008; Hong & Runnalls, 2020; Kamii & Kysh, 2006; Piaget et al., 1960/1970). Younger children may display a lack of understanding of area conservation when a shape is changed into a non-congruent shape, such as a collection of square tiles in which one or more tiles have been moved to a new location (e.g., Kamii & Kysh, 2006; Piaget et al., 1960/1970). Likewise, Cavanagh (2008) found that even after decomposing and rearranging a parallelogram to form a rectangle, Grade 7 students were "completely mystified" (p. 56) by the idea that the two figures might have the same area.

Challenges involving conservation of area persist even for some post-secondary students (e.g., Baturo & Nason, 1996; Tierney, 1990). For example, Baturo and Nason (1996) found that many preservice teachers used an "informal cut-and-paste method" (p. 251) to compare the areas of a square and a rectilinear shape that resembled the capital letter T. However, the preservice teachers demonstrated a general mistrust in the method having "commented that the informal comparing techniques were not very accurate" (p. 252). Some of them lacked confidence in the concept of conservation of area. For example, one student commented, "This sounds stupid, but get that (the T shape) and put it on that (the square)" (p. 252).

Tierney (1990) found that some preservice teachers experienced a different kind of challenge that seemed connected both to the conservation and orientation of figures. They correctly claimed that a rectangle and parallelogram appeared to have the same area, but when the researchers rotated one figure 90 degrees, they changed their mind and decided that the shapes no longer had the same area.

Conflating Length and Area

Many researchers have found evidence that students, ranging from Grade 1 to post-secondary, incorrectly used length measures when attempting to find area measures (e.g., Barrett et al., 2017; Doig et al., 1995; Hirstein et al., 1978; Hong and Runnalls, 2020; Kamii & Kysh, 2006; Kordaki & Balomenou, 2006; Kospentaris et al., 2011; Miller, 2013; Outhred & Mitchelmore, 2000; Tierney et al., 1990). For students in Grades 1 through 8 this involved counting units around the border of a rectangle (i.e., the units around the perimeter) when trying to find an area (e.g., Cavanagh, 2008; Doig et al., 1995; Hirstein et al., 1978; Kamii & Kysh, 2006). Also, when comparing the area of two quadrilaterals or triangles, Grade 7 students (e.g., Cavanagh, 2008) and preservice teachers (e.g., Hong & Runnalls, 2000; Kospentaris et al., 2011; Tierney et al., 1990) indicated the shape that appeared to be longer must have a larger area.

Spatial Structuring

Another important concept of area understanding is spatial structuring (Battista & Clements, 1996; Battista et al., 1998; Sarama & Clements, 2009). The concept of *spatial structuring* encompasses organized thinking, abstraction, and mental actions on objects, and is a type of abstraction process whereby a person attempts to make sense of and coordinate multiple ideas (Battista et al., 1998). Battista et al. (1998) suggested that almost all geometry involves spatial structuring of space.

For example, Battista and Clements (1996) described two students in Grade 3 who were attempting to count the edges of a triangular prism. At first, they did so in an unstructured way and obtained differing answers. After the researchers asked the students to explain their process, one student "enumerate[d] the rods in an organized and correct way, counting three rods on each base, then three lateral rods" (p. 288). According to Battista and Clements, it was the moment when students were asked to explain their process that they began to formulate and verbalize their counting schemes. Just because spatial structuring involves abstraction, this type of thinking does not necessarily always lead to correct answers (e.g., Battista & Clements, 1996; Battista et al., 1998; Prolux & Pimm, 2008).

Outhred and Mitchelmore (2000) investigated area-covering strategies used by students in Grades 1–4, before having been formerly instructed about the concept. One of their findings included a warning about pre-structuring, by which they meant that using wooden unit tiles might hide the array structure: "The array structure is inherent in the materials and does not need to be apprehended by the learner" (p. 146). In other words, Outhred and Mitchelmore (2000) were cautioning that the usage of physical tiles for area covering may end up hiding some of the important aspects involved in understanding the area concept.

By intentionally designing area tasks to avoid pre-structuring them, researchers (e.g., Miller, 2013; Outhred & Mitchemore, 2000) have gained insights into students' understanding of unit concepts that may have otherwise been hidden. For example, in a task that involved covering a rectangle with units, Miller (2013) provided only a single non-movable unit (i.e., printed on paper) in the shape of an isosceles right triangle and no grid units. Although all the students drew in units for this task, some allowed gaps between them, and some of the drawn units became distorted to the point where many of the hand-drawn units were no longer triangles. These findings suggest that students have difficulty creating grid units, and that students' struggle may be difficult to detect in tasks where the units are provided by the researchers.

Summary

In this section, I have discussed some aspects of area understanding, which are relevant to my study. Broadly, the concept of area is challenging for many students. Much of the discussion in this section involved rectangular regions. Students struggle to understand how the area formulas correlate with their respective regions. Understanding formulae requires students to develop an understanding of unit concepts and spatial structuring.

Interpretations of Concepts Related to Triangular Area

In this section, I describe research involving (a) base and height (e.g., Gutiérrez & Jaime, 1999; Horzum & Ertekin, 2018), (b) unitizing complications involving triangles (e.g., Barrett et al., 2017; Kamii & Kysh, 2006, Reynolds & Wheatley, 1994), (c) the orientation of figures (e.g., Dağlı & Halat, 2016; Davis et al., 2005; Vinner & Hershkowitz, 1980; Ward, 2004), (d) the impacts of triangle type (e.g., Cunningham & Roberts, 2010; Krajcevski & Sears, 2019; Vinner

& Hershkowitz, 1980), and (e) the influence of nearby objects (e.g., Berman et al., 1974; Piaget and Inhelder, 1948/1999). These concepts are important and relevant to a student's ability to use the triangle area formula with understanding because, as will be discussed, students must grapple with any number of these concepts when thinking about triangle area.

Base and Height

First, I clarify the meanings of the geometric objects: base, height, and altitude. The *triangle base* is any side of the triangle (Herbel-Eisenmann & Otten, 2011). Herbel-Eisenmann and Otten (2011) identified examples of teachers and students implying dual meanings when using the phrase *height*. They proposed that, at times, teachers and students may use geometric terminology (e.g., height) to simultaneously mean both a measure and a geometric object.[6] Gutiérrez and Jaime (1999) defined the altitude of a triangle as "the segment drawn from a vertex perpendicular to the opposite side or its elongation" (p. 261). To clarify, notice that the definition of altitude contains three important components: (a) perpendicularity to the base, (b) an endpoint concurrent with a triangle vertex opposite a base, and (c) an endpoint on a line containing the base.

Researchers have identified students' understanding of base (e.g., Horzum & Ertekin, 2018), altitude (e.g., Gutiérrez & Jaime, 1999; Şengün & Yılmaz, 2021), and height (e.g., Gürefe & Gültekin, 2016; Hong & Runnalls, 2020). Additionally, researchers (e.g., Gürefe & Gültekin, 2016; Horzum & Ertekin, 2018) have found evidence of some students who formed a type of dependency between base and height. In this section, I describe various research literature that involved students' understanding of triangle base and height.

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[6] Based on the findings from Herbel-Eisenmann and Otten (2011) and to differentiate the two potential meanings of height in my study, when I used the word *height*, by itself, I meant a geometric object called altitude, and placed the phrase *measure of* before height when using it to describe a measure.

Interpretations of Base. Researchers (e.g., Herbel-Eisenmann & Otten, 2011; Vinner & Hershkowitz, 1980; Ward, 2004) have identified a tendency for some students to identify the "bottom" of a triangle as the base. Students' sense of "bottomness" may be related to a triangle with an approximately horizontal side as the base (Vinner & Hershkowitz, 1980). This idea that a base should be horizontal may be influenced by prototypical textbook images (Cunningham & Roberts, 2010; Kadarisma et al., 2020; Vinner & Hershkowitz, 1980).

Also relating to the idea of "bottomness," some researchers have found that if a triangle does not contain a horizontal side, a student might consider a triangle side to be a base if it is approximately horizontal, perhaps being influenced by a so-called gravitational factor (Vinner & Hershkowitz, 1980). They also may rotate the triangle so that a side appears to be horizontal (Horzum & Ertekin, 2018; Ward, 2004). Finally, they may even consider the side that is closest to their body as the base (Horzum & Ertekin, 2018).

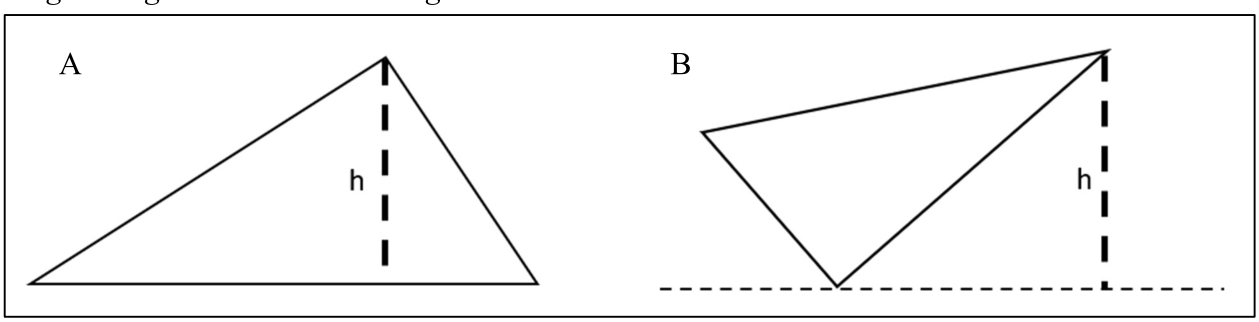
Horzum and Ertekin (2018) conducted a study, which began with a single written-response open-ended question followed by a 10-minute semi-structured interview of 139 preservice teachers. They found evidence that on rare occasions some students have a *height-dependent base* whereby they identify the height first, and then view the base as the triangle side that is perpendicular to that height. Their finding serves as evidence that some students have connected the ideas of base and height.

Interpretations of Height. Based on a 14-question survey of 190 preservice teachers, Gutiérrez and Jaime (1999) found evidence that some students consider the median of triangles to be the altitude. Interestingly, they also found that students in their study only drew altitudes as medians when the altitude should have been exterior to an obtuse triangle yet drew correct altitudes when they were inside the triangle. Additionally, Krajcevski and Sears (2019) found that some preservice teachers may confuse the concepts of median and altitude in some triangles and reasoned that the students "have not internalized the notion of an altitude in a triangle … although they know the concept definition of an altitude in a triangle" (p. 96). Researchers conjectured that some geometry textbooks offer too many prototypical examples of triangle altitudes, which contributes to students' challenges with understanding triangle altitude (Cunningham & Roberts, 2010; Vinner & Hershkowitz, 1980).

Other researchers (e.g., Barrett et al., 2012; Blanco, 2001; Gürefe & Gültekin, 2016; Kadarisma et al., 2020) found that some students draw triangle height in a vertical orientation with respect to their body.[7] Blanco (2001) described this as an error that may be caused by the "traditional representation of the altitude of a triangle in textbooks" (p. 6). Note that the segment, while remaining vertical, has one endpoint at the vertex (see Figure 3).

Figure 3

Height Imagined as a Vertical Segment



Note. Images adapted from "Errors in the teaching/learning of the basic concepts of geometry," by L. J. Blanco, 2001, May 24, *International Journal for Mathematics Teaching and Learning*, *24*, 1–11 (https://www.cimt.org.uk/journal/index.htm). Panel A: Stereotypical textbook example of height. Panel B: Height drawn vertically regardless of the orientation of the triangle.

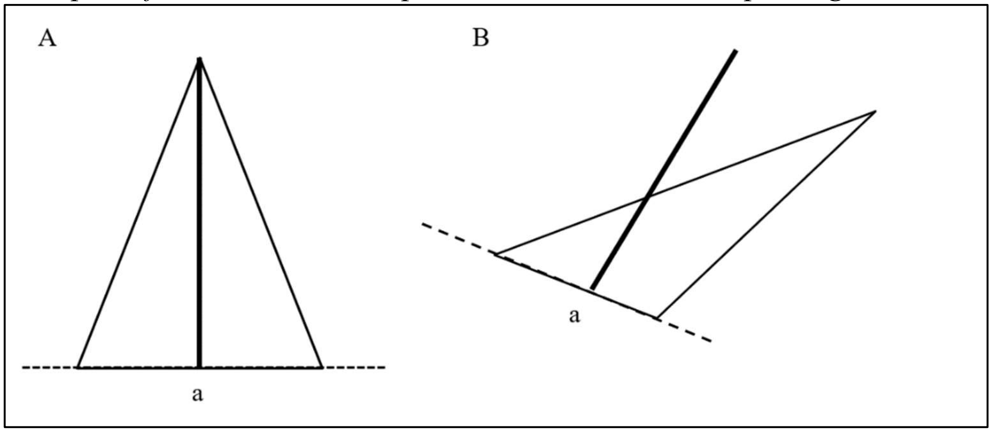
[7] Using the idea of body-centered midline (Pizzamiglio et al., 2000), I describe a vertical segment as a segment that is in the plane that bisects the body, with the line containing that segment passing through the midline of the body.

Some students view the height as the side of a triangle or parallelogram, regardless of perpendicularity with a base (e.g., Cavanagh, 2008; Gutiérrez & Jaime, 1999; Kospentaris et al., 2011; Krajcevski & Sears, 2019). For example, Kospentaris et al. (2011) found evidence of students using the triangle area formula, $A = 1/2 \cdot b \cdot h$, to calculate the area using two side lengths of a non-right triangle. Gutiérrez and Jaime (1999) found this to be a very rare occurrence, only having a single student in their study who drew altitudes as triangle sides.

Researchers (e.g., Gutiérrez & Jaime, 1999; Krajcevski & Sears, 2019) found evidence of preservice teachers' evocations of altitude that included perpendicularity with the base but not necessarily a connection with the vertex opposite the base. Gutiérrez and Jaime (1999) also identified an evocation of triangle altitude that they described as a blending "of the concepts of altitude and perpendicular bisector" (p. 269). In their study, this was quite rare, with less than 2% of the preservice teachers having been coded with this type of evocation on any one of their 14 tasks. They noted that when thinking of altitude, the preservice teachers were paying attention to the right angle and the connection to the base but not the connection to the vertex opposite the base. Of all the examples that Gutiérrez and Jaime (1999) provided for this, the 'altitude' either intersected with the vertex opposite the base (i.e., a correct altitude of an isosceles triangle; see Figure 4A or the altitude extended beyond the triangle so that it was approximately the same length as the triangle height (see Figure 4B). None of the examples from their article included the perpendicular symbol, and they did not address whether any student had drawn the symbol.

Figure 4

Examples of the Altitude as Perpendicular Bisector Concept Image

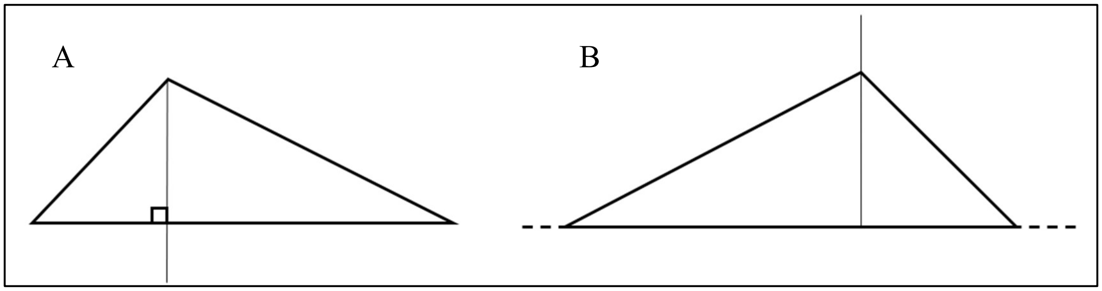


Note. Images adapted from "Preservice primary teachers' understanding of the concept of altitude of a triangle," by A. Gutiérrez and A. Jaime, 1999, *Journal of Mathematics Teacher Education*, *2*, 253–275 (https://doi.org/10.1023/A:1009900719800). The lower-case *a* in each panel represents the base that was pre-selected by the researchers. Panel A: Example of an altitude drawn in an isosceles triangle. Panel B: Example of an 'altitude' drawn in an obtuse triangle that is approximately the same length as, and parallel with, the triangle altitude to the given base.

Further, researchers (Gutiérrez & Jaime, 1999; Krajcevski & Sears, 2019) found that although some preservice teachers could construct altitudes that were perpendicular to the base of a triangle and contained the vertex opposite the base, their altitude lengths were incorrect. In Figure 5, I show a few possible ways that preservice teachers had not attended to the correct altitude length. Note that the preservice teachers from Krajcevski and Sears (2019) drew right angle symbols (Figure 5A), but the preservice teachers from Gutiérrez and Jaime (1999) did not (Figure 5B).

Figure 5

Altitude Understanding Without Correct Length

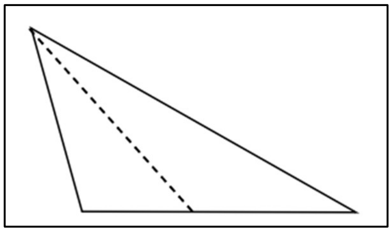


Note. Panel A: Image adapted from "Common visual representations as a source for misconceptions of preservice teachers in a geometry connection course," by M. Krajcevski and R. Sears, 2017, *International Journal of Mathematics Teaching and Learning*, *20*(1), 85–105. (https://eric.ed.gov/?id=EJ1216655). Panel B: Image adapted from "Preservice primary teachers' understanding of the concept of altitude of a triangle," by A. Gutiérrez, and A. Jaime, 1999, *Journal of Mathematics Teacher Education*, *2*(3), 253–275 (https://doi.org/10.1023/A:1009900719800).

Some students in Grades 7–9 (Şengün & Yılmaz, 2021; Vinner & Hershkowitz, 1980) and preservice teachers (Gutiérrez & Jaime, 1999) experienced challenges when attempting to draw exterior altitudes in obtuse triangles. For example, based on their survey of 550 students in Grades 7, 8, and 9, Vinner and Hershkowitz (1980) found that only 8% of their participants correctly drew an exterior altitude in an obtuse triangle. Viner and Hershkowitz hypothesized that this "can be a result of a (sometimes implicit) common belief that an altitude should always fall inside the triangle" (p. 182). This hypothesis was verified when Şengün and Yılmaz (2021) found that, instead of correctly drawing exterior altitudes, 7th Grade students drew interior "altitudes" (e.g., see Figure 6).

Figure 6

Example of Interior Altitude



Note. Image adapted from "Examining the efforts of middle school 7th Grade students to draw altitude in parallelogram and triangle," by K. Şengün and S. Yılmaz, 2021, *Osmangazi Journal of Educational Research, 8*(1), 220–238 (https://dergipark.org.tr/en/pub/ojer/issue/62900/878069).

Unitizing Complications Involving Triangles

The concept of unitizing (see Cullen, Eames, et al., 2018), which is already challenging for students in rectangular figures, becomes more challenging in triangles. First, from a purely mathematical perspective, any triangle that is subdivided into square units will always have some partial square units—the fractional units can cause difficulty for students who are attempting to count square units (Cavanagh, 2008). Second, many students do not view the square as a unit of measure (Kamii & Kysh, 2006), thus creating situations where they may count both whole and partial squares as units (e.g., Barrett et al., 2017; Reynolds & Wheatley, 1994). For example, found Reynolds and Wheatley (1994) that after subdividing a given triangle into approximately square units and right triangles, a student counted both squares and right triangles as whole units. Likewise, Barrett et al. (2017) found evidence of students counting "almost-whole squares each as wholes" (p. 117) in a gridded triangle.

Influential Factors on Students' Perception of Triangles and Angles

Figure orientation can influence students' perceptions of those figures. Some researchers have found that the orientation of shapes impacts students' perception of angles (e.g., Davis et al., 2005; Vinner & Hershkowitz, 1980) and their correct identification of triangles from among other shapes (Dağlı & Halat, 2016). Additionally, depending on the orientation of a figure, some researchers have observed some visual-type phenomena that they called a gravitational factor (Vinner & Hershkowitz, 1980).

Orientation and Perception of Angle. Researchers have found that when a problem or task contains a figure that has a horizontal or vertical component (i.e., an angle with a horizontal ray, or a triangle with a horizontal or vertical side), this tends to lead to improved student outcomes for that problem or task (e.g., Davis et al., 2005; Onley & Volkmann, 1958; Vinner & Hershkowitz, 1980). I offer four brief examples.

First, Onley and Volkmann (1958) asked college students to look at two intersecting line segments on a plexiglass disk such that one segment was fixed, and the other segment was able to rotate about the intersection point. The researchers had participants position their heads on a headrest and restricted their field of vision so that they could only see the disks. The participants were asked to rotate the moveable segment so that it became perpendicular to the fixed segment. Onley and Volkmann found that the angles, which students formed, were closest to perpendicular when the fixed line segment was either horizontally or vertically oriented on the screen.

Second, Vinner and Hershkowitz (1980) found that the students in Grades 7, 8, or 9 correctly identified straight angles, obtuse angles, and right triangles more often when one ray of the angle was horizontally oriented on the page. Likewise, students had greater success identifying right triangles when the triangle legs were positioned with one horizontal leg and one vertical leg than they did when attempting to identify right triangles with legs in other orientations.

Third, Davis et al. (2005) asked children, ages 5 and 6 years, to create a copy of a given angle (in various orientations) by drawing a line segment to connect with a second-given segment. They found that the children made fewer copying errors when the given segment was vertical.

Fourth, some students make determinations about angles based on perceived ray "length." For example, Cullen, Cullen, and O'Hanlon (2018) found evidence consistent with other studies (e.g., Bütüner & Filiz, 2017) that some students perceived angles with "longer" rays as having greater angle measures.

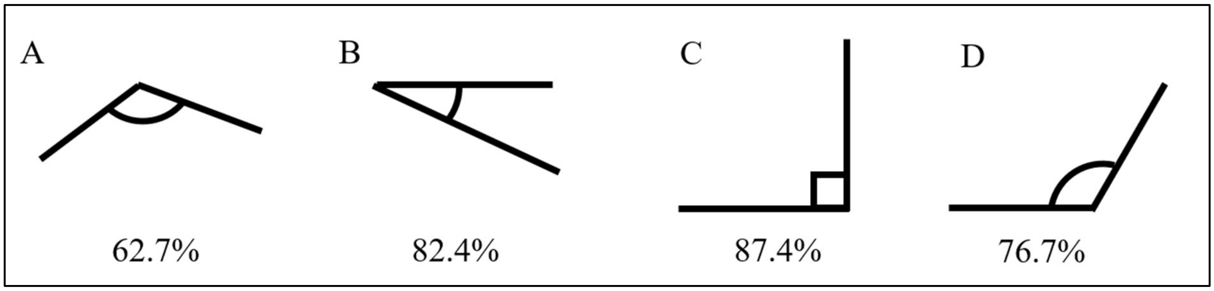
Gravity With Triangles. Researchers (e.g., Vinner & Hershkowitz, 1980; Ward, 2004) have used the idea of gravity to describe figures and to describe students' perceptions of figures. Here, I summarize both usages of gravity and clarify the distinctions and commonalities between them.

Ward (2004) used the term "gravity-based*"* (p. 41) to describe the position or orientation of figures that have one horizontal side. Ward observed preservice teachers, when attempting to identify triangles from among a collection of figures, rotating the paper to achieve a gravity-based orientation: "Upon viewing the triangles, a third teacher candidate also rotated the paper several times, viewing the triangles in different orientations forcing some to be gravity-based" (p. 44). To clarify, Ward (2004) used *gravity-based* to describe figures (e.g., triangles or hexagons) that either had a horizontal side or that students rotated until they appeared to have a horizontal side.

Vinner and Hershkowitz (1980) intended a similar, but slightly different, usage of the term *gravitational factor* to describe a person's visual perception of a geometric object (i.e., angle or triangle). For example, they asked middle school students to identify straight angles, obtuse angles, and right triangles from among a variety of figures. Vinner and Hershkowitz found that students had the highest rates of correct identification in figures that contained a horizontal or nearly horizontal side or ray. They hypothesized that students were influenced by a gravitational factor. Figure 7 shows one of their tasks involving students circling obtuse angles—the percentages indicate the correct choice to either circle or not circle the angle. For example, the percentage in Figure 7B indicates that 82.4% of students made the correct choice to not circle that angle. If gravity were to be applied to Figure 7B–D the angles in Figure 7C and D would not move at all, but the angle in Figure 7B would fall and land on an imaginary horizontal line causing slight counterclockwise rotation. In contrast, the angle in Figure 7A would fall and land with both rays touching an imaginary horizontal surface.

Figure 7

Identification of Obtuse Angles Task and Results From a Study



Note. Images adapted from "Concept images and common cognitive paths in the development of some simple geometrical concepts," by S. Vinner and R. Hershkowitz, 1980, *Proceedings of the 4th PME International Conference,* pp. 177–184 (https://weizmann.esploro.exlibris group.com/esploro/outputs/conferenceProceeding/Concept-images-and-common-cognitive-paths/993328410103596). Values below each panel indicate the percentage of study participants that correctly identified each angle as obtuse or not obtuse. Panel A: Obutse angle with no horizontal rays. Panel B: Acute angle with one horizontal ray. Panel C: Right angle with one horizontal ray. Panel D: Obtuse angle with one horizontal ray.

Additionally, Vinner and Hershkowitz (1980) hypothesized that a combination of typical textbook images and a gravitational factor may influence students' concept images of obtuse angles:

> Obtuse angles which have a horizontal ray are identified more easily than others. Some concept images contain only obtuse angles with horizontal rays. As we know teachers and textbooks, we can say that in both there is a tendency to draw obtuse angles with a horizontal ray. Possibly the "gravitational factor" has also some role in forming the above concept image. An angle is "stable" ("can stand") only if it has one horizontal ray (the other one being ascending rather than descending). This might cause people to draw

angles with a horizontal ray. As a result of this a concept image that contains only obtuse angles with a horizontal ray can be formed. (p. 181)

To summarize, Ward (2004) used the term gravity-based to describe a position or orientation of a figure, but also to describe when preservice teachers rotated triangles to a certain orientation. Vinner and Hershkowitz (1980) hypothesized a gravitational factor whereby students might imagine the figures falling onto an imagined horizontal line.

Interestingly, when Vinner and Hershkowitz asked students to sketch triangle altitudes to a pre-selected base, they found that the gravitational factor was "not dominant anymore" (p. 182) meaning that there seemed to be competing factors that influenced students' concept images. Although they hypothesized a gravitational factor for some circumstances (e.g., identification of obtuse angles), they found that other factors, including triangle type (e.g., isosceles, right) may be more influential on students' ability to correctly sketch triangle altitudes.

The Influence of Nearby Objects. In this section, I describe how nearby objects can impact students' perception of other objects. I use the phrase *nearby objects* broadly to include physical objects in the room nearby to the student (e.g., desk or wall) and nearby problems on the page.

Several authors (e.g., Berman et al., 1974; Piaget & Inhelder, 1948/1999) have found that objects adjacent to a geometric figure or a student's workspace can influence students' perception of the geometric objects. Researchers (e.g., Berman et al., 1974; Onley & Volkmann, 1958) posited that nearby objects may impact a person's perception of angles.

For example, if a person was looking at angles on a paper, which was placed on a square table versus a circular table, the shape of the table might influence the person's perception of or ability to correctly identify perpendicular segments on the paper (Berman et al., 1974). Likewise, Piaget and Inhelder (1948/1999) found that young children, when tasked with drawing trees on the side of a hill, tend to draw the trees perpendicular to the hill rather than to the horizontal surface on which the hill rested. Thus, the presence of the hill seemed to influence students' ability to show trees growing perpendicular to the horizon.

By broadening the definition of "nearby objects" to include nearby problems, then there may be further influences on student item responses. Researchers (Blessing & Ross, 1996; Karplus et al., 1983; Thompson & Thompson, 1996) found that when students identify a successful strategy in one problem, they tend to use that strategy in the next problem of a similar type. For example, Karplus et al. (1983) studied 11- to 13-year-olds' proportional reasoning in algebra. When comparing two groups of eighth-grade students who had worked on problems in a different order, they found with statistical significance that students who had worked on a problem that involved "within" type thinking (i.e., a particular proportional reasoning strategy that they had identified in their literature review) they were more apt to use that same strategy in the following problems.

Impacts of Triangle Type

Researchers have documented two types of triangles, isosceles and obtuse, as being influential in students' perception relating to altitude (Cunningham & Roberts, 2010; Krajcevski & Sears, 2019; Vinner & Hershkowitz, 1980). Students in Grades 7–9 seem to be most successful at drawing altitudes when triangles are closer to isosceles with an interior altitude and seem to struggle the most with the drawing of exterior altitudes in obtuse triangles (Vinner & Hershkowitz, 1980). Researchers have obtained similar results with preservice teachers (Cunningham & Roberts, 2010).

Chapter Summary

In this chapter, I identified concept image and definition as a helpful lens for describing students' understanding in response to various tasks (Vinner & Hershkowitz, 1980). I also discussed relevant aspects of understanding area in general, including (a) unit concepts (e.g., Barrett et al., 2017; Miller, 2013; Outhred & Mitchelmore, 2000; Sarama & Clements, 2003), (b) area and conservation (e.g., Cavanagh, 2008; Kamii & Kysh, 2006; Piaget et al., 1960/1970), (c) attending to the wrong attribute (e.g., Barrett et al., 2017; Doig et al., 1995; Hirstein et al., 1978; Kamii & Kysh, 2006; Kordaki & Balomenou, 2006; Miller, 2013; Outhred & Mitchelmore, 2000; Tierney et al., 1990), and (c) spatial structuring (e.g., Battista & Clements, 1996; Battista et al., 1998; Miller, 2013; Outhred & Mitchelmore, 2000; Prolux & Pimm, 2008).

Lastly, I discussed a variety of topics that either directly or indirectly impact understanding relating to triangular area: (a) students' interpretations of base, height, and altitude (e.g., Gutiérrez & Jaime, 1999; Horzum & Ertekin, 2018), (b) unitizing complications involving triangles (e.g., Barrett et al., 2017, pp. 117–118; Kamii & Kysh, 2006; Reynolds & Wheatley, 1994), (c) orientation of figures and the gravitational factor (e.g., Dağlı & Halat, 2016; Davis et al., 2005; Vinner & Hershkowitz, 1980), (d) the influence of nearby objects (e.g., Berman et al., 1974; Piaget and Inhelder, 1948/1999), and (e) the impacts of triangle type (Cunningham & Roberts, 2010; Krajcevski & Sears, 2019; Vinner & Hershkowitz, 1980). In the next chapter, I will discuss the methodology for my study, which fills critical gaps in research literature. Gaps include low quantities of studies involving high school students, needing to seek additional concept images for base, height, and area, needing to seek relationships among concept images of triangle base, height, and area, and understanding how students' perceptions of base height and area may be influenced by other factors.

CHAPTER III: DESIGN OF STUDY

In this chapter, I discuss all aspects of the study. First, I will provide a broad overview of the study. Then I provide details about the methodology including setting, participants, and instruments. Finally, I discuss how I analyzed the data through multiple phases of analysis.

Methodology and Study Overview

To gain insights into students' interpretations of triangle base, height, and area, I used the theoretical lens of concept image and concept definition due to its emphasis on individual concepts with multiple components or parts (Vinner & Hershkowitz, 1980). I implemented a qualitative study (Merriam, 2009). Broadly, I administered a survey to 95 high school geometry students, designed to assess students' concept images of triangle base, height, and area. Then I conducted semi-structured, task-based interviews with seven of the 95 students. Figure 8 shows the data collection and analysis timeline.

Figure 8

Timeline of Study

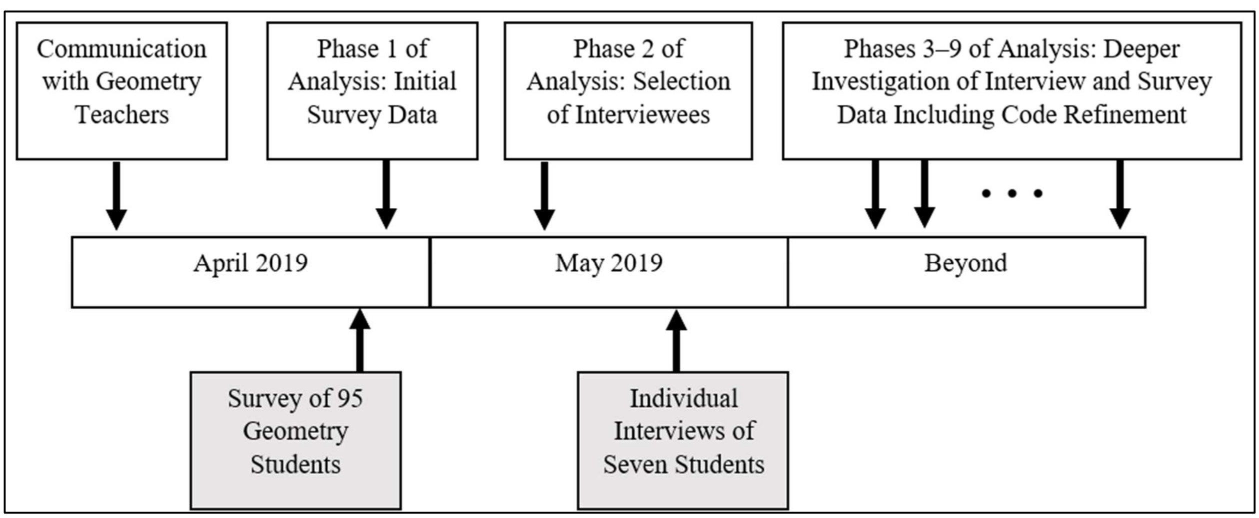


Figure 8 shows the timeline of the study, leading from participant recruitment to data collection and analysis. I will share additional details about both the participants and phases of analysis later in this chapter.

Qualitative analysis was appropriate for this study because I sought to describe students' evocations of triangle base, height, and area in the form of concept images (e.g., Gutiérrez and Jaime, 1999; Horizum & Ertekim, 2018), which could be verified, clarified, or expanded. Additionally, qualitative analysis allows for the interpretation of an individual student's concept images of triangle base, height, and area. Recall that my research questions were:

1. What are high school geometry students' concept images of triangle base, height, and area?
2. How are high school geometry students' concept images of triangle base, height, and area related to one another?
3. How are triangle orientation, gravity, and triangle type related to high school geometry students' concept images of triangle base, height, and area?

Setting and Participants

In this section, I begin with a brief positionality statement. Then, I discuss the setting of this study and narrow the focus to include a description of the study participants with some preliminary pre-study procedures.

Positionality Statement

At the time of writing this dissertation, I have taught high school mathematics for 17 years of which I have taught geometry for 10 years. Additionally, I have taught a wide range of classes from pre-algebra through calculus, including statistics. I was not one of the geometry teachers in this study and did not know any of the study participants.

Study Setting

This study took place in a public high school in the Midwest Region of the United States that enrolled approximately 2300 students. According to their 2019 report card (School Report Card, 2019), the top three school demographics by race were 65% White, 11% Black, and 11% Asian. This school also had 25% low-income families and a 91% graduation rate.

I surveyed the geometry classes immediately following their unit of study involving the area of two-dimensional regions—it just so happened that all three levels of geometry had finished studying that unit around the same time. The reason for the timing of my study was to obtain information about students' concept images at a time when they should, in theory, be the most closely aligned to the concept definitions—I reasoned that this would strengthen the study results if I were to find a wide variety of concept images even after instruction on the topic of area.

Participants and Pre-Study Procedures

After obtaining University Institutional Review Board approval, I emailed each of the five geometry teachers and shared a brief description of my study near the beginning of April 2019. I then asked the teachers if I could visit their classes to recruit student participants. Next, I briefly visited each of the 13 geometry classes to describe the study and to distribute both an information letter about the study and informed assent and consent letters (see Appendix A). I directed all students to return the forms—signed or unsigned to their teachers in sealed envelopes. In this way, teachers were not aware of who agreed to participate and who did not. I invited a total of 349 students to participate in the study and 95 responded affirmatively yielding a response rate of 27%.

Class Level Descriptions

The 13 geometry classes included two co-taught geometry classes (i.e., with each class including a general education and a special education teacher), eight regular geometry classes, and three honors geometry classes. A geometry teacher at this school reported that the geometry classes could be thought of as two different levels with co-taught and regular designed to be taught at a consistent level, and honors geometry designed to be at a more advanced level (Geometry Teacher, personal communication, December 2, 2022). In the following subsections, I will describe each of the two levels of geometry.

Regular and Co-Taught Geometry. I asked a geometry teacher who was teaching both regular and co-taught geometry classes some clarifying questions about those class types (Geometry Teacher, personal communication, December 2, 2022). They stated that the two classes were intended to include the same in content, pacing, and assessments. The geometry teacher informed me that the only notable difference was that, from time to time, the co-teacher would alter assessments in accordance with students' Individual Education Plans (IEPs). For example, the co-teacher brought some students to a smaller testing environment or sketched a triangle in a student's assessment as a visual support for a task in which the triangle was described only with letters. Otherwise, both class types covered the same content and used the same assessments.

In co-taught geometry classes, at most 30% of the students had identified disabilities served via IEPs, and the rest of the students were randomly placed into a co-taught or regular geometry class. This maximum percentage was set by state guidelines. Therefore, each year, the school district determines how many co-taught sections are needed to support the number of students with disabilities (Special Education Teacher, personal communication, May 9, 2023).

Honors Geometry. An honors geometry teacher at the school reported that the honors geometry course differed from both the regular and co-taught courses in pacing and content (Honors Geometry Teacher, personal communication, December 3, 2022). The additional content included two extra units of study, more complex tasks on homework, and additional assessments relating to the extra units of study as compared to regular geometry classes.

The two additional units in the honors geometry class focused on (a) logic and proof and (b) advanced triangle theorems. The unit on logic and proof incorporated analysis of conditional statements with hypotheses and conclusions, converse statements, and contrapositive statements. This unit on logic was positioned as the first unit of study in the honors geometry curriculum. Then, later in the Spring semester, the unit involving advanced triangle theorems included topics such as the law of sines, the law of cosines, and Heron's Formula.

The inclusion of two extra units of study was one factor that caused an increased pace in the honors geometry class as compared to both regular and co-taught geometry classes. Also, the honors geometry classes went deeper into each of the other respective units. Because of these two factors, the honors geometry teacher described the class as covering more content and with a faster pace.

According to the honors geometry teacher, students enrolled in that level of geometry by one of the following methods. First, they could have been recommended to move into honors geometry by a previous mathematics teacher. Second, students could request to move into honors classes. Third, parents or guardians could request that they move into the honors level.

Instruments

In this section, I discuss both study instruments—a survey and an interview—which I designed to address my research questions.

Survey Description

I created a survey, which contained 10 questions (see Appendix B). First, I discuss procedural considerations relating to the survey and then I discuss the specific survey content.

Survey Procedures

I gave the survey in a single day, with the help of another researcher, visiting one or two classrooms each period. The teacher used the survey as a warmup for all students in the class. After collecting the surveys, I copied the participant surveys and returned all originals to the teachers to use for their purposes.

When administering the surveys, I told students that a variety of tools were available if they chose to use them, including a calculator, ruler, scissors, a cut-out copy of the triangle from Survey Question 10, and centimeter grid paper. I designed the survey to take approximately 20 minutes to complete. Most students completed the survey in less than the given 20-minute timeframe.[8]

Survey Questions

In the following subsections, I describe each of the 10 survey questions based on the following groupings: (a) Questions 1–3, described as free-response questions; (b) Questions 4–8, described as questions involving base and height drawings; and (c) Questions 9–10, described as area questions. In Figure 9, I show how I had intended the survey questions to support the components of my research questions. I designed the survey so that each question within the survey had the potential to generate data related to at least one research question component. I accomplished this by including a variety of triangle types (i.e., acute, right, and obtuse) with a

[8] On the day of the survey, the honors geometry teacher requested that I restrict the survey to 10 minutes because she needed to accomplish a variety of other tasks. Therefore, many of the honors geometry students only partially completed their surveys. In all other classes, the teachers allowed the full 20 minutes, which seemed adequate for those students.

variety of orientations. Only one triangle contained a horizontal side and only one contained a vertical side. The remaining triangles were oriented such that no sides were horizontal or vertical—I intended to create opportunities to observe the influence of orientation and the gravitational factor (Vinner & Hershkowitz, 1980), and the influence of triangle type (Cunningham & Roberts, 2010; Vinner & Hershkowitz, 1980). I also asked participants to explain their methods when calculating area in the final two survey questions with the hope that I might be able to make inferences about their connection-making among the base, height, and area concept images.

Figure 9

Survey Questions' Connections to Concept Images and Influential Factors

Survey Question	CI of Area	CI of Base	CI of Height	Relationships Among CIs	Triangle Orientation	Gravity	Triangle Type
1	x						
2		x					
3			x				
4		x	x		x	x	x
5		x	x		x	x	x
6		x	x		x	x	x
7		x	x		x	x	x
8		x	x		x	x	x
9	x	x	x	x	x	x	x
10	x	x	x	x	x	x	x

Note. Each "x" indicates an intended connection between the survey question and the research question component. The abbreviation "CI" represents concept image.

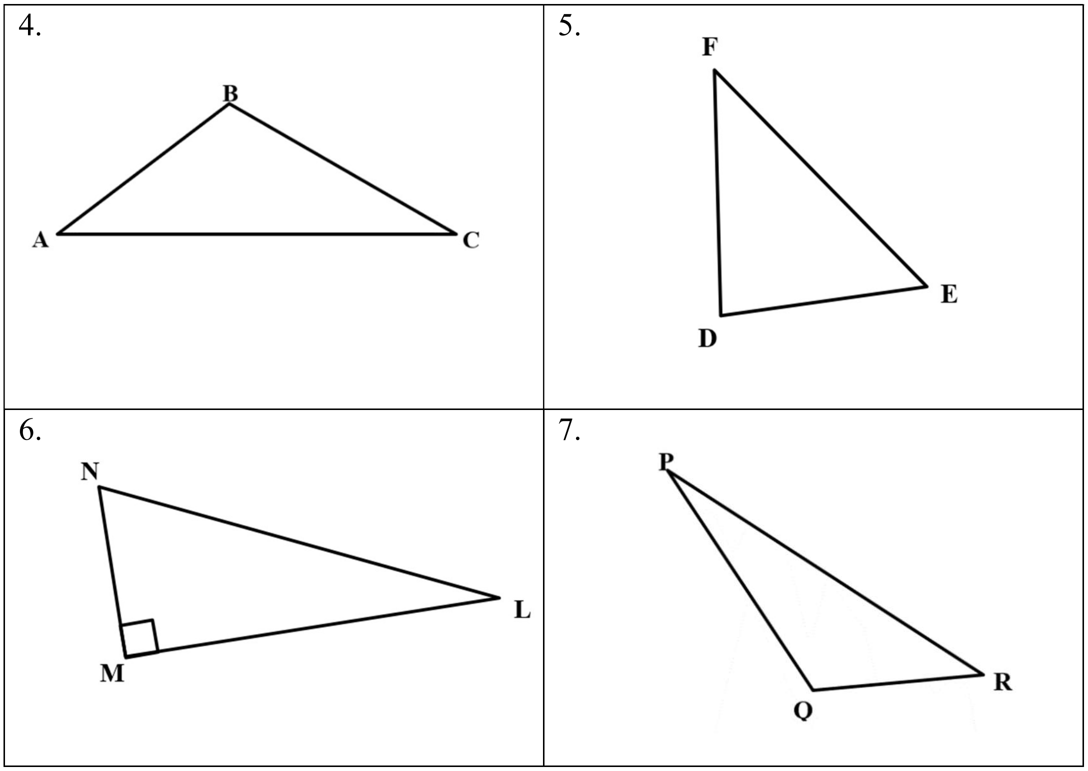
Free Response Questions. The first three questions asked participants to describe area, base, and height and contained no images of triangles. I intentionally used no triangle images on the first page because I did not want images to inadvertently influence participants' concept images of triangle area, base, and height for those questions. Also, I chose to place the area questions first in the survey so that students' evoked images would be less likely to be influenced by the questions about base and height which followed.

These questions were inspired by a study designed by Knuth et al. (2006) about students' understanding of the equal sign. In their study, Knuth et al. showed students an equation with an arrow pointing to the equal symbol and then asked students to identify the name and meaning of the symbol. Then they followed up by asking if the equal sign could mean anything else. They found that students often provided multiple answers when asked repeatedly about the same concept. Similarly, I used the first three survey questions to elicit evidence of students' evoked concept images for triangle attributes and measures (i.e., area, base, and height).

Questions Involving Base and Height Drawing. Questions 4–7 (see Figure 10) asked participants to draw a base and corresponding height in each of the four triangles. Then, in Question 8 students wrote about how they chose base and height for the right triangle in Question 6.

Figure 10

Triangles Used for Survey Questions 4–7



I constructed these triangles to ascertain the extent to which triangle type (e.g., Cunningham & Roberts, 2010; Heller, 2017; Krajcevski & Sears, 2019; Vinner & Hershkowitz, 1980) and triangle orientation (e.g., Dağlı & Halat, 2016; Davis et al., 2005; Vinner & Hershkowitz, 1980) influenced students' perceptions of triangle attributes. I included only one triangle with a horizontal base orientation to determine whether the non-horizontal base triangles elicited evidence of a gravitational factor (Vinner & Hershkowitz, 1980) in students' perceptions.

Survey Question 4 was a nearly isosceles obtuse triangle with the longest side nearly parallel to the bottom of the page. Such a triangle has an interior height when the bottom side is identified as the base. I created Survey Question 5 as an acute triangle, with no side parallel to the bottom edge of the paper. Because the triangle is acute, all altitudes are interior.

I created Survey Question 6 to be a right triangle to determine whether students would select the perpendicular edges of the triangle as the base and height in this situation (i.e., when height is measured along a side of the triangle) and because that triangle includes no horizontal or vertical sides. I created Survey Question 7 to be an obtuse triangle to determine whether students would create an exterior height to match a base that was close to being horizontal, or whether they would select a non-bottom base that would allow them to sketch an interior height.

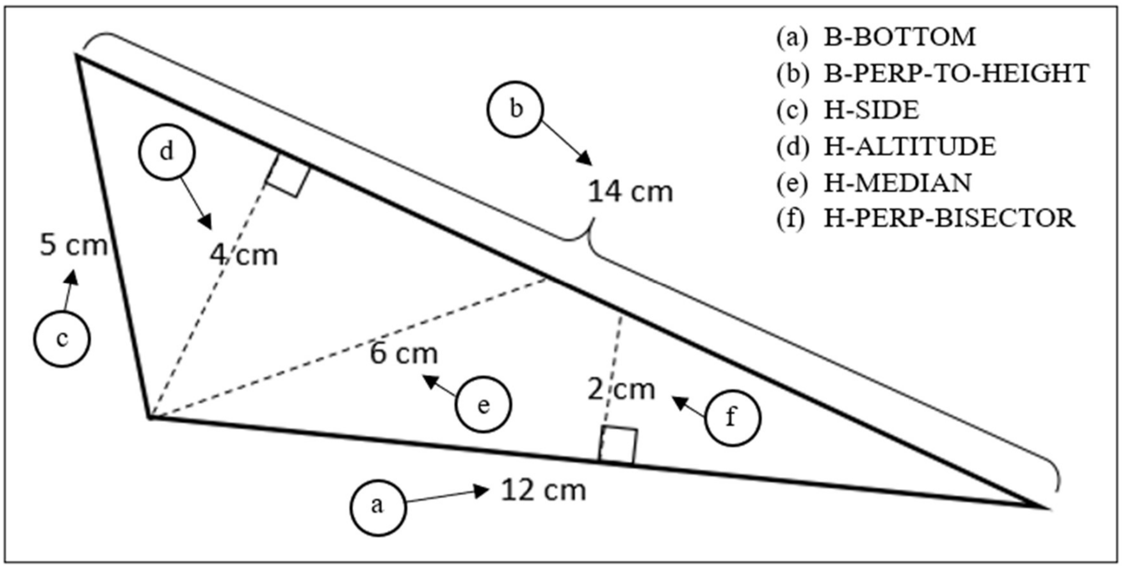
I created Survey Question 8 to gather more evidence about how students cope with right triangles by asking students to articulate how they would help a friend, over the phone, identify the base and height for the triangle in Survey Question 6. I created the phone call scenario for Survey Question 8 to elicit participants' verbal descriptions of those concepts. Also, I designed Survey Question 8 to augment the static labeling of the base and height from the previous four questions.

Area Questions. I designed the last two questions (i.e., Questions 9–10) of the survey to focus on triangle area. Specifically, these two questions assess students' ability to determine the areas of triangles in different contexts. Like Tossavainen et al. (2017), I reasoned that by allowing participants the opportunity to find area measures, I might be able to create a window into their concept image of area. Also, in both Questions 9 and 10, I asked participants to explain their reasoning because I wanted to create additional opportunities for them to explain their thought processes thereby gaining insight into how their base, height, and area concept images were connected.

Baturo and Nason (1996) found evidence that some preservice elementary teachers struggled with finding the area of triangles even when all appropriate lengths (i.e., base and height measures) were provided. Inspired by Baturo and Nason, I designed Survey Question 9 (see Figure 11) to determine which base and height pairing students identified among various sketched options. As indicated in the figure, the segments were purposefully placed to assess various concept images.

Figure 11

Triangle Used for Survey Question 9

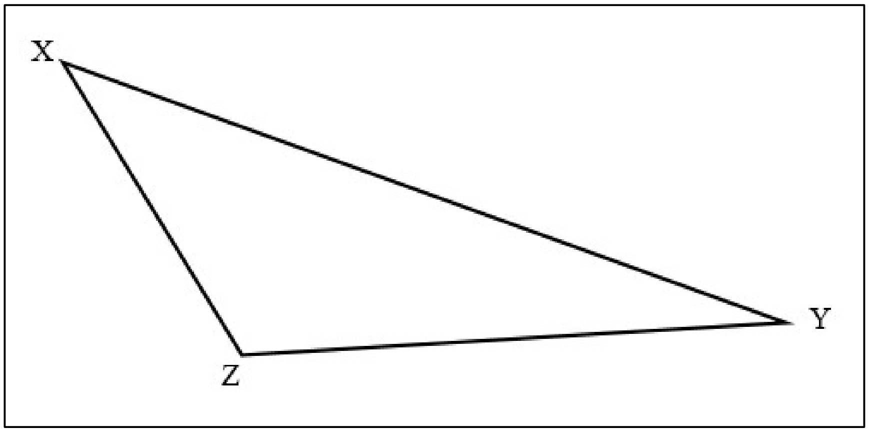


Note. This triangle was smaller in the survey. The triangle in the survey did not contain the concept image list (a–f) or the corresponding labels on the figure.

For Question 10 (see Figure 12), I created an obtuse triangle with vertices X, Y, and Z but without side measure labels. I oriented the triangle so that the longest side was not parallel to the bottom of the page. I decided to use an obtuse triangle to allow the possibility of an exterior altitude, and I used an orientation in which none of the sides was vertical to avoid the prototypical triangle shape (Vinner & Hershkowitz, 1980).

Figure 12

Triangle Used for Survey Question 10



Summary of Survey

In summary, my survey contained 10 questions, and I administered it to 95 participants. Overall, I designed the survey to address my three research questions. I was intentional in the design of each question to generate data related to components of the research questions and also to help with the interviewee selection process.

Interview Description

Broadly, the interviews were individual, semi-structured, and task-based (Goldin, 2000; Merriam, 2009). Goldin (2000) described task-based interviews as involving a "subject (the problem solver) and an interviewer (the clinician), interacting in relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way" (p. 519). Goldin argued that task-based interviews "make it possible to focus research attention more directly on subjects' processes of addressing mathematical tasks, rather than just on the patterns of correct and incorrect answers in the results they produce" (p. 520). Goldin also created ten principles for task-based interviews, including, (a) designing tasks to address research questions, (b) careful selection of tasks that are both accessible to the subjects and enable rich discussion, and (c) encouragement of open-ended problem-solving.

Interview Procedures

The interviews for the present study took place over the span of 2 weeks outside of school hours (i.e., before or after school) in a vacant classroom. Additionally, to compensate students for their time, I offered each student a $10 gift card to a local coffee shop. Each interview lasted about 30 to 40 minutes. I used two video cameras—one positioned over the interviewee's shoulder aiming downward at their paper and the second one directly across from their desk also aiming downward towards their paper. For redundancy, I also used an audio recording device.

Before the start of each interview, I obtained verbal assent for the interview during a brief pre-interview interaction with the students (see Appendix C). Then, before administering the five interview tasks, I asked demographic and informal questions to help the students feel comfortable with the interview process (Merriam, 2009).

Throughout the interviews, I focused on each student's evocations by asking them to elaborate when their thinking was unclear, using think-aloud techniques (Charters, 2003; see also Battista et al., 1998; Goldin, 2000). For example, I asked questions such as, "How did you decide to draw the height there?" and, "Talk to me about the base in this triangle." Such questioning techniques have been described as *think-aloud* and can provide a valid source of data relating to participants' thinking (Charters, 2003). In this way, I probed students' thoughts and motivations regarding triangle attributes. I also asked some questions that involved altering the goals within some tasks such as, "Would the base change if we rotate the triangle this way?"

Interview Task Descriptions

I created five interview tasks (see Appendix D) so that interviewees worked through the tasks in the same order. In this way, I was able to generate data that was comparable across participants. In Figure 13, I show how I had intended each interview task to support the components of my research questions. In a manner like the survey design, I included a variety of triangle types and orientations of triangles within the interview tasks.

Figure 13

Interview Tasks' Connections to Concept Images and Other Influential Factors

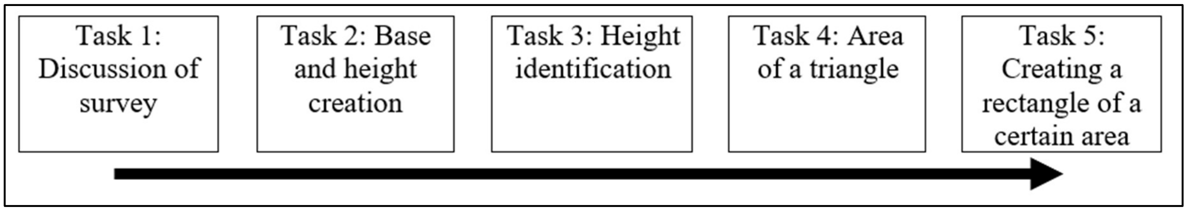
Interview Task	CI of Area	CI of Base	CI of Height	Relationships Among CIs	Triangle Orientation	Gravity	Triangle Type
1	x	x	x	x	x	x	x
2		x	x		x	x	x
3			x		x	x	x
4	x	x	x	x	x	x	x
5	x	x	x	x	x	x	x

Note. Each "x" indicates an intended connection between each interview task question and each research question component. The abbreviation "CI" represents concept image.

The sequence of the interview tasks is shown in Figure 14. The arrow within Figure 14 indicates how each interviewee progressed through the tasks in the same order. I describe each of the five tasks in the subsections which follow.

Figure 14

The sequence of Interview Tasks



As I began Task 1, I told the students that I was more interested in their thinking than whether their answers were correct or incorrect. Inspired by Goldin's (2000) emphasis on "free problem solving" (p. 523) and neutral question types, I encouraged each student to think aloud as they worked through the tasks (e.g., Dursken et al., 2021).

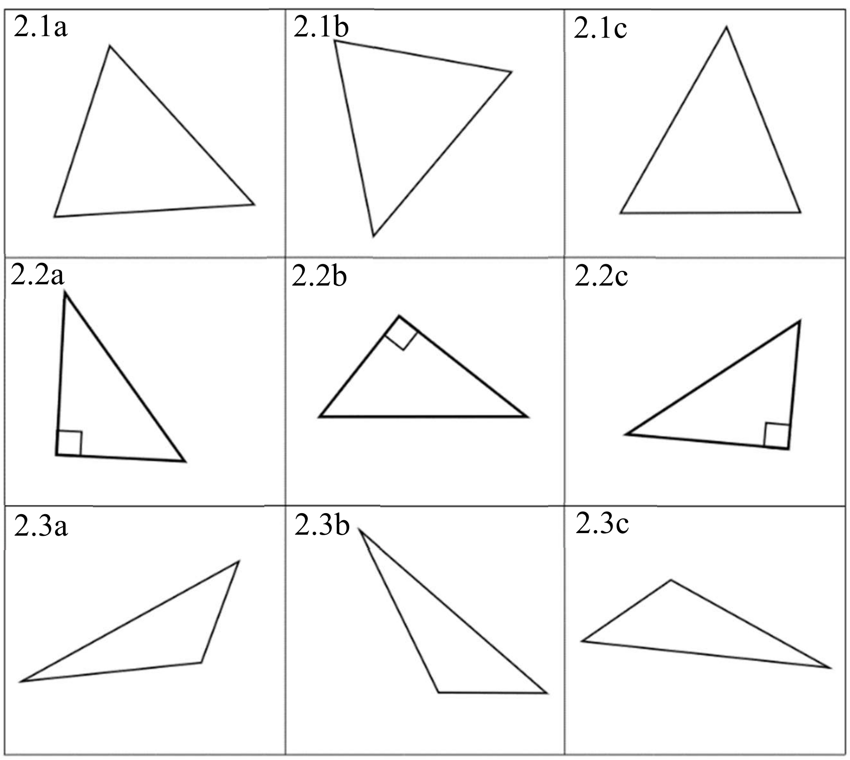
I had intended all interview tasks to provide some data related to the influence of triangle type, gravity, and triangle orientation. As such, I paid attention to participants' concept images as compared to triangle type, any responses that seemed connected to gravity, and their rotation or lack of rotation of the paper within each task. In the following subsections, I briefly describe each of the interview tasks.

Task 1: Discussion of Student Survey. My intention in Task 1 was to ask clarifying questions about the interviewees' surveys before I asked students about the other tasks. Asking clarifying questions gave each interviewee a chance to say more about their responses to the survey questions. For example, regarding Survey Questions 4–7, I asked, "Can you tell me how you identified the base and height for these questions?" I wanted to know if there was something about the survey questions that might have influenced the evocation of one concept image in one question compared to another.

Task 2: Unstructured Base and Height Identification. In Task 2, I created nine unique triangles (see Figure 15) for each interviewee to examine, with three different types of triangles (i.e., acute, right, and obtuse), each with three orientations. This task is like the survey questions that Gutiérrez and Jaime (1999) used where they presented preservice teachers with 14 triangles, each labeled with a base. They asked the preservice teachers in their study to “draw the ALTITUDE with the side marked with the letter a” (p. 261). In contrast, I created triangles without any designated base to collect data about interviewees’ concept images of base as well as height.

Figure 15

Triangles From Interview Task 2



Note. I scaled the triangles down to condense space for this figure. The labels are shown for convenience but were not present during the interview.

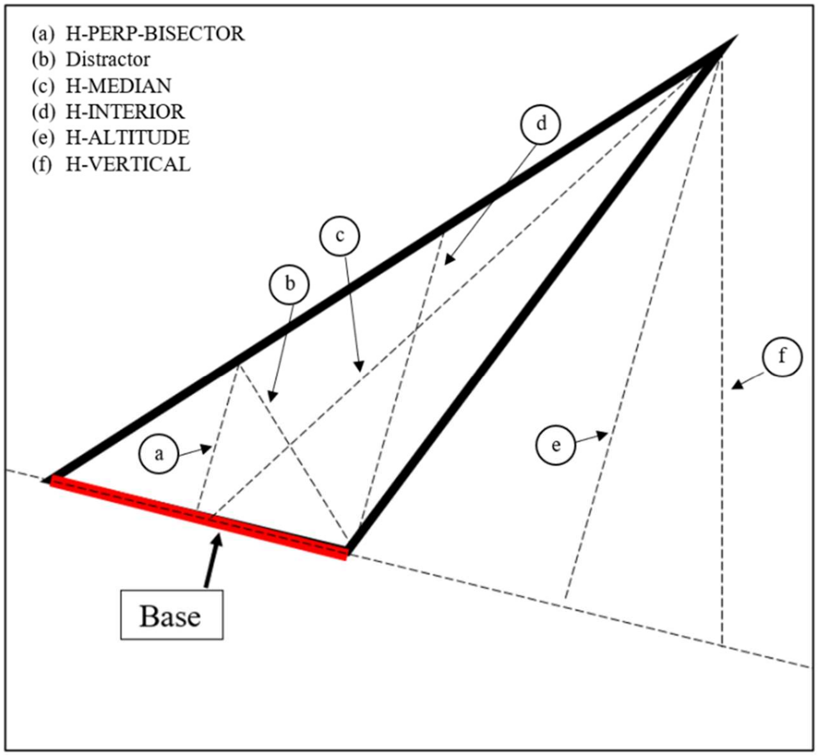
I intended this to be a task in which interviewees would identify a base and height of each triangle by marking a base with a blue marker and marking a height with a red marker.[9] As they worked, I asked them to describe the reasoning behind their choices and I asked clarifying questions, as needed. I included both height and base identification in these tasks because prior researchers (e.g., Gutiérrez & Jaime, 1999; Vinner & Hershkowitz, 1980) found that many students struggle with drawing altitudes in triangles—especially exterior altitudes of obtuse triangles.

Task 3: Scaffolded Height Identification. I created three distinct triangle types (i.e., one of each acute, right, and obtuse) in this height identification task. For each triangle, I identified one side as a base using a bolded red segment and asked students to identify the height that corresponded to the identified base. As illustrated in Figure 16, the triangle included several dashed auxiliary segments. Each auxiliary segment was designed to match an a priori concept image of height thus indicating possible heights, which could be associated with that base. For example, if an interviewee identified the segment labeled "f" as the height corresponding to the given base, that may have indicated that they evoked a concept image of H-VERTICAL

[9] My interview protocol calls for using a red marker for base and blue for height. However, one of my first interviewees had used blue for base and commented about the alliteration between the words blue and base. Therefore, I modified my spoken instructions for Task 2 to ask for blue marker for base and red for height.

Figure 16

A Priori Concept Images Embedded in Interview Task 3



Note. This triangle was larger on the interview task page. The interview task did not contain the concept image list (a–f) or the corresponding auxiliary segment labels on the figure.

Inspired by the findings that previous mathematical tasks can impact students' thinking on a subsequent task (Blessing & Ross, 1996; Karplus et al., 1983; Thompson & Thompson, 1996), I created Task 3 as a follow-up to Task 2. Specifically, I designed Task 3 to determine whether the auxiliary segments might serve as a support for identifying a triangle's height. I wondered whether having the correct height option among several auxiliary segment choices would prompt a type of cognitive dissonance if the interviewees had drawn an incorrect height in Task 2.

Task 4: Area of a Triangle. I created Task 4 based on a task administered by Barrett et al. (2012). Researchers in that study gave pairs of students an obtuse triangle and asked them to find the area of the triangle. They offered students the use of a ruler and calculator. Later in the interview, they asked the students about the meaning of their numerical answers. Similarly, in my study, I offered interviewees a variety of tools such as a ruler, a square centimeter made of card-stock paper, a duplicate triangle cut out of paper, square centimeter graph paper, a pair of scissors, a calculator, and a ruler that had both U.S. customary and metric units. I gave interviewees the instructions, "Measure a base and height, then determine the area of the triangle." Like Tossavainen et al. (2017), I had hoped to use interviewees' area strategies as a window into their concept images of area and to create an opportunity for me to observe interviewees relating the base, height, and area concept images together. I used an obtuse triangle in this task to assess interviewees' concept images of area, base, and height, and to create an opportunity to discuss exterior altitudes.

After the interviewees provided an area measure, I asked them to interpret various components of their answers. Generally, if they answered that the area measure represented the number of square units that could fit inside the triangle, I followed up with questions about the reasonability of their area measures.

Task 5: Creating a Rectangle of Double the Triangle Area. For Task 5, I gave interviewees an acute triangle and asked them to create a rectangle with an area of two times the area of the triangle. I offered them the same tools as described in Task 4. My intentions of creating opportunities for interviewees to reveal their concept images and connections among concept images were the same with this task as with Task 4. Also, I had hoped to create opportunities to discuss the role of the *half* in the standard triangle area formula—a discussion that has proven difficult even for preservice teachers (e.g., Baturo & Nason, 1996; Tierney et al., 1990).

Summary of Interview Tasks

To summarize, after asking demographic questions, I began by discussing each interviewee's survey and then moved into Task 2 by asking them to draw a base and height in each of nine triangles. Then, in Task 3, I asked each interviewee to identify a height, from among several auxiliary segments, in a triangle with a side preselected to be the base. In Task 4, I asked interviewees to find the area of a given obtuse triangle, and then lastly in Task 5, I asked them to create a rectangle with double the area of a given acute triangle. In the sections which follow, I describe each of the phases of data analysis.

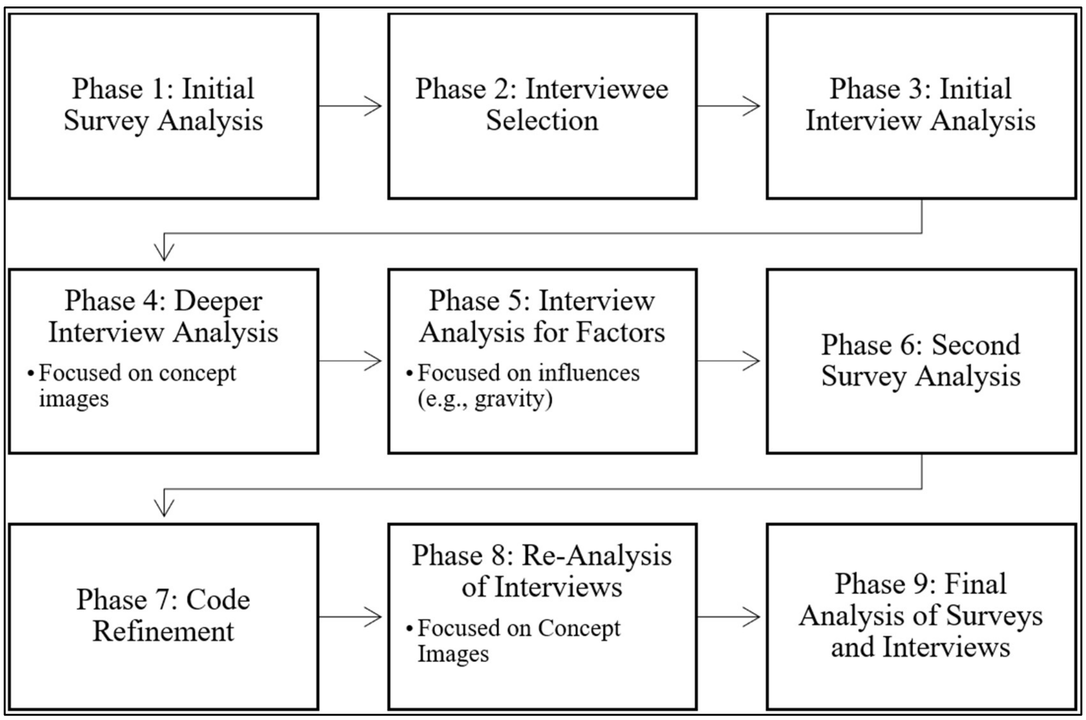
Data Analysis Overview

In this section, I discuss a multi-phased approach to data analysis. By starting broadly and then focusing, I continued to find new information with each phase of analysis. My data analysis process was inspired by a study conducted by Kobiela and Lehrer (2015). They described their analysis of 4 weeks' worth of interviews as occurring in three phases. In their first phase, they selected a portion of the full data set to obtain a smaller set of interviews on which to focus. In their second phase, they strategically selected 16 episodes for analysis, which included transcription, and parsed those episodes further into units of analysis. In their third phase, they selected another, smaller, set of episodes to compare with their proposed pathways. Clement (2000) argued that "in even a 10-minute section of videotape, we are faced with a continuous stream of behavior that can be extremely rich" (p. 572) and that when determining what is relevant to the research questions, "investigators must narrow their focus gradually" (p. 572).

I adapted and expanded upon the three-phase process from Kobiela and Lehrer (2015). Figure 17 depicts the phases of analysis for this study. In the remainder of this section, I describe each phase of analysis. Note that Phases 1 and 2 occurred after the survey and before the interviews. Then, Phases 3–10 occurred after the interviews.

Figure 17

Nine Phases of Dissertation Analysis



Units of Analysis

In the present study, I considered two different units of analysis—those generated from participants' surveys and those generated from interviews. Following recommendations from Roller and Lavakas (2015), I was careful to select units of analysis that were appropriate for this study and that allowed for rich analysis.

Recall that each survey consisted of 10 numbered questions, but some had multiple parts. Therefore, by counting each part separately, each survey can be thought of as having 17 questions. Some question parts were coded with multiple concept image evocations (e.g., a base and a height evocation), and some were left blank. For example, I often identified one base and one height evocation within each of Survey Questions 4–8. Some participants' responses did not contain enough information for me to be able to confidently identify an evoked concept image. This resulted in a total of 1635 units of analysis, where each unit contained one evoked concept image from a single survey question part for one participant. Then, when I analyzed my interview data, the units of analysis were the 315 episodes within the interviews. Although the units of analysis come from different sources (i.e., surveys and interviews), one commonality is that each unit is connected to a single evoked concept image.

Phase 1: Initial Survey Analysis

First, I analyzed the surveys by trying to identify evidence of a priori concept images of base (e.g., Horzum & Ertekin, 2018) and height (e.g., Gutiérrez & Jaime, 1999; Şengün & Yılmaz, 2021). I noted participants' area strategies as indicators of their area concept images (Tossavainen et al., 2017). I also identified any potentially new concept images.

I recorded my findings in a spreadsheet. Each row corresponded to an individual participant's survey, and I used columns for notes. Within 2 days after the participants' completion of the survey, I reviewed my spreadsheet notes to classify possible concept images evident in participants' work.

Phase 2: Selection of Interview Participants

I used my initial survey analysis, from Phase 1 to determine which students to invite to participate in interviews. In the two subsections below, I describe the interview selection process in more detail.

Interview Participant Selection Details

I examined 95 surveys to identify interview candidates whose written work indicated that they had elicited a wide variety of concept images. I looked for evidence of the a priori concept images, which I had identified before the study. I also documented any potentially new concept images as well as any evidence of links between orientation, gravity, or triangle type with and participants' concept images.

In some surveys, the participants' concept images for base and height seemed consistent throughout. In other surveys, the participants' responses to the base and height drawing questions (i.e., Questions 4–8) aligned with their responses to the describing questions (i.e., Questions 1–3), but then their evoked concept images seemed to change in Questions 9 and 10. In such cases, I made notes about the changing of evoked concept images in the survey spreadsheet.

Also, I identified participants for whom I noticed something unique about their survey responses. For example, in an obtuse triangle, a participant drew an interior height (connected to an acute angle vertex) and drew a right-angle symbol, even when their "height" looked like it formed an acute angle with the base. I considered this a unique response because I had not seen this in other participants' responses.

By identifying participants who had written unique answers and those participants whose concept images for base and height had changed at least once within their surveys, I narrowed the group of potential interviewees down to 22 candidates. I verified that they had evoked many of the a priori concept images and had evoked some new concept images within their surveys. Then, to help me narrow my search, I identified every participant among those 22 who also seemed to switch strategies for finding area at some point during the survey. This narrowed my list down to 10 potential interviewees. Of the 10 students I invited to be interviewed, seven of them agreed and completed the interviews.

When I discuss survey results from a participant who did not become an interviewee, I use a code that includes letters and numbers to communicate the type of geometry class and a randomly assigned number for that participant. Specifically, I abbreviate co-taught (CT), regular (R), and honors (H) geometry class types. For example, I may share survey results from H3 and CT4 meaning a student from an honors geometry class and a co-taught geometry class, respectively. In contrast, whenever I share survey or interview data from an interviewee, I use their pseudonyms with class-type abbreviations. For example, Odessa(CT10) represents an interviewee who was the tenth student from the cotaught geometry class who participated in my study. In Table 1, I share the list of the seven interviewees, using pseudonyms, in the order that I interviewed them, along with each participant's geometry class.

Table 1

Interviewees: Order and Geometry Classes

Interview Order	Pseudonym	Class Level
1	Odessa(CT10)	Co-Taught
2	David(R26)	Regular
3	Lisa(H16)	Honors
4	Nate(R21)	Regular
5	Alyssa(R23)	Regular
6	Erica(H24)	Honors
7	Phoebe(CT11)	Co-Taught

Phase 3: Initial Analysis of Interviews

I performed constant comparative analysis (Merriam, 2009) on my interview videos to see if any new concept images emerged and to refine my descriptions of existing concept images. Specifically, I watched each interview on the same evening or the day following that interview. While I watched, I typed notes in a digital document. I noted anything that stood out to me as a clear example of any concept image for base or height. I also noted area strategies and observations related to orientation or triangle type.

With each note, I included a timestamp so that I could find that part of the interview later, as needed (see Figure 18). Additionally, I noted when an interviewee's evoked concept image varied across tasks. After examining the range of concept images evident across all interviewees and conferring with another mathematics education researcher, I decided that I had gathered a suitable amount of interview data to conclude interviews and move forward with further analysis. The researcher and I considered several factors including (a) the wide variety of concept images present in the initial analysis of interviews, (b) instances of interviewees seeming to evoke a variety of concept images throughout their interviews, and (c) evidence of new concept images.

Figure 18

Sample of Initial Interview Notes

Interview began at 4:00pm
1. (5:00) Task 1 begins.
2. (5:36): Student described base [*evocation of B-BOTTOM*].
3. (5:51) Height goes from the top to the base.
4. (6:30) Kept paper as is. Recognized height first.

Note. The time values in parentheses represent minutes and seconds into the interview.

Phase 4: Deeper Interview Analysis for Concept Images

After completing Phase 3 of the analysis with all the interviews, I noticed that there seemed to be definable episodes during the interviews, which coalesced around particular concept image discussions or other interesting events. These episodes were emergent and varied across participants, as opposed to being tied to specific tasks. Therefore, I watched each interview again and recorded notes about each episode within a digital spreadsheet. This process yielded 315 unique episodes across the 7 interviewees—each of which ranged from 5 seconds to 2 minutes—regarding either an evoked concept image for base, height, area, or a note about something interesting, which I observed. For example, in Erica(H24)'s summary notes, I wrote, "Student explained how the base and height can switch in right triangles, but not in other triangles."

I determined the beginning and end of an episode based on several criteria. First, I would always begin an episode at the beginning of a new problem (e.g., the second right triangle in Interview Task 2). Also, I would begin a new episode whenever the interviewee evoked a concept image of base, height, or area. Therefore, at times, the episodes were quite short. For example, an interviewee might have described how they determined a base and then less than 5 seconds later begin describing how they determined a height. In this example, the conversation resulted in a very short episode relating to base and another episode relating to height.

To keep track of all my episode notes, I continued working within the existing spreadsheet. For each episode, I made notes including (a) the start time of the episode, (b) the evoked concept image I observed, (c) if that episode seemed to include an example of a connection (e.g., between two concept images or between a concept image and triangle type), (d) if the observed statement or action could be judged as mathematically correct or incorrect, and

(e) any other details that stood out to me at the time (e.g., an interviewee's gesture, or a certain phrase that they used).

As I analyzed interviews during this phase, I had opportunities to revise my a priori concept image descriptions or the new ones, which I had developed during Phases 1 and 3. Additionally, I found more new concept images during this phase of analysis. Next, I created copies of the spreadsheet data so that I could sort it based on different characteristics. For example, in one copy I sorted the data by concept image and in another copy by connections. By sorting the pages, I was able to analyze the data more deeply than I had in previous phases.

Phase 5: Interview Analysis for Factors: Orientation, Gravity, and Triangle Type

Using the same 315 episodes, which I had identified earlier, I extended the analysis of interviews to focus on new details. This time, I focused on any observable relationships among concept images, triangle orientation, gravity, and triangle type. Using additional columns in my data spreadsheet, I made a record of such relationships. For example, I typed, "B—T" to indicate an observed relationship that an interviewee made between triangle base and triangle type. I also wrote brief descriptions of the observed relationships.

Phase 6: Second Pass Through Surveys: Identifying Additional Concept Images

Up to this point in my analysis, I had spent a good deal of time analyzing the interviews and deepening my understanding of participants' concept images and the connections that seemed evident between triangle type and orientation. Using my new insights (i.e., newly observed concept images and area strategies), I re-coded the survey data again using all available concept image codes. I identified as many evoked concept images as I could within each participant's survey—it was during this phase that I identified a total of 1635 evocations of concept images where each evocation was associated with a survey question of one participant.

Phase 7: Improving Concept Image Descriptions

During this phase of analysis, I met with an experienced mathematics education researcher who had expertise in geometry and measurement-related topics as well as coding and analyzing qualitative data to discuss my concept image descriptions and improve those descriptions. Before our meeting, I provided him with my existing descriptions and an example of each concept image from the surveys and interview episodes.

Then, we met on two occasions to discuss the intended meaning of each concept image code and potential ways to improve the descriptions of each concept image. We also considered how the newly phrased codes would or would not apply to the various examples, which I had shared with the researcher. Specifically, I described the important characteristics of the a priori concept images and of the new concept images. Then the researcher and I discussed ways to improve my existing descriptions to attend to those characteristics. For example, I described the three important characteristics of altitude in the H-ALTITUDE concept image and the researcher helped me to revise my description of that concept image. As a second example, I had originally described one concept image, B-BOTTOM as:

> The student describes base as the 'bottom side' and chooses a side which is closest to the bottom edge of the paper (i.e., the edge of the paper that is closest to the student). Student may also describe base as the side where the triangle "sits."

The researcher indicated that he wanted to classify one of the examples as an evocation of B-BOTTOM but felt restricted by the way which I had described the concept image. Because the participant did not specifically use the phrase "bottom side" in their description, felt that he needed to claim "not enough information" for that example's base code. After hearing that he felt restricted, I clarified that based on existing research (e.g., Herbel-Eisenmann & Otten, 2011;

Vinner & Hershkowitz, 1980), I intended the description to include a variety of phrases like "bottom side" or that a triangle "sits" or could include the selection of a side closest to the student. After discussing my intended description of B-BOTTOM, the researcher suggested that I broaden the description to allow for more phrases than just "bottom side" and to change the word *and* to *or*. Therefore, I broadened the description of B-BOTTOM as follows:

> Student *may* describe base as the "bottom side" *or* choose a side which is closest to the bottom edge of the paper (i.e., the edge of the paper that is closest to the student). Student may also describe base as the side where the triangle "sits."

In this way, I was able to clarify why I believed certain examples should be coded with one concept image or another. Through this process, we discussed all code descriptions, and we were able to resolve our disagreements as well as improve the concept image codes and their descriptions.

Phase 8: Re-Analysis of Interviews for Concept Images

In this phase of analysis, I re-coded every interview episode using my updated concept image descriptions and area strategies. Then I compared those new codes with the original ones I had identified. When I refined the concept image descriptions for base and height, some of the classifications became broader. This meant that, in some instances, multiple codes were appropriate based on the information from the interview episode data.

Phase 9: Final Analysis of Interviews and Surveys for Relationships

Finally, I re-examined the raw data in the surveys and interviews. I searched for evidence of participants having demonstrated any relationships among base, height, and area concept images. I created codes to reflect different types of base-height relationships and relationships to area. For example, I found evidence that some participants described height as dependent on the base, so I coded that with “height depends on base.” This process involved my searching through the surveys and interviews twice more. I identified several examples of participants having demonstrated relationships among base, height, and area concept images. Within those examples, I noted the concept images that I had identified during previous phases of analysis.

Summary of Data Analysis

Upon completing my nine phases of data analysis, I was able to produce partial or complete answers to each of my research questions. In summary, I began with a set of a priori concept images for base, height, and area. Then, I verified existing concept images, refined existing descriptions of concept images, and generated new ones. I was also able to find examples within my interviews of relationships among the concept images, triangle type, and triangle orientation.

Chapter Summary

My study was qualitative in nature. I surveyed 95 students from 13 geometry classes in a public high school in the Midwest. Then, I interviewed seven of those students with a semi-structured task-based interview. Upon completion of a nine-phase analysis process, I determined whether evoked concept images matched a priori concept images and identified new concept images. I was also able to identify many examples of relationships among concept images, triangle orientation, and triangle type. In the following chapter, I discuss the results of my analyses.

CHAPTER IV: RESULTS

In this section, I present my results, organized by concept images for triangle base, height, and area, by describing each concept image in detail. Then I discuss relationships among those concept images. Lastly, I discuss the influence of triangle orientation, gravity, and triangle type, on those concept images. The choice to organize my result in this way was driven by my research questions. Recall that they were:

1. What are high school geometry students' concept images of triangle base, height, and area?
2. How are high school geometry students' concept images of triangle base, height, and area related to one another?
3. How are triangle orientation, gravity, and triangle type related to high school geometry students' concept images of triangle base, height, and area?

Participants' Concept Images of Base, Height, and Area

In this section, I discuss results related to Research Question 1. First, I share a broad comparison between survey questions and interview tasks. Then, I share all the concept images that I found within this study, and I describe the evidence for my conclusions about students' concept images of base, height, and area.

Comparison Between Survey Questions and Interview Tasks

One way to compare survey questions and interview tasks is by considering how many concept image evocations are associated with each of them. The only group of students who engaged with both were the seven interviewees. Therefore, in Figure 19 below, I show how many total evocations the group of seven interviewees produced per survey question and interview task.

Figure 19

Interviewees' Concept Image Evocations Across Survey and Interview Items

Survey Question	Base	Height	Area
1a	0	0	6
1b	0	0	5
1c	0	0	5
2a	3	0	0
2b	2	0	0
2c	1	0	0
3a	0	2	0
3b	0	0	0
3c	0	3	0
4	7	6	0
5	6	6	0
6	9	7	0
7	6	6	0
8	6	4	0
9	5	3	7
10a	5	7	2
10b	1	4	3

Interview Task	Base	Height	Area
1.1a	0	0	3
1.1b	0	0	0
1.1c	0	0	0
1.2a	5	0	0
1.2b	0	0	0
1.2c	1	0	0
1.3a	0	3	0
1.3b	0	0	0
1.3c	0	0	0
1.4	4	6	0
1.5	4	2	0
1.6	6	5	0
1.7	2	3	0
1.8	1	0	0
1.9	4	4	8
1.10a	3	4	5
1.10b	1	3	5

Interview Task	Base	Height	Area
2.1a	6	7	0
2.1b	8	8	0
2.1c	7	8	0
2.2a	6	6	0
2.2b	6	6	0
2.2c	5	6	0
2.3a	5	9	0
2.3b	6	9	0
2.3c	6	6	0
3a	1	8	0
3b	1	6	0
3c	3	7	0
4	7	12	12
5	8	5	3

≈ Indicates Similar Questions or Tasks

Note. Different values, ranging from 0 to 12, are shaded so that darker shading indicates a greater number of evocations.

In Figure 19, there are two important details to discuss. First, symbols indicate similar items between the survey questions and interview tasks. To clarify, by *similar* I mean identical questions with different modes of communication, or as an indication of questions or tasks with identical goals. Therefore, questions and tasks that I designated as similar may have elicited different concept images or may have provided varying opportunities for evidence of extra influences on concept images (e.g., gravity).

For example, Interview Task 1.4 refers to the portion of the interview during Task 1 in which the interviewee and I were discussing their written response to Survey Question 4. The reason I consider these to be similar, as opposed to identical, is that interviewees were not answering the survey question again. Rather, they discussed many of their responses to the survey. I also consider Interview Tasks 2.1a–2.3c to be similar to Interview Tasks 1.4–1.8 and, therefore, to Survey Questions 4–8. In all three sets of items, the students were asked to sketch bases and heights for a variety of triangles, or to discuss their identification of bases and heights in previous tasks.

Second, there are patterns visible in the shading in Figure 12 that indicate the frequency of concept image evocations for base, height, and area. Notice that the frequency of identifiable evocations for Survey Questions 1–10 is consistent with the frequency of evocations for Interview Tasks 1.1–1.10b. However, there is a relatively greater number of concept image evocations in similar Interview Tasks 2 and 4 as compared to either Survey Questions 1–10 or Interview Tasks 1.1–1.10b. The greater number of identifiable evocations may be explained by the nature of the semi-structured interview process. I was able to ask interviewees clarifying questions while they were working and was also able to backtrack to previous questions during the interview to ask additional questions about some of the tasks. It seems that participants provided some details in their surveys, but interviewees offered a more details in their interviews.

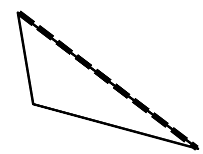
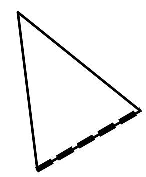
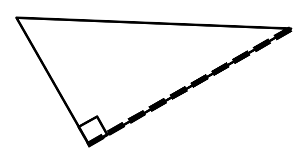
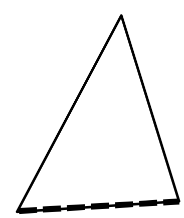
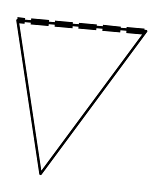
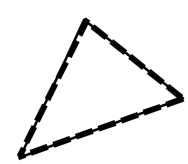
In the next section, I return to the larger data set including all survey and interview data. I discuss all concept images for base, height, and area that I identified in participants' responses to the survey questions and interview tasks. These concept images include most of the a priori concept images and several new ones.

Concept Images of Base

Across all surveys and interviews, I identified 10 distinct concept images of triangle base—three of which were among the a priori concept images I identified from previous research. I begin this section by sharing a brief description of all concept images, whcih I separated into three broad categories based on commonalities among the concept images (see Table 2).

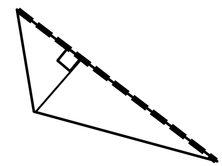
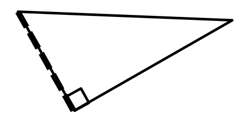
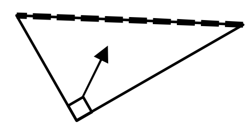
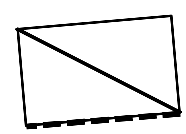
Table 2

Base Concept Images Present in the Survey and Interviews

Concept Image (Code)	Description	Example (base dashed)
	Base by Length	
Base as Longest Side (B-LONG)	Either in writing, or spoken word, the student describes base as being the longest side. Or, without such a description, student is consistent in choosing the longest side of the triangle as the base.	
Base as Different Length (B-DIFFERENT)	Students with this code select a certain side as the base because that side appeared to be a different length than the other two (i.e., the base of an approximately isosceles triangle).	
Base as Longer Leg of Right Triangle (B-LONG-LEG)	The student recognizes a right triangle and chooses the longer leg as the base. This code was only given when the student specifically stated choosing that leg because it was the longer one.	
	Base by Orientation	
Base as Bottom Side [a] (B-BOTTOM)	Student may describe base as the "bottom side" or chooses a side which is closest to the bottom edge of the paper (i.e., the edge of the paper that is closest to the student). Student may also describe base as the side where the triangle "sits."	
Base as Horizontal (B-HORIZONTAL)	When no side appears to be the bottom, student turns the triangle so that a side becomes the "bottom." The side which they select is the one which was closest to horizontal in the original orientation. Student may describe this side as the "flattest."	
Base as Any Side [a] (B-ANY)	Student communicates either by speaking, or writing, or by multiple labels that the base can be any side.	

(Table Continues)

(Table Continued)

Concept Image (Code)	Description	Example (base dashed)
	Base as Connection to Other Attributes	
Base as Perpendicular to Height[a] (B-PERP-TO-HEIGHT)	In either order, the student identifies the height, and they view the base as being perpendicular to that height. In some cases, the student clearly identifies the height first, and the base second. Student attends to the perpendicularity between height and base.	
Base as Leg of a Right Triangle (B-LEG)	Student chooses the base because it is the leg of a right triangle and not for some other reason.	
Base as Opposite an Angle (B-OPPOSITE-ANGLE)	Student describes base as being opposite a particular angle (e.g., opposite the right angle).	
Base as Side of a Rectangle (B-SIDE-RECT)	Student considers an approximate rectangle formed by a triangle and a congruent-rotated image and considers the side of that rectangle to be both a base of the rectangle and base of the triangle.	

[a] Indicates an a priori concept image.

In Table 3 I report frequencies associated with each of the observed base concept images from the survey. Table 3 includes the total number of participants who evoked each concept image at least once and information regarding the frequency of those evocations. The purpose of Table 3 is not to suggest generalizations of the broader population of geometry students but rather to give the reader a sense of how common or rare each concept image was among my participants using my instruments.

Table 3

Frequency of Base Concept Image Evocations by Survey Participants

Concept Image	No. of Participants (n = 95)	No. of Participants		
		1–3 evocations	4–6 evocations	7–9 evocations
B-LONG	26	19	5	2
B-DIFFERENT	0	0	0	0
B-LONG-LEG	6	6	0	0
B-BOTTOM [a]	89	27	35	27
B-HORIZONTAL	27	17	10	0
B-ANY [a]	8	8	0	0
B-PERP-TO-HEIGHT [a]	23	22	1	0
B-LEG	24	24	0	0
B-OPPOSITE-ANGLE	1	1	0	0
B-SIDE-RECT	0	0	0	0

[a] Indicates an a priori concept image.

Table 3 provides a broad perspective of evoked concept images within participants' surveys. Participants evoked a total of eight unique base concept images within their surveys. These included all three concept images from the a priori set and five of the seven new concept images identified through this present study. I did not find evidence of the concept images B-DIFFERENT and B-SIDE-RECT within participants' surveys (though they were evident in task-based interviews). Notice that most participants evoked the B-BOTTOM concept image within their surveys with over 90% of the participants evoking this concept image at least once. Also, B-LONG-LEG, B-ANY, and B-OPPOSITE-ANGLE were evoked by the fewest participants, with each concept image being evoked by fewer than 10% of the survey participants.

Table 4 includes information on how many interviewees evoked each base concept image and the frequency of those evocations. Like the previous table, this table provides a broad perspective of evoked concept images among interviewees' episodes. I found evidence of 10 different base concept images within the interview episodes. These included all eight of the concept images found within the surveys and two additional concept images, which were not found in the surveys—namely B-DIFFERENT and B-SIDE-RECT. Notice that B-LONG, B-BOTTOM, B-HORIZONTAL, and B-LEG were evoked at least once during their interview by a large proportion of interviewees (i.e., six or seven). In contrast, only one interviewee evoked each of B-LONG-LEG, B-ANY, B-PERP-TO-HEIGHT, and B-SIDE-RECT within their interviews.

Table 4

Frequency of Base Concept Image Evocations by Interview Participants

Concept Image	No. of Interviewees (n = 7)	No. of Interviewees		
		1–3 episodes	4–6 episodes	7–9 episodes
B-LONG	6	3	2	1
B-DIFFERENT	3	3	0	0
B-LONG-LEG	1	1	0	0
B-BOTTOM [a]	7	4	2	1
B-HORIZONTAL	6	3	2	1
B-ANY [a]	1	1	0	0
B-PERP-TO-HEIGHT [a]	1	0	0	1
B-LEG	6	4	2	0
B-OPPOSITE-ANGLE	2	2	0	0
B-SIDE-RECT	1	1	0	0

[a] Indicates an a priori concept image.

Notice in Table 4 that two of the a priori concept images—B-ANY and B-PERP-TO-HEIGHT—were evoked somewhat infrequently (i.e., by one interviewee) and the third a priori concept image—B-BOTTOM—was evoked by all interviewees. Similarly, B-BOTTOM was evoked by most participants, and B-ANY was evoked by a few participants, at least once during their surveys. In contrast, B-PERP-TO-HEIGHT was evoked by about 25% of participants at least once during their surveys.

In the rest of this section, I discuss each base concept image that I found from my analysis of participants' surveys and interview episodes. This discussion includes some examples from participants' surveys and interview episodes.

Base by Length

I grouped three concept images for the base as having some connection to length: (a) base as longest side (B-LONG), (b) base as a different side (B-DIFFERENT), and (c) base as the longest leg (B-LONG-LEG). These three concept images involved some type of emphasis on length.

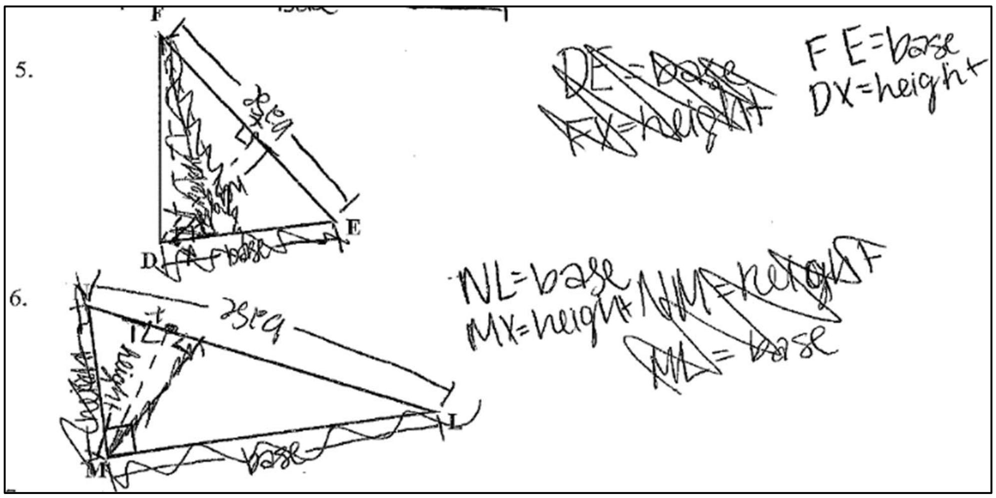
Base as Longest Side. I found evidence of the B-LONG concept image within participants' surveys and among the interviewees. For example, in her survey responses, R17 consistently chose the longest side. Notice that in her response to Survey Questions 5 and 6 (see Figure 20), she originally wrote "base" along the side closest to the bottom of the sheet of paper on which the figure was printed. Then, she crossed off several labels, and her final selection was the longest side as the base. Also, in Survey Question 8, R17 wrote "Look for the longest side and create a perpendicular line to the opposite [vertex[10]]" as an explanation for how she

[10] Non-italicized words in brackets are used to clarify meaning.

determined base and height in Survey Question 6. Notice also that R17 seemed to have rotated the paper to write the word "base" for each triangle.

Figure 20

R17's Questions 5 and 6: B-LONG

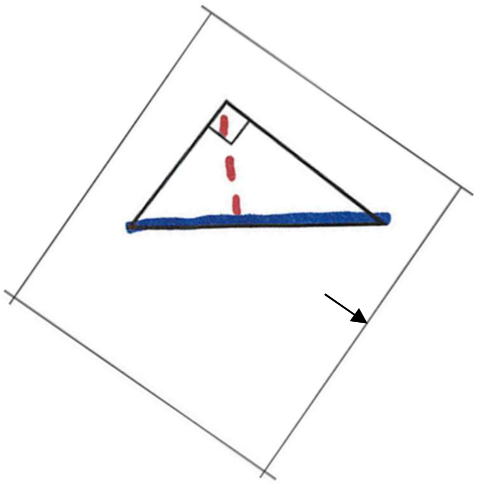


Note. Participants' work from surveys will be presented in the portrait orientation of the page on which the survey was printed and handed to the students (i.e., bottom, 8.5-inch edge of the paper closest to the person and approximately horizontal).

Among my interviewees, those who evoked the B-LONG concept image tended to rotate the paper so that the longest side of the triangle was closest horizontal. By turning the paper in this way, the longest side effectively became the "bottom" side. For example, in Interview Task 2.2a, Phoebe(CT11) rotated the paper so that one side was approximately horizontal from her perspective, traced over the base, and said, "Here, I would probably, like, flip it, so I could see the base right here" (see Figure 21).

Figure 21

Phoebe(CT11)'s Task 2.2a: B-LONG



Note. Any time I show an image of an interviewee's work, it will be from their perspective. Unless otherwise specified, each interview figure includes a small black arrow that points toward the top of the original sheet on which tasks were printed. This arrow is meant to help the reader imagine the interviewee's rotation of the figure.

Because of the open-ended nature of the interview process, I was able to ask Phoebe(CT11) to clarify why that side seemed like the base. She said, "I'm not sure. I always liked the longer side a bit, it kind of divides it equally." It was because of her additional explanation that I was able to classify her evoked concept image as B-LONG.

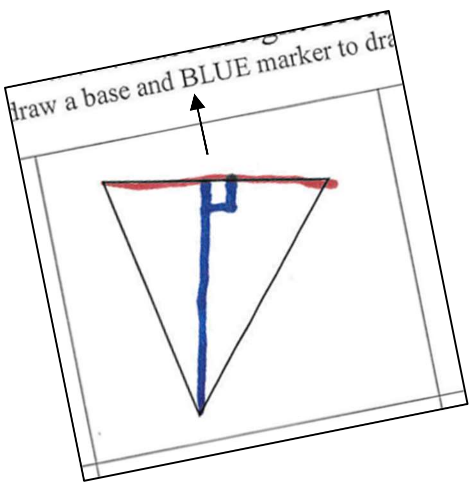
Base as Different Length. This concept image was not evident in my survey analyses. Therefore, it was the case that I created this code during my interview analysis process. I only identified the B-DIFFERENT concept image among the approximately isosceles triangles in Task 2. For example, on Interview Task 2.1b (see Figure 22), David(R26) said:

On this one … I would say the base would be this right here [*traced a side that was furthest from him*[11]]. The reason why is because it is … because the other sides are congruent … so making this one … or at least they appear to be congruent … this one to me would be the base because it's different.

David(R26) had rotated the paper slightly (i.e., approximately 20 degrees) such that his chosen base was approximately horizontal with respect to the desk (as opposed to a paper edge). During his interview, this was the only time that he described choosing a base because it appeared different than the other two sides.

Figure 22

David(R26)'s Task 2.1b: B-DIFFERENT



Base as Longer Leg of Right Triangle. In a right triangle, the two sides that intersect at the right angle are called legs. Thus, this concept image is only associated with right triangles. Sometimes participants would describe the base as the longest leg without directly identifying

[11] Italicized words in brackets used to describe meaningful non-verbal communication.

the triangle type as a right triangle. In other words, they would either directly state or imply that they thought the triangle was a right triangle.

For example, in Survey Question 8, H3 wrote, “The longest of the two legs of the triangle and what leg the triangle appears to be sitting on is the base and the other leg is the height.” H3 did not directly identify the triangle as right. Instead, she focused on the two legs. Another participant, R12, also did not directly describe the triangle as right but did use other terminology related to right triangles. She wrote, “The longest line that is part of the 90-degree angle is the base and the shorter one that isn’t the hypotenuse is the height.” In both examples, I had to infer that the participants’ evoked concept images were related to right triangles.

In contrast, my interview with David(R26) allowed me to have greater confidence in my classification of his evoked concept image. He evoked the B-LONG-LEG concept image during our discussion of interview Tasks 2.2a, 2.2b, and 2.2c. For these triangles, David(R26) rotated the paper so that the base, which he identified, was closest to his body and approximately horizontal. I asked him why he rotated the paper. He said, “I think it’s because there’s a hypotenuse and because it’s a right triangle and has one, I see that this is a shorter side, and therefore, the base would be the longest leg.” David(R26)’s description of why he rotated the paper and when choosing a base enabled me to determine that this thinking was more aligned with a concept image of B-LONG-LEG, as compared to B-BOTTOM, which I may have inferred without his description.

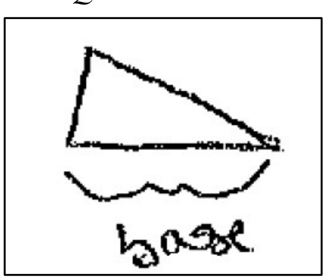
Base by Orientation

I grouped three concept images for base together as having a connection to triangle orientation. They were: (a) base as bottom side (B-BOTTOM), (b) base as horizontal (B-HORIZONTAL), and (c) base as any side (B-ANY). Recall that both B-BOTTOM and B-ANY were a priori concept images. In this section, I discuss the three concept images of base that I grouped as having a connection to orientation. To clarify, for B-BOTTOM, the participant needed a sense of "bottomness," which involves the perception of a side closest to their body. The B-HORIZONTAL concept image involved rotation (i.e., reorienting the triangle) so that the side became approximately horizontal, and B-ANY involved the explicit idea that the orientation of the triangle was not a critical component of determination of base.

Base as Bottom Side. The concept image, B-BOTTOM, was evoked by the most participants within their surveys and all seven interviewees. The following are representative examples of the B-BOTTOM concept image from the surveys. In H3's survey, she wrote that to find the base of a triangle, "measure the bottom of the triangle," and she drew a triangle and labeled the bottom side as base (see Figure 23). Other participants wrote similar descriptions in their surveys. For example, CT1 wrote, "Measure the bottom of the triangle" and "the bottom of an object/what the object stands on the width." R2 wrote, "The bottom of something" to describe a triangle base. Lastly, H9 wrote, "The base of the triangle is exactly how it sounds like it's the foundation of the triangle."

Figure 23

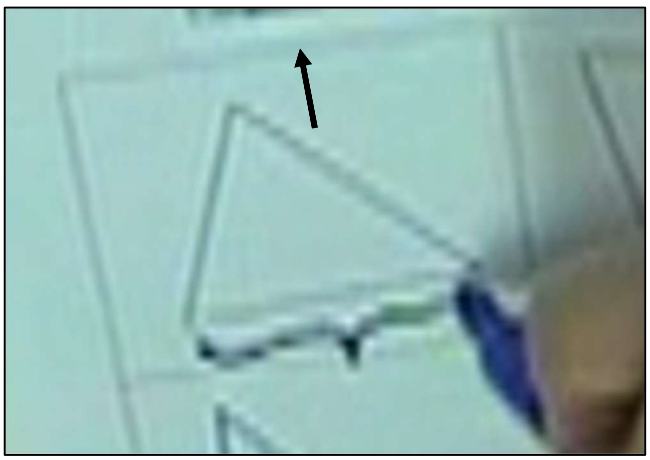
H3's Question 2c: B-BOTTOM



My interviews produced some additional insight into the B-BOTTOM concept image. In describing Interview Task 2.1a, Lisa(H16) said, “I’m [going to] call this one my base, because it’s where the—how the triangle is sitting.” In Lisa(H16)’s case, the description of sitting was my clue about her evoked concept image as B-BOTTOM (see Figure 24).

Figure 24

Lisa(H16)’s Task 2.1a: B-BOTTOM

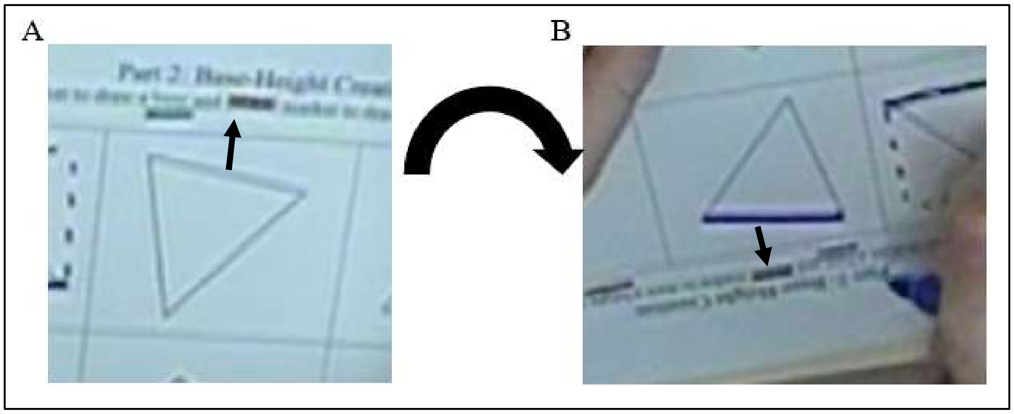


Base as Horizontal. The concept image B-HORIZONTAL is like the B-BOTTOM concept image. The first time I distinguished the two was during my analysis of interviews, and then later I found some evidence of B-HORIZONTAL within the surveys. Participants tended to rotate the page so that this side became horizontal. The rotation of the paper, with the goal of making one side become horizontal, became one of the key components of the B-HORIZONTAL concept image.

For example, when Alyssa(R23) identified the base in Interview Task 2.1b, she rotated the triangle nearly 180 degrees so that the side, which had appeared almost horizontal in the original orientation (see Figure 25A), became approximately horizontal (see Figure 25B). After she identified the base in Task 2.1b, I asked her how she was choosing her base. She said, "I'm mostly choosing it as, like, the flattest surface."

Figure 25

Alyssa(R23)'s Task 2.1b: B-HORIZONTAL

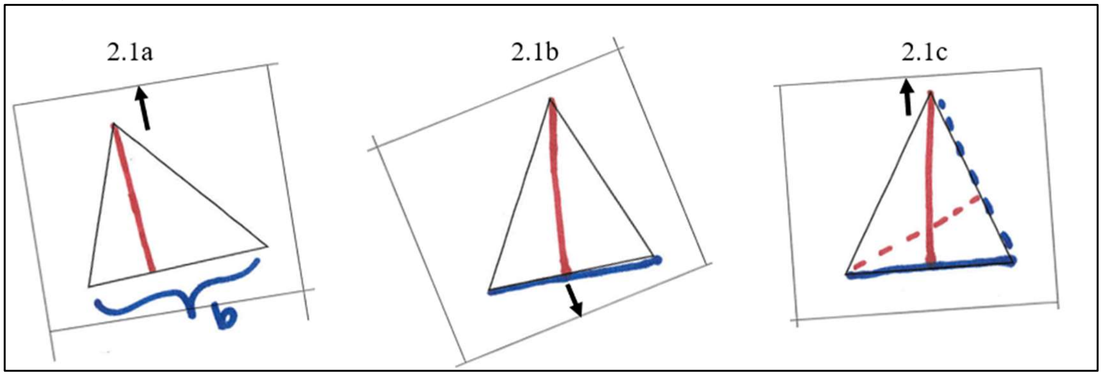


Note. The curved black arrow represents Alyssa(R23)'s rotation of the paper in the clockwise direction during the episode. Panel A: Paper orientation when Alyssa(R23) first looked at Task 2.1b. Panel B: The moment after Alyssa(R23) had rotated the paper and identified the base.

In this next example, I contrast B-BOTTOM and B-HORIZONTAL. While working on Interview Tasks 2.1a–2.1c, Lisa(H16) evoked the concept images B-BOTTOM, B-HORIZONTAL, and B-BOTTOM respectively (see Figure 26). For Task 2.1a Lisa(H16) said, "I'm going to call this one my base because it's where … [*slight pause*] … how the triangle is sitting." Her choice of base along with her description of base as *sitting* caused me to classify her concept image in that episode as B-BOTTOM.

Figure 26

Lisa(H16)'s Tasks 2.1a–c: B-HORIZONTAL and B-BOTTOM



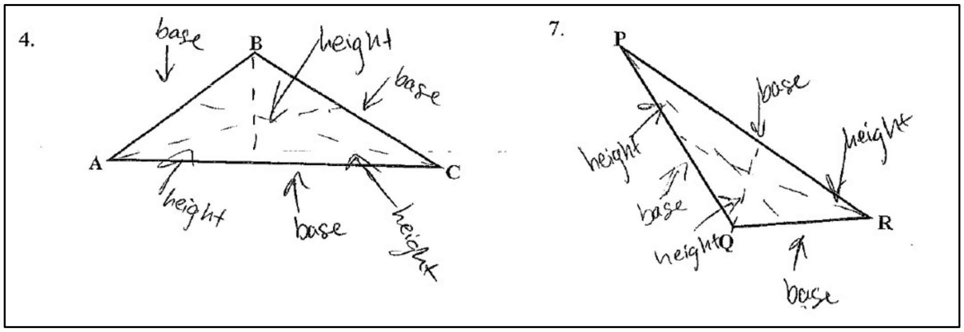
Note. Lisa(H16) drew the dashed red and blue segments in 2.1c later in the interview.

Then, when Lisa(H16) chose a base in Task 2.1b, she rotated the paper around such that her chosen base was approximately horizontal. She said, "I kinda saw that [side] first [*ran finger along base*] because it seemed to be a little more like turned, and so I turned it around." Her rotation of the paper caused me to classify her evoked concept image as B-HORIZONTAL for Task 2.1b. However, for Task 2.1c, she rotated the paper back nearly to its original orientation and said, "I'm going to draw how the triangle sits." When Lisa used the language of sitting, once again I coded her evoked concept image as B-BOTTOM.

Base as Any Side. Among all the concept images for triangle base, B-ANY seems most closely aligned with the concept definition. The clearest example, which I can offer, is one where a participant, R2, wrote, "any side of the triangle" in response to Survey Question 2a. In contrast, in another participant's survey, R30 evoked the B-ANY concept image without specifically writing that description of base "any side." She had labeled three sides as base options for Survey Questions 4 and 7 (see Figure 27). I classified those responses as evocations of B-ANY.

Figure 27

R30's Questions 4 and 7: B-ANY



The following example demonstrates a benefit of the interview process, because I did not see this kind of explanation within other participants' survey responses When discussing Interview Task 4, Lisa(H16) said, "Any side can be the base," and, "Each base will have its own height [*touched each of the triangle sides with marker cap*]." I classified that episode with the B-ANY concept image. It seemed that Lisa(H16) had connected the concept of height with base.

Base as Connection to Other Attributes

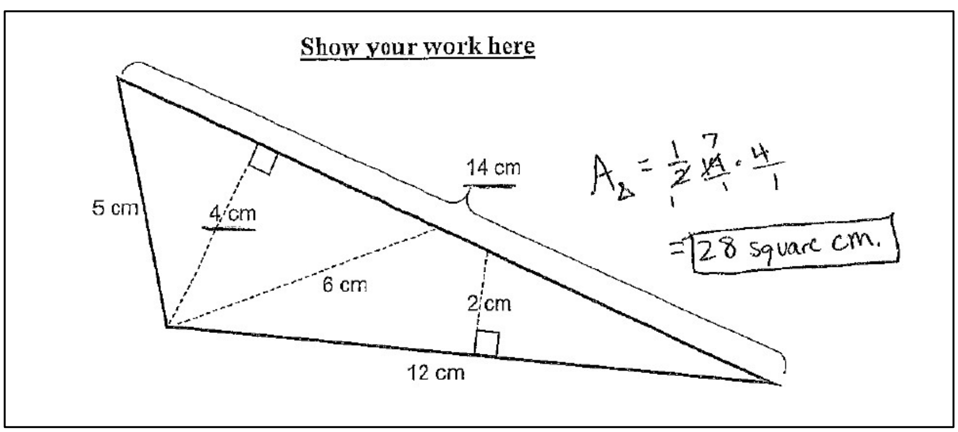
I grouped together four concept images of base as having a connection to other triangle attributes. They were: (a) base as perpendicular with height (B-PERP-TO-HEIGHT), (b) base as leg of a right triangle (B-LEG), (c) base as opposite an angle (B-OPPOSITE-ANGLE), and (d) base as side of a rectangle (B-SIDE-RECT). Recall that B-PERP-TO-HEIGHT was an a priori concept image. I grouped these concept images together as being connected to other attributes because each of them involved a relationship to some other component of the triangle (e.g., height, triangle type, an angle).

Generally, the participants with these four concept images (i.e., B-PERP-TO-HEIGHT, B-LEG, B-OPPOSITE-ANGLE, B-SIDE-RECT) seemed to make decisions about which base was appropriate after noticing nearby triangle attributes. For example, some participants decided about the existence of or location of the triangle height before they decided which side was the "correct" triangle base.

Base as Perpendicular to Height. The following two examples are representative of B-PERP-TO-HEIGHT—first from a participant's survey and second from an interview. H27 found the area in Survey Question 9 (see Figure 28), and explained her method by writing, "I used the base as 14 cm because by using that side as the base; I could use the 4 cm dotted line as the height since it is drawn from the vertex to the base as a perpendicular line." Although subtle, she revealed that her decisions about base were dependent upon the initial identification of a perpendicular line segment (i.e., the height). Without her written explanation, I would not have known that she was thinking of the base as being the segment that was perpendicular to the height.

Figure 28

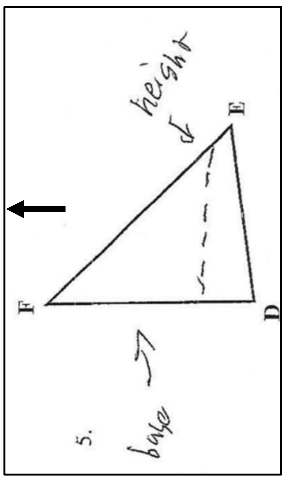
H27's Question 9: B-PERP-TO-HEIGHT



The interviews allowed me to gain additional insights into this concept image as compared to survey work alone. For example, during Interview Task 1 with Nate(R21), I asked him to tell me about the base. He replied, "Base is um … perpendicular to the height." Although this was a general description of base, this indicated to me that when he imagined a base, he was envisioning a height. Again, while discussing his survey, Nate(R21) described the base, which he drew in Survey Question 5, by saying, “this would be the base right here, because it would be perpendicular, and under the height,” (see Figure 29).

Figure 29

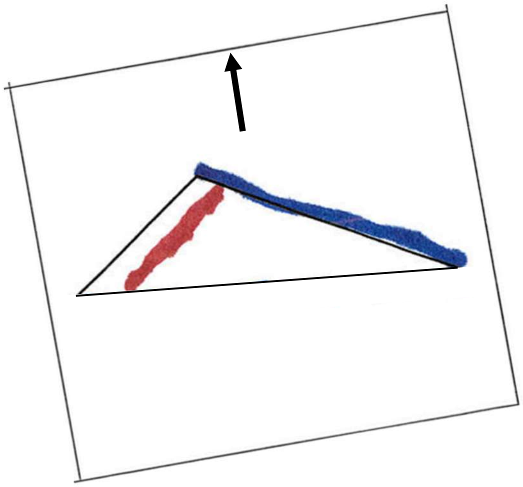
Nate(R21)’s Task 1.5: B-PERP-TO-HEIGHT



During Interview Task 2.3c, Nate(R21) identified the height first, and the base second (see Figure 30). After drawing the height, Nate(R21) drew the triangle base and said, “I found the base by going … it being perpendicular, so it would be perpendicular to right here, I believe.”

Figure 30

Nate(R21)'s Interview Task 2.3c: B-PERP-TO-HEIGHT



Based on those three brief examples from Nate(R21)'s interview, I infer that he understood the base to be a triangle side that was perpendicular to the height. The interview allowed me to witness the order in which he drew base and height—namely height first and base second. I would not have been able to observe this within the survey responses.

Base as Leg of a Right Triangle. The B-LEG and B-LONG-LEG concept images are similar. Recall that the B-LONG-LEG concept image involved a connection to length. In contrast, a participant who evoked the B-LEG concept image seemed to focus only on the attribute of a segment being a triangle leg.

For example, in Survey Question 8, R44 wrote, "So the base can be either ML or MN and the height would be the other one you didn't pick." Her description was referring to the right triangle in Survey Question 6 with legs labeled ML and MN. Although R44 did not specifically describe those sides as legs of a right triangle, I classified her response as an evocation of B-LEG because of the implied identification of base as a leg.

Interestingly, I found evidence of the B-LEG concept image in some participants' responses to non-right triangles. In response to Survey Question 9 (i.e., an obtuse triangle), R4 wrote, "I used 5 as the height and 12 as the base. I used the two numbers because they are legs of the triangle." In this case, I cannot say for sure if R4 considered the triangle to be a right triangle, but she did clearly identify the sides as legs.

I learned more about the B-LEG concept image from my analysis of interview episodes. For example, during Interview Task 1, Alyssa(R23) described the base from Survey Question 6 when she said, "This is just basically like a right triangle. That's the hypotenuse [*pointed to the hypotenuse*], so this is the base [*traced finger over one of the legs*]." From her response, I found that she first identified the hypotenuse and then classified a non-hypotenuse side as the base. Although she did not specifically describe triangle legs, I inferred that she was thinking of the sides as legs because her chosen base was not the hypotenuse.

Also, the interview discussions allowed me to improve my confidence in classifying evoked concept images. For example, during Task 2.2c, Alyssa(R23) rotated the triangle so that her chosen base was approximately horizontal and said:

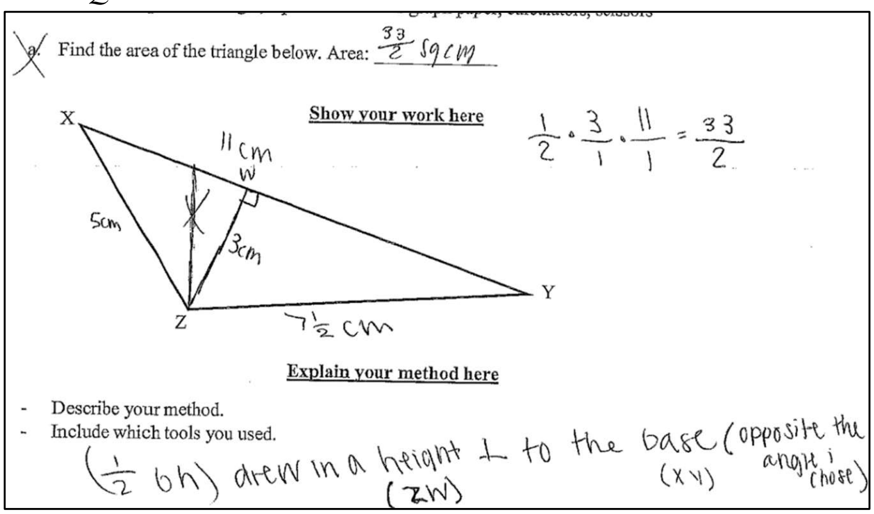
> Basically, all the right triangle ones are relatively the same to me … like these two [*traced marker cap over the legs*] are, like, both the legs, and this is the hypotenuse, but like you could choose either one of these [legs] as your base or height [*tapped each leg with marker cap*].

Because she explained that either leg can be the base or height, it seemed to me that the length of the leg was not the deciding factor for Alyssa(R23). Her statements led me to believe that she had identified the triangle as right first, and then determined the location of the base and height second.

Base as Opposite an Angle. I did not discover the B-OPPOSITE-ANGLE until my interview analysis process, and then later found one example during my final analysis of the surveys. First, I present the single example of B-OPPOSITE-ANGLE. In her survey, H29 had drawn an interior altitude in Survey Question 10a (see Figure 31), and had written that the base was, “opposite the angle I chose.” This indicated, to me, that her identification of base was influenced by her perception of it being opposite a particular angle of her choice.

Figure 31

H29’s Question 10a: B-OPPOSITE-ANGLE

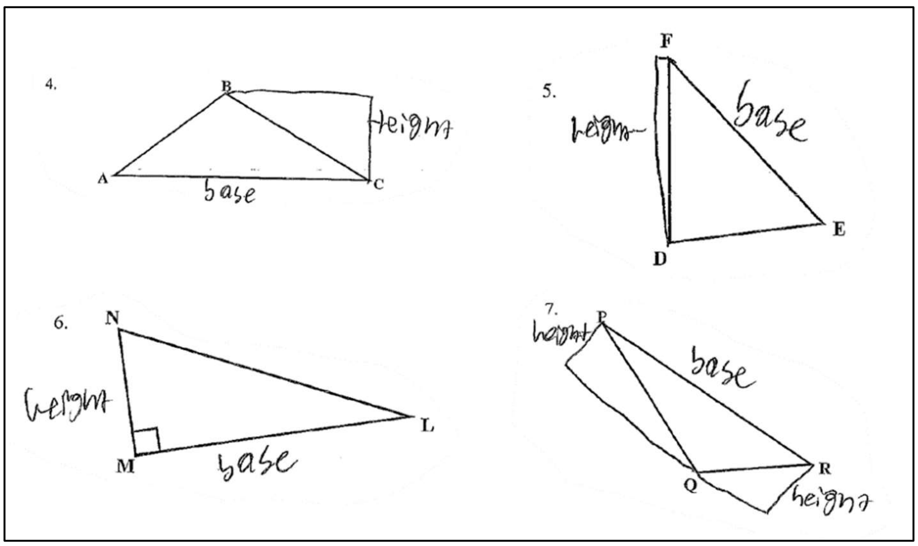


At times, the interview process allowed me to gain clarification about interviewees’ written work from their surveys. In the following example, I discuss how I initially coded some responses within an interviewee’s survey as B-LONG. Then, during our interview, I found that he had evoked the B-OPPOSITE-ANGLE concept image. This concept image was difficult to detect within my study, and I believe that I would not have identified it without the benefit of the interview discussion.

When assigning codes to David(R26)'s survey, I had originally coded Survey Questions 2a, 4, 5, and 7 with B-LONG. In Figure 32, I show David(R26)'s responses for Survey Questions 4–7. Notice that in Survey Question 6, David(R26) did not select the longest side as the base. Also, it seems that because of the orientation of his handwriting, he probably did not rotate the paper much when identifying base and height for the triangles.

Figure 32

David(R26)'s Questions 4, 5, and 7: B-OPPOSITE-ANGLE



However, during Interview Task 1 when David(R26) discussed Survey Questions 4 and 5, he explained two things. First, David(R26) stated that he pictured the base as the longest side, as I had suspected. Second, he described how he envisioned right angles within the triangles and how those right angles also informed his decision about triangle base. Referring to Survey Question 4, David(R26) said:

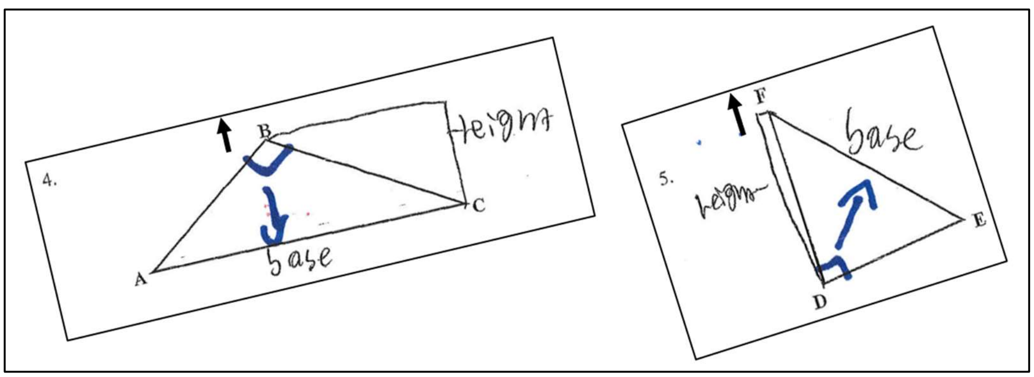
> Well, I think on this one [*slid finger along segment AC*], I chose it because it was, like, the longest one. I think for me, like, something I definitely see is, with the exception of

this one [*pointed to the right triangle in Survey Question 6*], is kinda like where the right angle would be [*pointed to the obtuse angle*]. If you know what I mean. That kinda looks like it could be … would be a right angle there.

I had asked him to draw his imagined right angle in the diagram (see Figure 33). Regarding Survey Question 4, David(R26) said, "Like that [*drew a right-angle symbol*] and then because it [the right angle] points out [*drew an arrow pointing to side AC*], it would be like the base." He drew a similar right angle with an arrow in Survey Question 5.

Figure 33

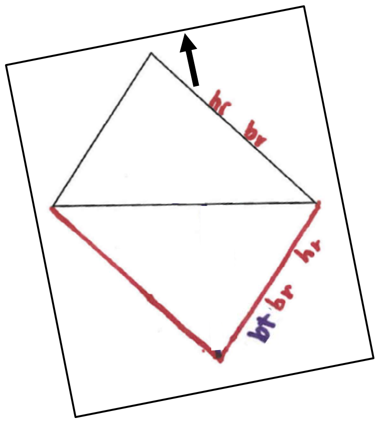
David(R26)'s Tasks 1.4 and 1.5: B-OPPOSITE-ANGLE



Base as Side of a Rectangle. When Phoebe(CT11) was working on Interview Task 5, she had identified one side of the triangle as a base. Then, she created a copy of the triangle and joined it together with the original to form a "rectangle." At this point, she described the base and height of the rectangle by writing "br" and "hr" along both sides of the rectangle, indicating that the rectangle base and height are interchangeable. Next, she evoked the B-SIDE-RECT concept image by also marking "bt" to indicate the rectangle side as the triangle base (see Figure 34).

Figure 34

Phoebe(CT11)'s Task 5: B-SIDE-RECT



Note. The red "br" and "hr" marks represent Phoebe(CT11)'s identification of rectangle base and height respectively. The blue "bt" represents triangle base.

Summary of Base Concept Images

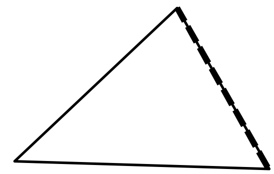
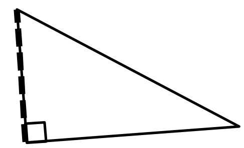
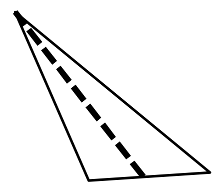
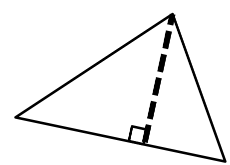
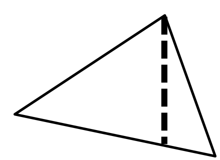
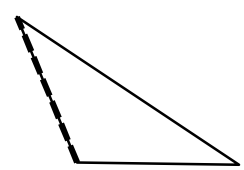
In this section, I have described 10 concept images of triangle base. Three of those were a priori and seven of the concept images were new. I also offered detailed examples to illustrate each concept image. I also shared frequency data from surveys and interviews. Some concept images were not present in surveys, but all were present in at least one interview. In the next section, I will repeat the same organizational structure but with a focus on concept images of triangle height.

Concept Images of Height

Based on my analysis of survey and interview data, I found 12 distinct concept images of triangle height—six of which were a priori and six of which were new. In this section, I start by briefly describing each of the 12 concept images grouped into three broad categories (see Table 5).

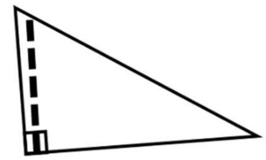
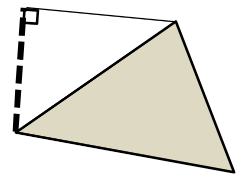
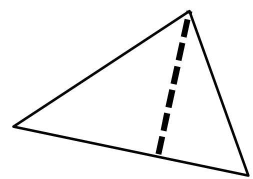
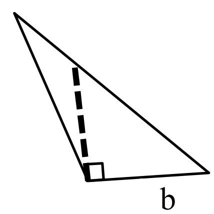
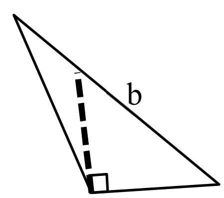
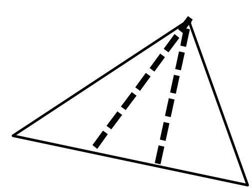
Table 5

Concept Images for Height Present in the Surveys and Interviews

Concept Image (Code)	Description	Example (height dashed)
	Height as Triangle Attribute	
Height as Side [a] (H-SIDE)	Height is the side of a triangle. The perpendicularity of the height with base may or may not play a role in the student's decision making.	
Height as Leg of a Right Triangle (H-LEG)	Height is the leg of a right triangle. The student seems to be influenced more by the identification of leg rather than the perpendicular connection of height and base.	
Height as Median [a] (H-MEDIAN)	Height is the median. Students may use the phrase such as "down the middle" or draw an approximate median that visually differs from the triangle altitude.	
Height as Altitude [a] (H-ALTITUDE)	This concept image best aligns with the concept definition. Student may describe the height as the altitude or describes the characteristics of the altitude. Alternately, without description (i.e., verbally or written), student draws all necessary components of the altitude.	
	Height as Visually Oriented	
Height as Vertical [a] (H-VERTICAL)	The student draws height approximately vertically rather than perpendicular to the base.	
Height as Vertical side (H-VERT-SIDE)	Height is the side that seems most vertical from the student's perspective. Although similar to H-SIDE, in H-VERT-SIDE the student indicates that their choice of side was influenced by the "verticalness" of the side.	

(Table Continues)

(Table Continued)

Concept Image (Code)	Description	Example (height dashed)
	Height as Visually Oriented	
Height as Interior [a] (H-INTERIOR)	Height is a segment that must exist in the interior of a triangle even if that means drawing a slightly shorter segment approximately parallel to a triangle side (i.e., in cases when the side could be considered the height).	
Height as Exterior (H-EXTERIOR)	Height is drawn outside the given triangle and as part of a right triangle where the hypotenuse of the right triangle is a side of the original triangle. This type of height may or may not be perpendicular with the chosen base.	
	Height as Connected to Other Attributes	
Height as Straight to base (H-STRAIGHT-TO-BASE)	Height is described as "straight to base." Alternately, the student drew height that was approximately perpendicular with the base but did not specifically mention nor draw the right angle.	
Height as Perpendicular to the base [a] (H-PERP-TO-BASE)	Student draws a height that is perpendicular with the base, but which may or may not coincide with the vertex opposite the base.	
Height as Forms a Right Angle (H-RIGHT)	Student identifies or draws a height with a right angle, but the right angle is not connected to the triangle base.	
Height as Connected to Vertex (H-VERTEX)	Student focuses only on a connection between height and vertex, thereby allowing for multiple height options.	

[a] Indicates an a priori concept image.

I identified one or more height concept images within each participant's survey. In Table 6, I report frequencies associated with each of the observed height concept images from the survey. These include the total number of participants who evoked each concept image and the frequency of those evocations.

Table 6

Frequency of Height Concept Image Evocations by Survey Participants

Concept Image	No. of Participants (n = 95)	Total Evocations Across the Survey Questions		
		1–3 evocations	4 – 6 evocations	7 or 8 evocations
H-SIDE [a]	32	24	4	4
H-LEG	29	29	0	0
H-MEDIAN [a]	6	6	0	0
H-ALTITUDE [a]	43	34	8	1
H-VERTICAL [a]	13	11	2	0
H-VERTICAL-SIDE	25	23	2	0
H-INTERIOR [a]	7	38	0	0
H-EXTERIOR	43	41	2	0
H-STRAIGHT-TO-BASE	38	6	1	0
H-PERP-TO-BASE [a]	43	36	6	1
H-RIGHT	3	3	0	0
H-VERTEX	0	0	0	0

[a] Indicates an a priori concept image.

The survey participants, collectively, evoked 11 unique height concept images within their surveys. Six of those were a priori and five were new. I did not find evidence of the H-VERTEX concept image within the surveys. The four most frequently evoked height concept images were H-ALTITUDE, H-EXTERIOR, H-STRAIGHT-TO-BASE, and H-PERP-TO-BASE with 40–45% of participants evoking each of these concept images at least once. The least commonly evoked height concept images were H-MEDIAN, H-INTERIOR, and H-RIGHT and were each evoked by fewer than 8% of the participants. Notice that, as compared to the corresponding table relating to evoked base concept images (i.e, Table 3), the height evocations have generally lower percentages and greater variability. Next, Table 7 includes a focus on frequency of each concept image among interview episodes. It also includes how many interviewees are represented among those episodes.

Table 7

Frequency of Height Concept Image Evocations by Interview Participants

Concept Image	No. of Interviewees (n = 7)	No. of Interviewees		
		1–4 episodes	5–8 episodes	9–12 episodes
H-SIDE [a]	2	2	0	0
H-LEG	4	4	0	0
H-MEDIAN [a]	2	1	1	0
H-ALTITUDE [a]	5	3	2	0
H-VERTICAL [a]	3	3	0	0
H-VERTICAL-SIDE	0	0	0	0
H-INTERIOR [a]	2	1	0	1
H-EXTERIOR	3	2	0	1
H-STRAIGHT-TO-BASE	4	2	1	1
H-PERP-TO-BASE [a]	5	2	0	3
H-RIGHT	0	0	0	0
H-VERTEX	1	1	0	0

[a] Indicates an a priori concept image.

The interviewees, collectively, evoked 10 unique height concept images. Six of those were a priori and four were new. The H-VERTEX concept image was evoked by one interviewee, but I did not find evidence of participants having evoked that concept image within their surveys. In contrast, I found that some participants evoked the concept images H-VERTICAL-SIDE and H-RIGHT within their surveys, but I did not find evidence of interviewees having evoked those two concept images. From Table 7, notice that five interviewees evoked the H-ALTITUDE and H-PERP-TO-BASE concept images at least once during their interviews.

Next, I discuss each height concept image. This discussion includes specific examples of each concept image from participants surveys and interview episodes.

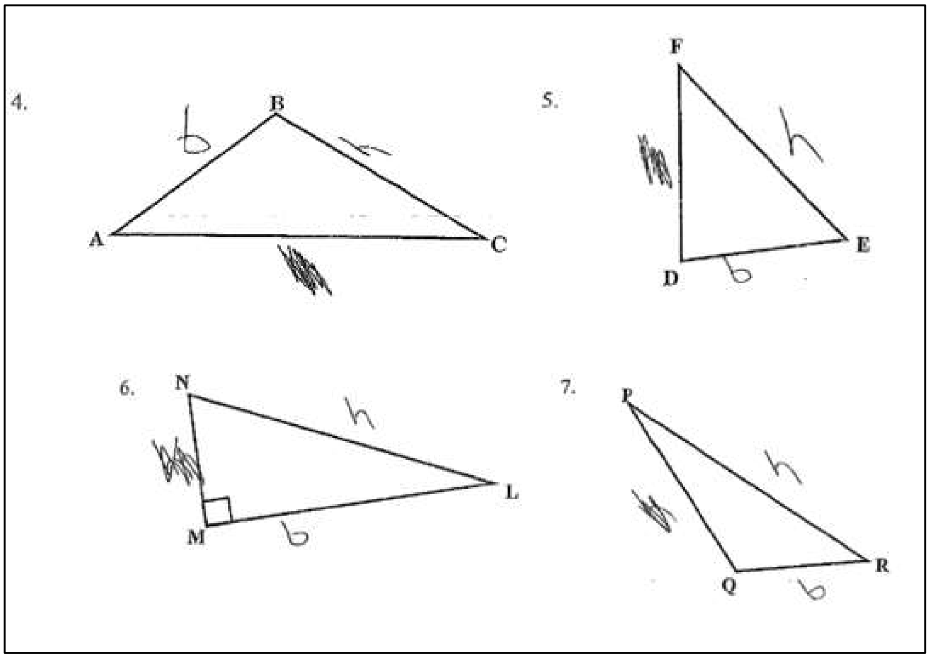
Height as Triangle Attribute

I grouped together four concept images that were associated with triangle attributes: (a) height as side (H-SIDE), (b) height as leg of a right triangle (H-LEG), height as median (H-MEDIAN), and (d) height as altitude (H-ALTITUDE). Essentially participants with these concept images viewed the height as those objects described, rather than relying on other visual perceptions such as “verticalness” or “straightness.”

Height as Side. In his survey, R18 consistently demonstrated the H-SIDE concept image. In Survey Questions 4–7, R18 labeled the base as one side, and the height as another (see Figure 35). His work is representative of the other participants’ surveys and interviewees’ work that contained evidence of the H-SIDE concept image.

Figure 35

R18's Questions 4–7: H-SIDE



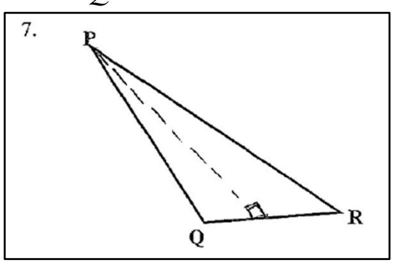
Height as Leg of Right Triangle. Although most participants who evoked this concept image used the term *leg*, some did not. Some participants used the term *hypotenuse* but did not specifically call the other sides *legs*. In such cases, I classified their concept image as H-LEG because they named a different part of the right triangle. For example, when discussing Survey Question 6 during Interview Task 1, Alyssa(R23) described the triangle as a right triangle. She then identified the hypotenuse and showed me that her base and height were the other two *sides*. Even though she did call the sides legs, I took this as implied because she used another term *hypotenuse*, which related only to right triangles. This example is representative of other participants' evocations of H-LEG.

Height as Median. In this study I found evidence of participants evoking the H-MEDIAN concept image for triangles with external altitudes, triangles with altitudes that coincided with a side, and for triangles with internal altitudes. I describe three evocations of H-MEDIAN for which the altitude should have been exterior, interior, and a side.

First, in her survey, H31 evoked the H-MEDIAN concept image based on her drawing an approximate median in three out of five of the questions, which she attempted. As shown in Figure 36, H31 identified the height for Survey Question 7. Notice how she drew the approximate median and included a right-angle symbol.

Figure 36

H31's Question 7: H-MEDIAN



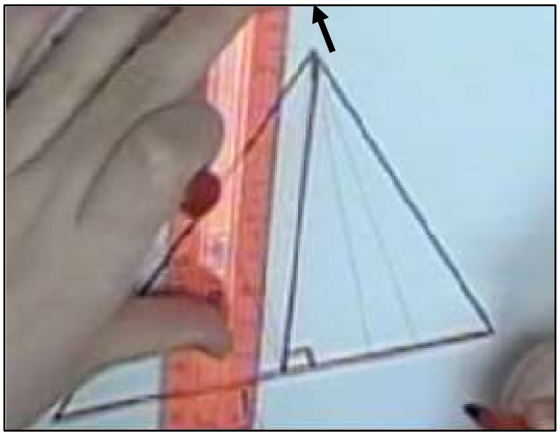
As my second example of H-MEDIAN, I classified Alyssa(R23)'s concept image as H-MEDIAN based on her work in Task 3a. She spoke about how she visualized the height by saying:

> So, the height would probably have to connect to one of, to this point [*pointed to vertex*] … but there's not really, like, a straight line [*slid marker cap along median*], which the height should be a straight line and should be perpendicular to this line [*pointed to base*].

Then, she traced the median of the triangle and drew in a right-angle symbol (see Figure 37).

Figure 37

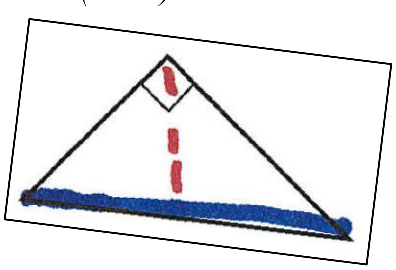
Alyssa(R23)'s Task 3a: H-MEDIAN



Lastly, I also found evidence of H-MEDIAN among some of Phoebe(CT11)'s interview tasks. For example, in Task 2.2a, she described the height by saying, "Divide that 90-degree angle." Then when drawing the height for Task 2.2b, she said, "Then divide it again," when drawing the height (see Figure 38). When she said this, she sketched a segment, which she called the height. Because her height seemed to bisect the base, which she had previously identified, I infer that she was drawing the triangle median. I classified Phoebe(CT11)'s concept image as H-MEDIAN for the episodes involving Tasks 2.2a–c.

Figure 38

Phoebe(CT11)'s Task 2.2b: H-MEDIAN



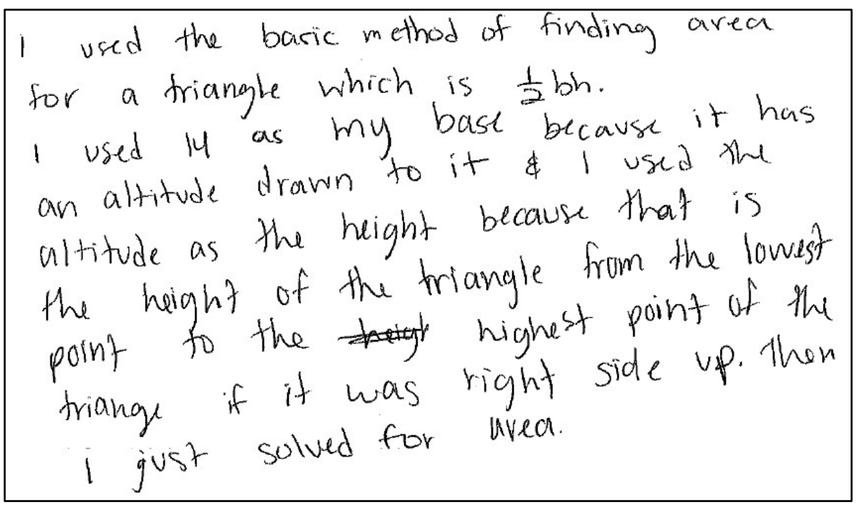
Height as Altitude. Although the terms altitude and apothem refer to different geometric objects, some interviewees used the word "apothem" while drawing an altitude. In such cases, I still classified that interviewee's evoked concept image as H-ALTITUDE.

In Survey Question 9 (see Figure 39), H5 referred to height as an altitude when explaining her method of finding triangle area as follows:

> I used the basic method of finding area for a triangle which is 1/2 *b h*. I used 14 as my base because it has an altitude drawn to it [and] I used the altitude as the height because that is the height of the triangle from the lowest point to the highest point of the triangle if it was right side up. Then I just solved for area.

Figure 39

H5's Question 9: H-ALTITUDE



I used the basic method of finding area
for a triangle which is $\frac{1}{2}$bh.
I used 14 as my base because it has
an altitude drawn to it & I used the
altitude as the height because that is
the height of the triangle from the lowest
point to the ~~heigh~~ highest point of the
triange if it was right side up. Then
I just solved for area.

As another example, H2 did not specifically use the word altitude but did describe height in Survey Question 9 as follows, "I knew the 4 cm length was the height because it started on the opposite angle to the base and formed a right angle [with the base]." Noticing that H2 described characteristics of an altitude, I classified this response as evidence of the H-ALTITUDE concept image.

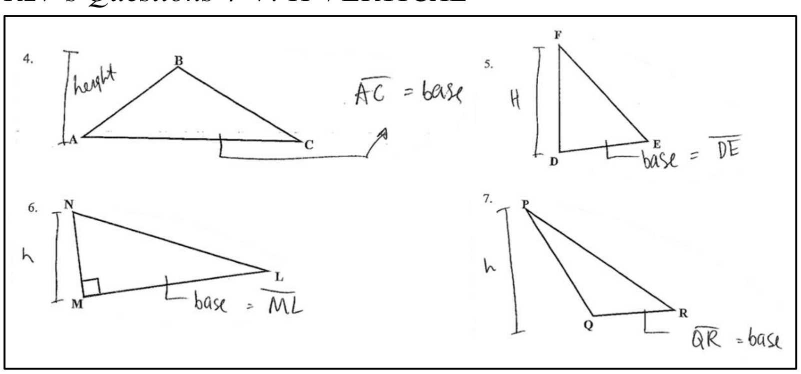
Height as Visually Oriented

I categorized four concept images of height that seemed to be visually oriented or located by the participants: (a) height as vertical (H-VERTICAL), (b) height as a vertical side (H-VERT-SIDE), (c) height as interior (H-INTERIOR), and (d) height as exterior (H-EXTERIOR). Recall that both H-VERTICAL and H-INTERIOR were a priori concept images. I considered these four concept images to be visually oriented because they were based on the perception of the participant. For example, the perception of "verticalness" of a side or an imagined interior altitude in an obtuse triangle, which should have an exterior altitude.

Height as Vertical. I had inferred this concept image based on existing research (e.g., Barrett et al., 2012; Blanco, 2001; Gürefe & Gültekin, 2016; Kadarisma et al., 2020). Therefore, I offer examples to extend existing research. I found evidence of H-VERTICAL within the survey of R27 (see Figure 40). R27 had drawn nearly vertical bracket symbols outside of each of the four triangles in Survey Questions 4–7.

Figure 40

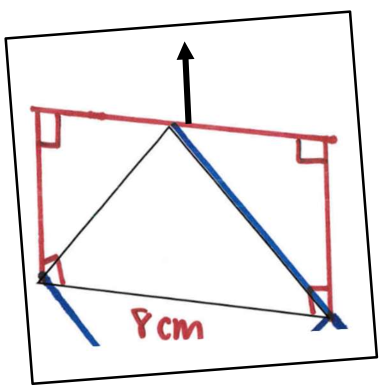
R27's Questions 4–7: H-VERTICAL



This second example from an interview offered me additional insights about the H-VERTICAL concept image. Because of the nature of the interview process, I was able to witness Alyssa(R23) during her process of drawing the height rather than just witnessing the final product, as was the case in surveys. When Alyssa(R23) began Interview Task 5, she described the base as being slanted, and then she grappled with the positioning of the ruler to draw the height. She oriented the ruler vertically, then approximately perpendicular with the base, then back vertically. After adjusting the ruler orientation at least eight times, she modified her strategy for drawing a height, which was approximately vertical from her perspective (see Figure 41).

Figure 41

Alyssa(R23)'s Task 5: H-VERTICAL

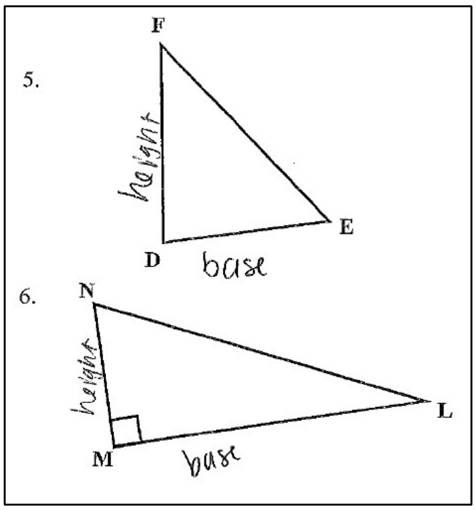


Note. Alyssa(R23) drew in the blue marks later in the interview. The nearly vertical red lines represent Alyssa(R23)'s height for Interview Task 5.

Height as Vertical Side. Although none of the interviewees evoked H-VERT-SIDE during their interviews, several students provided evidence of this concept image on their surveys. For example, CT2 evoked the H-VERT-SIDE concept image for Survey Questions 5 and 6 (see Figure 42). Also, earlier in her survey (i.e., Survey Question 3), she described the concept of triangle height as "the vertical line on a triangle" and sketched a sample height by drawing a right triangle and pointing to the leg, which was nearly vertical. Additionally, in Survey Question 8, she wrote, “The vertical side with the right angle is the height.” These examples are representative of other participants who evoked H-VERT-SIDE within their surveys.

Figure 42

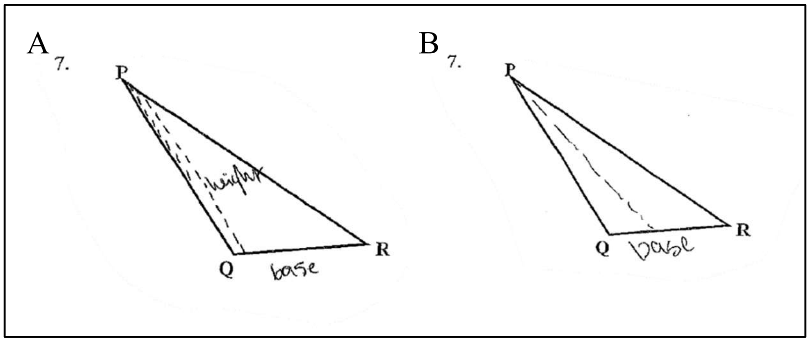
CT2’s Questions 5 and 6: H-VERT-SIDE



Height as Interior. Some participants who I had classified with the H-INTERIOR concept image had connected the heights to a vertex, and others had connected the height to a side other than the base. For example, both R39 and H17 demonstrated evidence of the H-INTERIOR concept image (see Figure 43). They both drew a segment, which was inside the triangle, without any emphasis on perpendicularity to the base. It is possible that H17 was evoking H-MEDIAN (see Figure 43B) but without any written indication that she intended the height to intersect the midpoint of the base, I felt more confident with a coding of H-INTERIOR.

Figure 43

R39's and H17's Question 7: H-INTERIOR

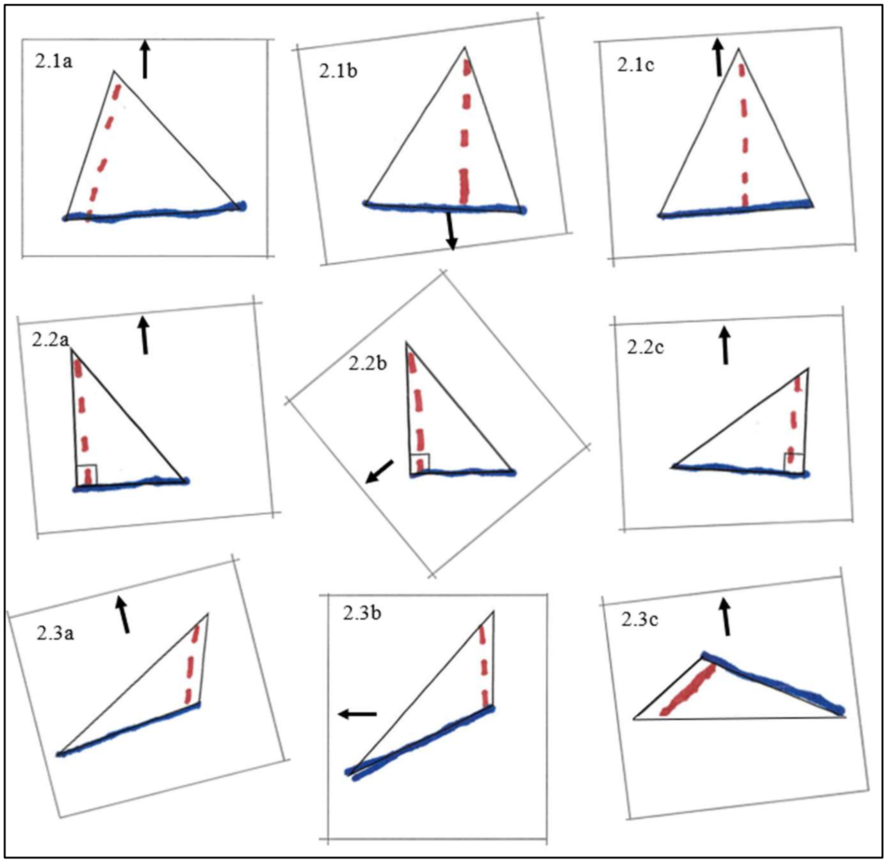


Panel A: Response from R39. Panel B: Response from H17.

Additionally, Nate(R21) offered a clear example of H-INTERIOR that extended my understanding of the concept image. When he drew the bases and heights for the triangles in Task 2, he exhibited the H-INTERIOR concept image in all nine triangles (see Figure 44). I asked Nate(R21) if his marked heights were intended to be the side or a segment that was inside of the triangle, and he told me that it was intended to be inside the triangle.

Figure 44

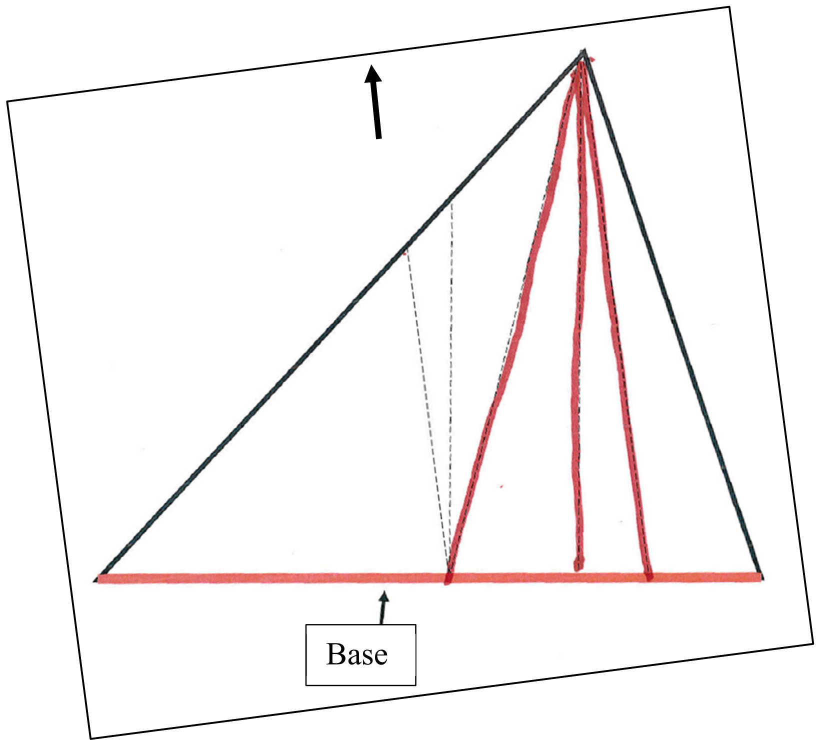
Nate(R21)'s Task 2: H-INTERIOR



I observed that for most of the triangles, Nate(R21) rotated them so that the height was nearly vertical from his perspective—the only exception being Task 2.3c. Later in the interview, Nate(R21) identified three possible heights in the larger acute triangle in Task 3a (see Figure 45). In this task, all his heights connected to the vertex opposite the base, and all were interior to the triangle. This was a slight variation compared to the nine triangles in Task 2 where he had just drawn in the heights disconnected from a vertex. This is perhaps because I had created segments in Task 3, which he had to choose from, as compared to Task 2 where I did not have segments identified as base.

Figure 45

Nate(R21)'s Task 3a: H-INTERIOR

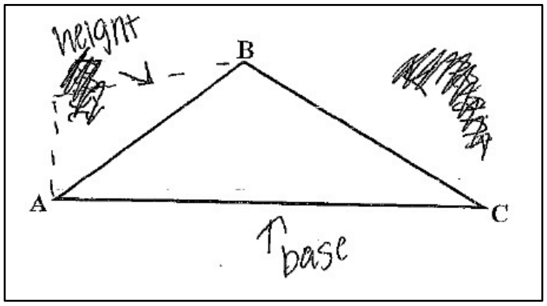


Height as Exterior. It was not until my interview analysis that was able to describe the H-EXTERIOR concept image. Therefore, I share some examples from an interviewee—Alyssa(R23)—which help explain this concept image. Each of the examples demonstrates additional subtleties related to the H-EXTERIOR concept image.

This first example demonstrates how H-EXTERIOR does not require that the two segments, drawn outside the triangle, to be perpendicular. Alyssa(R23) evoked the H-EXTERIOR concept image in both her survey and during her interview. For Alyssa(R23), the way to draw height was to create a triangle that was exterior to the given triangle and then use one of the sides of that triangle as the height. In Survey Question 4 she had considered one of the legs of the exterior triangle to be the height of the original triangle (see Figure 46). When I was originally analyzing her survey, I did not know if she intended the height to be represented by the nearly horizontal dashed segment, the nearly vertical dashed segment, or both segments together.

Figure 46

Alyssa(R23)'s Question 4: H-EXTERIOR



Therefore, during Interview Task 1, I asked her about Survey Question 4. Alyssa(R23) said:

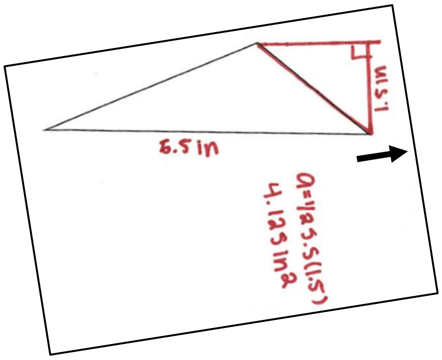
> Yeah, so the base in this one, I just put as the flat surface [*pointed finger to segment AC*] because all these [other two sides] are slanted, and the height I put from this to here [*traced finger along the two dashed segments starting at A and ending at B*] because it's how tall the triangle is from where my base is.

Because Alyssa(R23) traced her finger along the two dashed segments, it was still unclear to me if she was picturing the height to be represented by one or both segments.

As a second example, during Interview Task 4, when Alyssa(R23) measured one of the two sides of an exterior triangle-height, she labeled the height as the nearly vertical segment of the exterior right triangle but did not vocalize her perception of height. Therefore, the best I could say was that her writing potentially indicating that she interpreted one of the exterior segments as the height (see Figure 47). A few moments later, I confirmed that she was indeed thinking of the 1.5 cm labeled segment as the height and not the perpendicular segment.

Figure 47

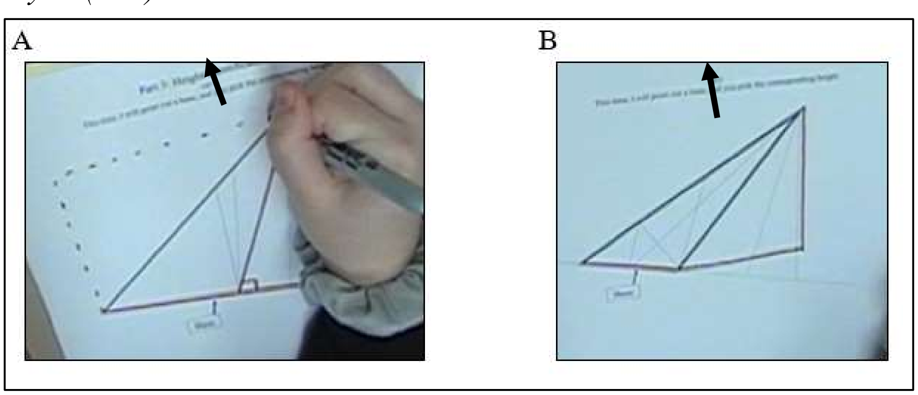
Alyssa(R23)'s Task 4: H-EXTERIOR



This final example demonstrates how the exterior height need not be perpendicular to the line containing the base. One of the most notable instances of this was her work on Interview Task 3a. After Alyssa(R23) had identified one of the segments as the height, I asked her where she would have drawn the height if I had not created the possible heights, or if she would have drawn it somewhere else. In response, she evoked the H-EXTERIOR concept image by drawing a right triangle such that one leg was nearly vertical (see Figure 48A). Alyssa(R23) evoked the H-EXTERIOR concept image again when drawing the height in Task 3c (see Figure 48B).

Figure 48

Alyssa(R23)'s Tasks 3a and 3c: H-EXTERIOR



Note. Panel A: Interview Task 3a. Panel B: Interview Task 3c.

Alyssa(R23) appeared to draw her exterior heights in a nearly vertical orientation from her perspective. Notice how Alyssa(R23)'s height in Interview Task 3c was not perpendicular to the line containing the base.

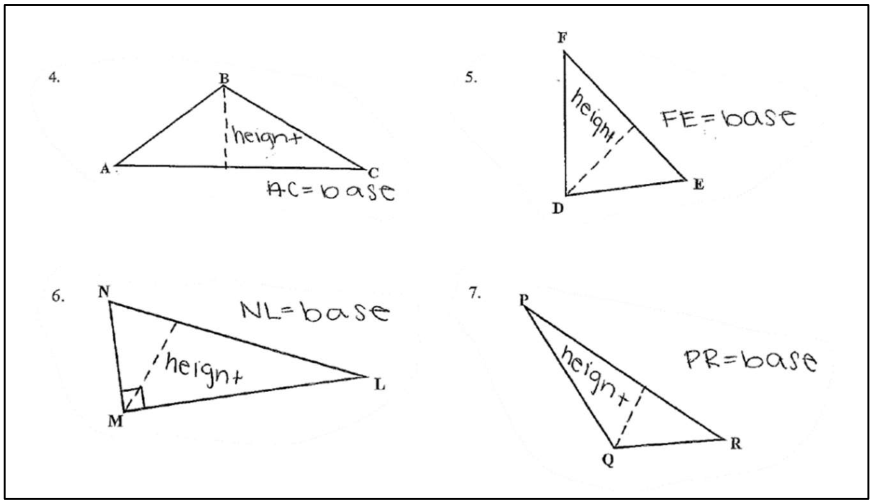
Height as Connected to Other Attributes

Some participants' concept image of height seemed to be influenced by other triangle attributes. I grouped four concept images into this category: (a) height as straight to base (H-STRAIGHT-TO-BASE), (b) height as perpendicular to the base (H-PERP-TO-BASE), (c) height as forms a right angle (H-RIGHT), and (d) height as connects to vertex (H-VERTEX). Recall that H-PERP-TO-BASE was an a priori concept image. These four concept images involved participants' focus on a triangle component (e.g., base, right angle, vertex opposite the base). Both H-RIGHT and H-VERTEX were extremely rare in this study. I only found evidence of H-RIGHT in three surveys, and only found evidence of H-VERTEX in a single interview episode.

Height as Straight to Base. For this concept image, I share one participant's survey responses. They are representative of other participants' evocations of H-STRAIGHT-TO-BASE. In R15's survey, she seemed to choose the longest side as the base, and then drew a height that was approximately perpendicular to the base. However, R15 never described the height as being perpendicular to the base. In Figure 49, I show her responses to Survey Questions 4–7.

Figure 49

R15's Questions 4–7: H-STRAIGHT-TO-BASE



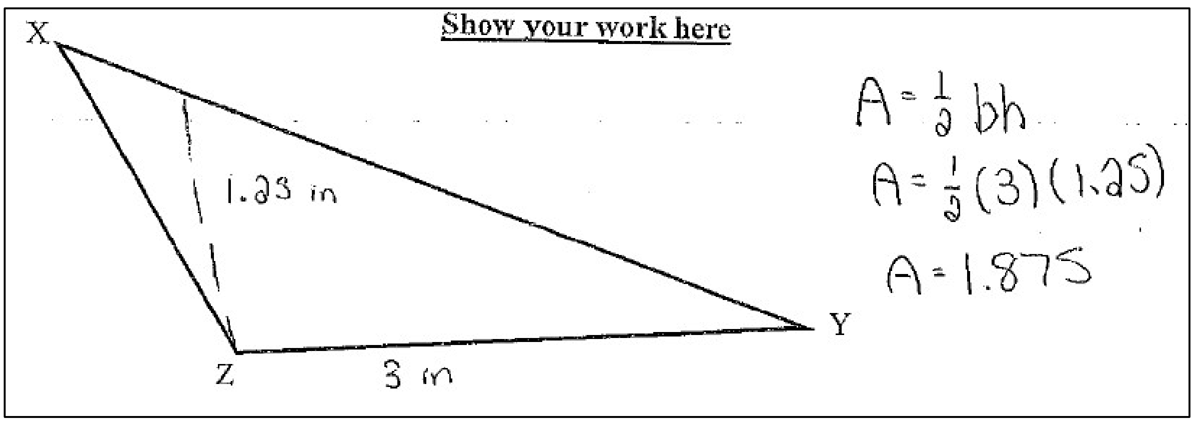
Height as Perpendicular to the Base. Most participants who evoked this concept image drew a segment that was perpendicular to the base and had an endpoint as the vertex opposite the base. Some participants evoked this concept image in an interesting way by drawing a segment that had an endpoint connected to the side opposite the base. Now, I share two evocations of H-PERP-TO-BASE from one participant's survey to illustrate both representations of the concept image.

In Survey Question 8, R7 wrote, "The height must create a 90-degree angle with the bottom." Then for Survey Question 9, she wrote, "I used 14 cm as my base because it seemed to be the base of the overall triangle and 4 cm as my height because it created a 90-degree angle with the base I chose." In both explanations, R7 emphasized the perpendicular connection between height and base but made no mention about the connection to the vertex opposite the base.

R7 also drew a height in Question 10 (see Figure 50), which was approximately perpendicular to her chosen base. However, her height terminated when it intersected the opposite side of the triangle rather than the vertex opposite the base. R7's survey responses are representative of other participants' evocations of H-PERP-TO-BASE in the sense that they emphasized a perpendicularity with the base but de-emphasized a connection to the vertex.

Figure 50

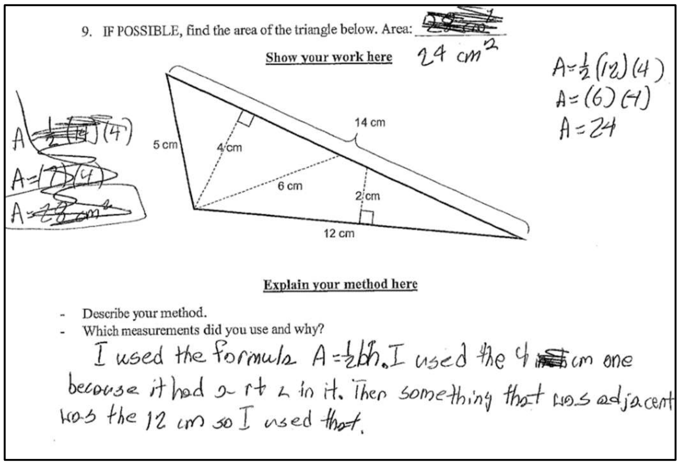
R7's Question 10: H-PERP-TO-BASE



Height as Forms a Right Angle. Participants who evoked this concept image seemed to place an emphasis on the existence of a right angle to identify the height. For example, in Survey Question 9 (see Figure 51), R20 wrote, "I used the formula A=1/2bh. I used the 4 cm one [the height] because it had a rt [angle] in it. Then something that was adjacent was the 12 cm, so I used that [as base]." R20 seemed to place an emphasis on the existence of a right angle to identify the height but did not place any emphasis on the height forming a right angle with the base or connecting the height to the vertex opposite the base.

Figure 51

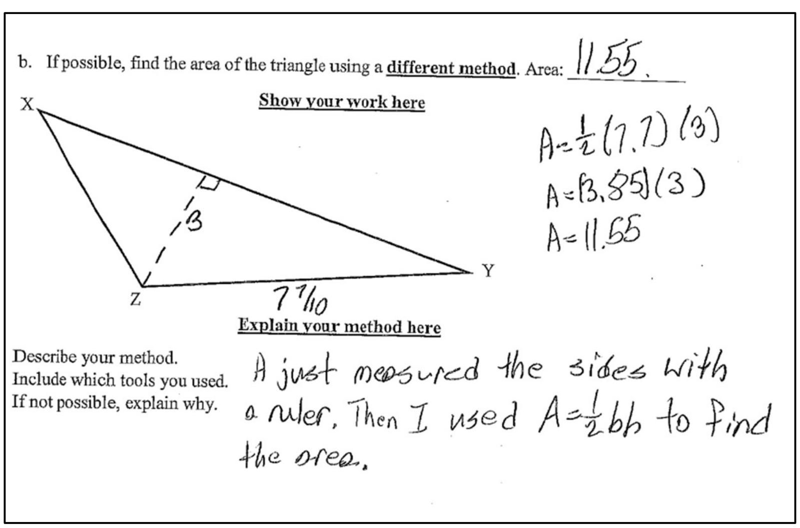
R20's Question 9: H-RIGHT



Also, in Survey Question 10b, R20 evoked the H-RIGHT concept image but with less written explanation this time (see Figure 52). I determined his concept image based on his written formula in conjunction with their diagram. He drew a segment and labeled it "3," such that the segment formed a right angle—but not with the base. These examples are representative of the other two participants' evocations of H-RIGHT.

Figure 52

H20's Question 10b: H-RIGHT



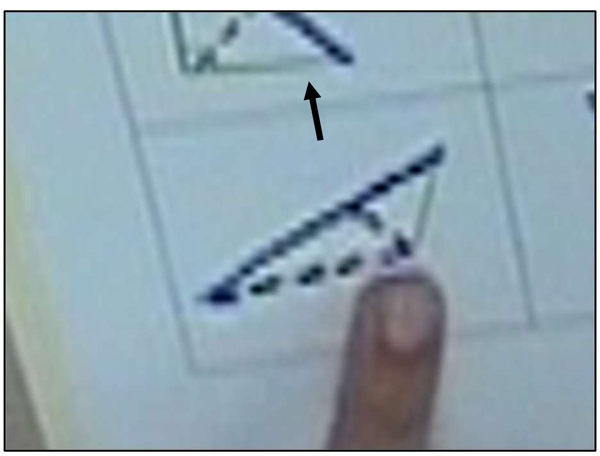
Height as Connected to Vertex. Recall that I found only one evocation of the H-VERTEX concept image within one interview episode. After Phoebe(CT11) and I had finished Task 2.3c, I asked her about Task 2.3a and drew a dashed line on one of the sides connected to the obtuse angle and asked if that could possibly be the base. Phoebe(CT11) replied:

> Umm, maybe yeah. I think I could keep the height the same right here. I know it's not like straight, but I think that'd be the height, because you can't really … I feel like you would have to put the it [height] where the angles meet [*pointed to a vertex connected to the base*].

Based on when Phoebe(CT11) pointed to the vertex and said, "where the angles meet," I inferred that she was focusing on a connection to a vertex a defining component of triangle height (see Figure 53).

Figure 53

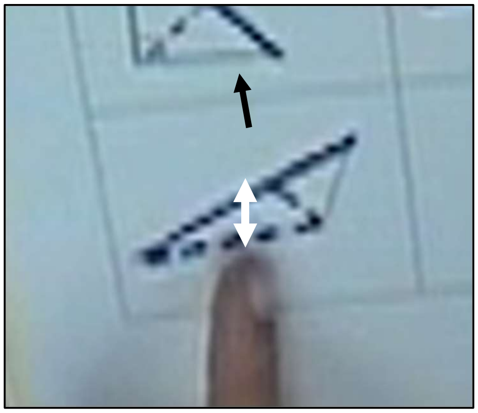
Phoebe(CT11)'s Task 2.3a: H-VERTEX



Then, Phoebe(CT11) clarified where the height could *not* be when she said, "So I couldn't, like, put it [the height] right there [*sliding motion up and down with finger*] because it would be different at different points" (see Figure 54). I believe she was indicating sliding the segment left and right from her finger would cause the segment to become different lengths and to become disconnected from the vertex, and therefore would disqualify it as a height.

Figure 54

Phoebe(CT11)'s Task 2.3a: Explanation of Non-Height



Note: The white arrow indicates Phoebe(CT11)'s finger sliding up and down the paper as she said, "So, I couldn't put it [the height] right here."

Summary of Height Concept Images

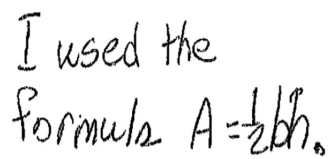
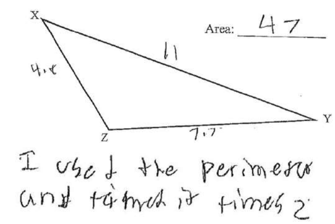
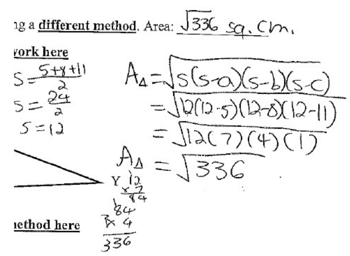
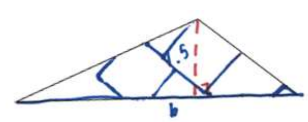
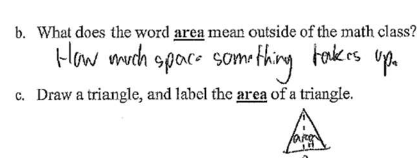
In this section, I have described 12 concept images of the triangle height. Six of those were a priori and six of the concept images were new. I offered detailed examples to illustrate each concept image and shared frequency data from surveys and interviews. Recall that I did not find evidence of H-VERTEX within surveys, and I did not find evidence of H-VERTICAL-SIDE and H-RIGHT within interviews. In the next section, I will repeat the same organizational structure but with a focus on concept images of triangle area.

Concept Images of Area

I found six distinct concept images of triangle area—four of which were a priori to this study and two of which were new. I separated those six concept images into two broad categories—area as formulaic and area as visual. In this section, I discuss participants' concept images of triangle area by giving examples from surveys and interviews. First, I begin with a brief description of each concept image (see Table 8).

Table 8

Concept Images of Area Present in the Surveys and Interviews

Concept Image (Code)	Description	Example
	Area Formulaic in Nature	
Area as Standard Formula (A-STANDARD-FORMULA) [a]	Student either speaks or writes the "one half base times height" formula for triangles—student may use variables for base and height or may just use numerical triangle measures.	
Area as All Sides (A-ALL-SIDES) [a]	Student evaluates the perimeter of a triangle as a strategy for finding area of the triangle—alternately student attempts to use some other formula (e.g., standard formula) but uses all three sides as part of their calculation.	
Area as Heron's Formula (A-HERONS)	Student attempts to use Heron's Formula either by naming it as such, or by attempting the Heron's formula calculation. Must include usage of semi-perimeter and square root.	
	Area as Visual	
Area as Decomposition (A-DECOMPOSITION)	Student attempts to find the area of a triangle by separating the triangle into parts and attempting to find the area of each part by any method.	"I decided to find the area of the three small triangles then add them together."
Area as Square Units (A-SQUARE-UNITS) [a]	Student estimates the area of a triangle by using square units (e.g., square centimeters) to approximately fill the triangle.	
Area as Size (A-SIZE) [a]	Student writes or discusses (a) 2-dimensional nature of area, (b) bounded nature of area, or (c) size or amount in general.	

[a] Indicates an a priori concept image.

I found evidence of one or more area concept images within each participants' surveys. In Table 9, I report frequencies associated with each of the observed area concept images from the survey. Table 9 includes a total number of participants who evoked each concept image and frequency information relating to those evocations.

Table 9

Frequency of Area Concept Image Evocations by Survey Participants

Concept Image	No. of Participants (n = 95)	No. of Participants		
		1 or 2 evocations	3 or 4 evocations	5 evocations
A-STANDARD-FORMULA [a]	93	65	27	1
A-ALL-SIDES [a]	4	4	0	0
A-HERONS	8	8	0	0
A-DECOMPOSITION	17	17	0	0
A-SQUARE-UNITS [a]	11	11	0	0
A-SIZE [a]	71	71	0	0

[a] Indicates an a priori concept image.

Most participants evoked A-STANDARD-FORMULA within their surveys with over 97% of participants evoking this concept image at least once. Also, fewer than 10% of participants evoked A-HERONS and A-ALL-SIDES at least once during their surveys. The generally low frequency of evocations for area concept images for most concept images (i.e., all except for A-STANDARD-FORMULA) may be explained by fact that the survey only contained one question asking about the meaning of area and two questions asking participants to find the area of triangles.

Next, I report frequencies related to interviewees' concept images (see Table 10). Table 10 includes the frequency of each concept image and the number of interviewees represented among those episodes.

Table 10

Frequency of Area Concept Image Evocations by Interview Participants

Concept Image	No. of Interviewees (n = 7)	No. of Interviewees		
		1–3 episodes	4–6 episodes	7–9 episodes
A-STANDARD-FORMULA [a]	7	3	2	2
A-ALL-SIDES [a]	1	1	0	0
A-HERONS	1	1	0	0
A-DECOMPOSITION	4	4	0	0
A-SQUARE-UNITS [a]	1	1	0	0
A-SIZE [a]	1	1	0	0

[a] Indicates an a priori concept image.

The same six area concept images were evoked within the interviews. All seven interviewees evoked A-STANDARD-FORMULA at least once during their interviews. In contrast, A-ALL-SIDES, A-HERONS, A-SQUARE-UNITS, and A-SIZE were evoked by only one interviewee each. In the following sections, I discuss each area concept image. Within those sections, I include examples from participants' surveys and interview episodes.

Area as Formulaic

Based on my survey and interview analysis, I found the following triangle area concept images having to do with a focus on formulas: (a) area as standard formula (A-STANDARD-FORULA), (b) area as all sides (A-ALL-SIDES), and (c) area as Heron's Formula (A-

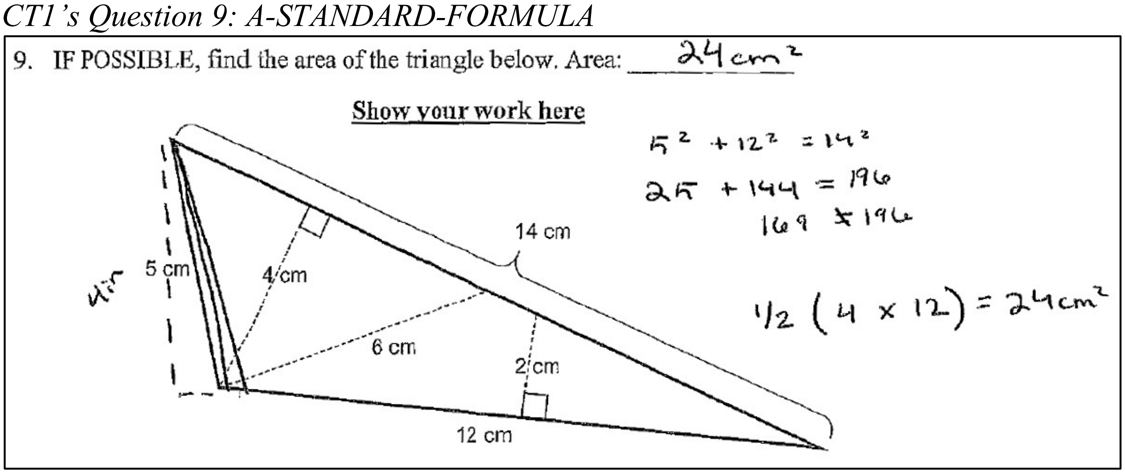
HERONS). Broadly, these concept images describe a type of thinking whereby the participants thought of area as a quantity to be calculated by substituting various measurement values in place of the formula's parameters. Recall that A-STANDARD-FORMULA and A-ALL-SIDES were a priori for this study.

Area as Standard Formula. Considering that participants had just finished learning about area in their geometry classes, I had anticipated the large representation of participants who evoked the A-STANDARD-FORMULA concept image.

For example, CT1 evoked the A-STANDARD-FORMULA concept image in response to the Survey Question 1a when she wrote "*1/2* (*b* x *h*)." Then, in Survey Question 9, she used the standard formula to attempt to determine the area of a given triangle (see Figure 55). Although CT1 incorrectly recorded the length of an exterior auxiliary segment to be 4 in., I suspect that she had measured using centimeters. She also used the standard triangle area formula in Survey Question 10. So, I considered each of those examples to be evidence of the A-STANDARD-FORMULA concept image. The other participants, both for surveys and interviews, similarly evoked the A-STANDARD-FORMULA concept image.

Figure 55

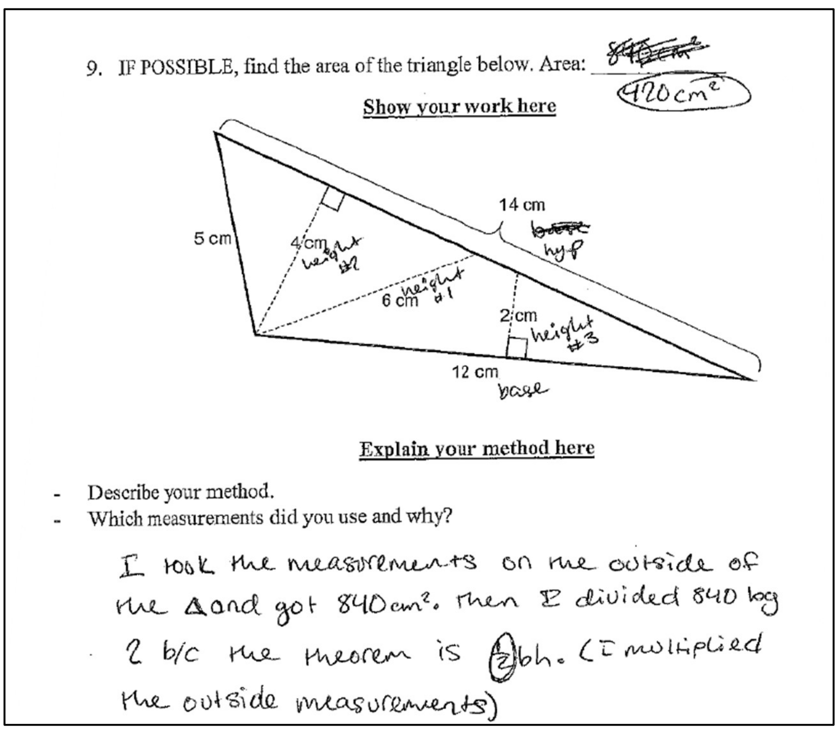
CT1's Question 9: A-STANDARD-FORMULA



Area as All Sides. I found that participants who evoked the A-ALL-SIDES concept image seemed to be attempting to use a formula but used all three sides instead of identifying a base and height. For example, in Survey Question 9, R39 evoked the concept image A-ALL-SIDES. She obtained an area measure of 420 square centimeters (see Figure 56) and described her strategy below the triangle diagram: "I took the measurements on the outside of the [triangle] and got 840 centimeters squared then I divided 840 by 2 b/c the theorem is $1/2bh$. (I multiplied the outside measurements)." Indeed, if you multiply 5, 12, and 14 together you will get 840. It is because of the final statement about multiplying *all* the outside measurements together that I classified her concept image as A-ALL-SIDES as opposed to A-STANDARD-FORMULA.

Figure 56

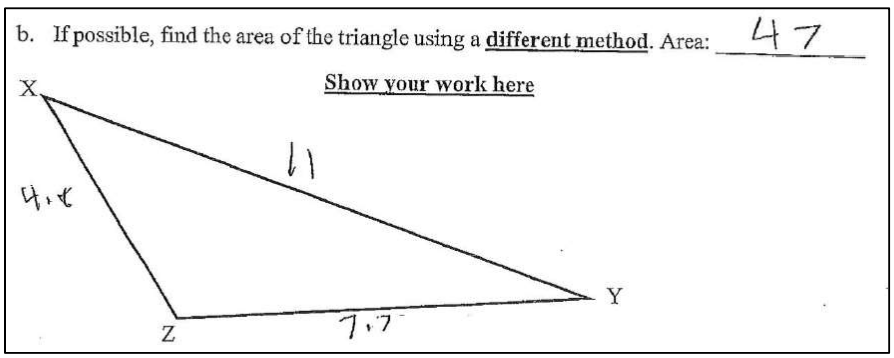
R39's Question 9: A-ALL-SIDES



I found another variation of A-ALL-SIDES within an interview. During Interview Task 1.10b, Nate(R21) evoked the A-ALL-SIDES concept image when he said, "I found the perimeter. I don't know why, um … I don't know why but I timesed [multiplied] it by two and I thought that was the area." Nate(R21) had measured the side lengths as 4.8, 7.7, and 11 (see Figure 57), added them together to obtain 23.5, and doubled that to obtain 47. This is an example of A-ALL-SIDES because of Nate(R21)'s usage of all three sides of the triangle.

Figure 57

Nate(R21)'s Task 1.10b: A-ALL-SIDES

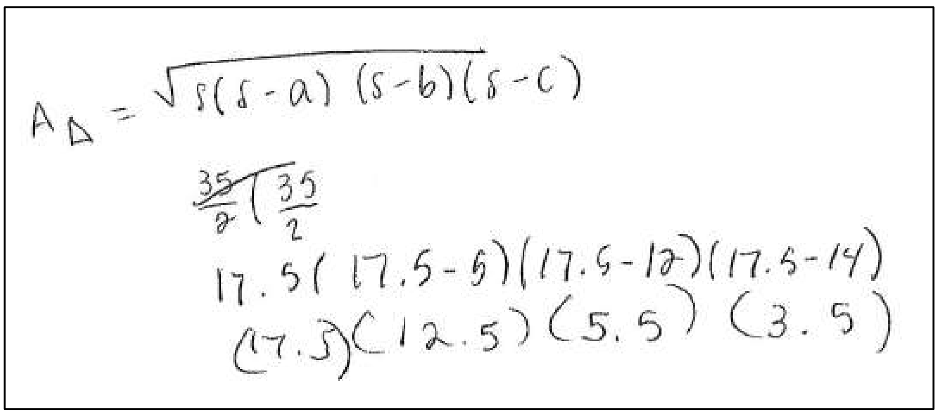


Area as Heron's Formula. Although this study was not focused on participants' class level as a unit of analysis, I noticed that all eight participants who evoked the A-HERONS concept image were part of an honors geometry class and taught by the same honors geometry teacher. The following examples are representative of the participants' evocations of A-HERONS.

Lisa(H16) evoked the A-HERONS concept image in two episodes during her interview. In one episode, Lisa(H16) was completing Task 1, during which she was discussing Survey Question 9. In that question, she began using Heron's formula but prematurely stopped (see Figure 58). Lisa(H16) had correctly written Heron's formula using the parameters *a*, *b*, *c*, for the sides, and *s*, for the semi-perimeter. Then, she substituted the given side lengths into the correct formula and described her work by saying, “This is the Hero’s formula … so I found the semi-perimeter, which is half of the perimeter of the triangle and then you multiply it by each side of the triangle.”

Figure 58

Lisa(H16)’s Question 9: A-HERONS



Later in the interview, after Lisa(H16) had attempted to find area using the standard area formula in Task 4, I had asked her if there was any other way that she could find the area without using “one half times base times height.” Lisa(H16) described the possibility of using Heron’s formula as another option for calculating the area of a triangle.

Visual Strategies for Determining Area

I found evidence of the following triangle area concept images that can be categorized as visual approaches: (a) area as decomposition (A-DECOMPOSITION), (b) area as square units (A-SQUARE-UNITS), and (c) area as size (A-SIZE). Each of these concept images involved a visual type of understanding of area. Recall that A-SIZE and A-SQUARE-UNITS were a priori concept images.

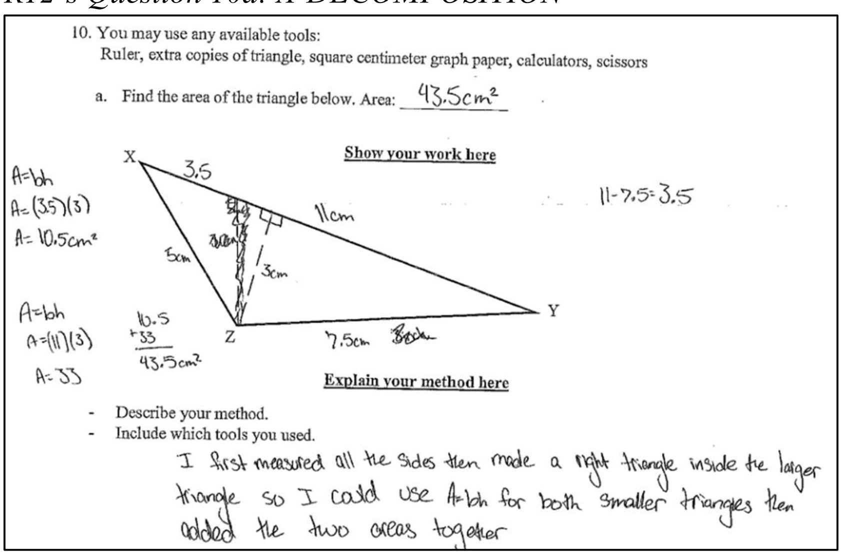
Area as Decomposition. The A-DECOMPOSITION concept image was a visual strategy for finding area. Interestingly, all participants who evoked A-DECOMPOSITION went on to attempt to apply the standard formula when computing the areas of the component parts—all triangles.

For example, R12 evoked the A-DECOMPOSITION concept image in Survey Questions 9 and 10. In Survey Question 9, she evoked the A-DECOMPOSITION concept image when she wrote "I first looked at the triangle to find any right angles. Once I saw one, I split up the whole triangle into smaller ones … then I added them [the areas] together to get the whole thing."

She used the same method to complete Question 10 (see Figure 59) by drawing a triangle altitude as a way of dividing the given obtuse triangle into two smaller triangles. To explain her method, R12 wrote, "I first measured all the sides then made a right triangle inside the larger triangle so I could use *A=bh* for both smaller triangles then added the two areas together."

Figure 59

R12's Question 10a: A-DECOMPOSITION



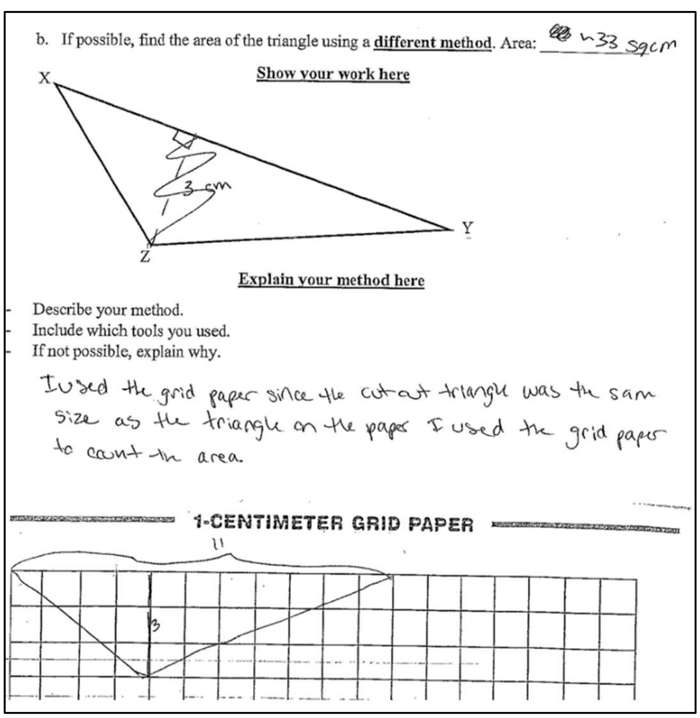
Note. R12 had decomposed the obtuse triangle into two right triangles, computed the area of each, and then summed those area values to determine the area of the larger obtuse triangle.

R12's responses were representative of the other participants' A-DECOMPOSITION concept images. Other participants who evoked A-DECOMPOSITION also subdivide a triangle into smaller triangles and then attempted to compute the area of those smaller triangles using the standard area formula.

Area as Square Units. For example, H27 evoked the A-STANDARD-FORMULA concept image while completed Survey Question 10a using. Then, in Question 10b, H27 described using the grid paper in conjunction with the triangle cutout. She wrote, "I used the grid paper since the cutout triangle was the same size as the triangle on the paper. I used the grid paper to count the area." As shown in Figure 60, H27 had traced the triangle cutout onto the grid paper. Also, based on her measures of 11 and 3, it seems like H27 had counted 33 square units that make up the rectangle that surrounds the triangle rather than the squares within the triangle.

Figure 60

H27's Question 10b: A-SQUARE-UNITS



During my interviews, I was able to elicit additional interpretations related to A-SQUARE-UNITS. For example, when Lisa(H16) completed Task 4 of the interview and obtained an area of 4.125 using the standard area formula. Then, I asked her about the meaning of her answer. She evoked A-SIZE and then A-SQUARE-UNITS and I share the transcript of our conversation during that part of the interview:

Researcher (R): So, 4.125 … 4.125 … that's the answer?

Lisa(H16; S): Um hm.

R: So, what does area even mean?

S: Area is, um, all of the space [*sweeping motion over triangle region with marker cap*] that is inside this triangle [*evocation of A-SIZE*].

R: So, in the picture, I can see that this [*finger traced over a side length*] is the 5.5.

S: Um hm [*indicating yes*].

R: And this [*finger traced over the height*] is the 1.5.

S: Yes.

R: And I'm having a hard time picturing 4.125 of what?

S: Um. It would be inch… square inches. Well, I used inches right? So, it would be square inches.

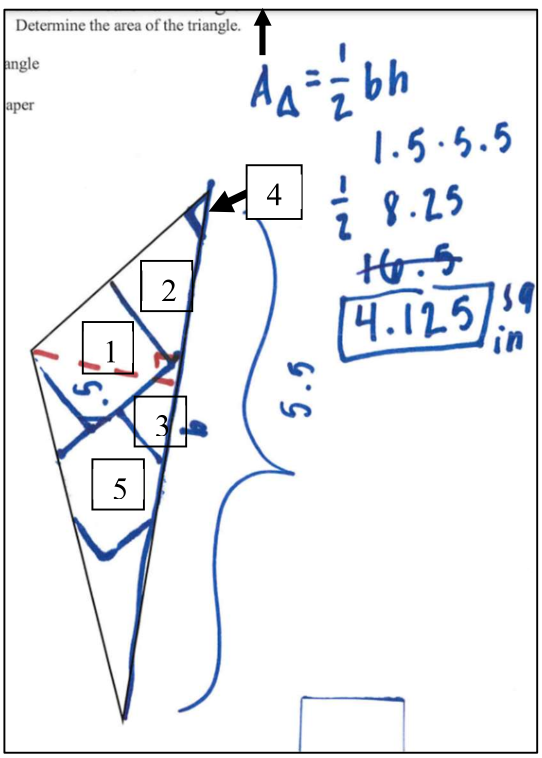
S: So, it would be. Um. If you put this in [*placed square centimeter cutout inside triangle*], it would be 4.125 square inches [*evocation of A-SQUARE-UNITS*]. That doesn't seem right.

It seems that Lisa(H16) had begun to describe how many square inches would fit inside the triangle, but she had laid a square centimeter inside the triangle when she said, "That doesn't seem right." What happened next was quite intriguing to me and further revealed her A-SQUARE-UNITS concept image.

Next, Lisa(H16) created a square inch using her ruler and cut it out. She traced one full square inch, and then several partial squares in the order indicated in Figure 61. After outlining 3 square inches (i.e., one complete squares and two decomposed squares) within the triangle, she talked about how the empty space, which was not yet covered with square units, would likely be enough to complete the partial squares, totaling up to 4.125 squares. In summary, Lisa(H16) started this task by evoking the A-STANDARD-FORMULA concept image, but when I asked her about the meaning of her answer, she evoked A-SIZE and then A-SQUARE-UNITS.

Figure 61

Lisa(H16)'s Task 4: A-SQUARE-UNITS



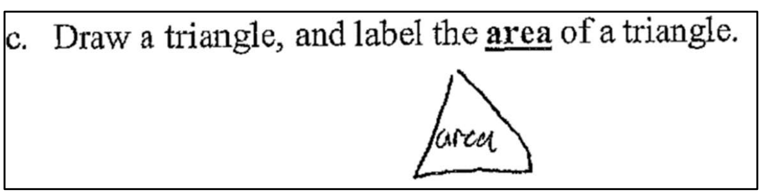
Note. This was the orientation of the paper when Lisa(H16) began filling the triangle with square units. Numbers indicate the order of Lisa(H16)'s square and partial squares.

Area as Size. All the participants evocations of A-SIZE occurred in Survey Question 1 except for a single evocation of A-SIZE within Lisa(H16)'s interview. I just shared Lisa(H16)'s evocation of A-SIZE within the previous section regarding when she and I were discussing Interview Task 4. Now, I will share a few brief examples of the A-SIZE concept image from surveys to reveal some subtle differences in participants concept images. For example, CT4 described area in Survey Question 1b by writing, "How much of an object's space exist." Answering the same question, R3 wrote, "How much space something takes up." I infer subtle difference between CT4's and R3's responses in that in the former area simply exists, but in the latter area is viewed as a portion of some larger quantity of available space.

Sometimes, I found participants' evocations of A-SIZE in their drawings, such as when H1 drew a triangle and wrote the word area inside, in response to Survey Question 1c (see Figure 62).

Figure 62

CT4's Question 1c: A-SIZE


c. Draw a triangle, and label the **area** of a triangle.

Summary of Concept Images of Base, Height, and Area

In this section, I began by comparing the frequency of concept image evocations from interviewees' surveys versus interview episodes. It appeared that Interview Tasks 2–4 generated more evocations as compared to the survey questions and Interview Task 1. Then, I discussed 10 concept images of base, 12 concept images of height, and six concept images of area. Of those three types of concept images, I identified seven, six, and three new concept images respectively. The concept images that the highest number of participants evoked were B-BOTTOM, H-ALTITUDE, H-EXTERIOR, H-PERP-TO-BASE, and A-STANDARD-FORMULA. Of those, H-EXTERIOR was new, and the rest were a priori. In the next section, I discuss relationships among base, height, and area that participants revealed in their surveys and interviews.

Relationships Among Base, Height, and Area

In this section, I discuss results related to Research Question 2. I analyzed both the survey and interview data for evidence of participants connecting the ideas of base, height, and area (i.e., Phase 9 of data analysis). Broadly, I found that about half of the participants' surveys contained evidence that, within at least one question or part, there was some relationship between their concept image of base and their concept image of height. In some cases, I found evidence of multiple kinds of relationships within one survey. Also, every interviewee demonstrated evidence of viewing base and height as related to one another during their interviews. In this section, I will discuss the different types of relationships and how participants typically related triangle base and height to area.

Relationships between Base and Height

I found three different ways that participants related base concept images and height concept images: (a) height depends on base, (b) base depends on height, and (c) base and height are interrelated. To determine the existence of relationships, I examined written responses from surveys and observed the phrasing and hand gestures used by interviewees. In Table 11, I show the frequencies of the three base-height connection types from surveys and interviews.

Table 11

Frequency of Base-Height Connection Types Within Surveys and Interviews

Connection Type	No. of Participants' Surveys (n = 95)	No. of Interviewees (n = 7)
Height Depends on Base	28	6
Base Depends on Height	3	2
Base and Height Interrelated	30	4

Height Depends on Base

Participants who demonstrated this kind of relationship between height and base concept images tended to describe the base on its own by evoking one of the base concept images. Then, they described the height as a side or segment that connects to the base or line containing the base, usually perpendicularly. I found evidence of this kind of relationship within 28 participants' surveys and among six interviewees.

For example, in her response to Survey Question 8, CT1 wrote, "The base is just the bottom of the triangle, and if the side of the triangle forms a 90-degree angle with the bottom that side is the height." Initially, CT1 evoked B-BOTTOM and did not reveal any relationship of the base to the triangle height. Then, when describing the triangle height, she referred to both the formation of a 90-degree angle and to the bottom, which she just identified as the base. From these statements, I inferred that part of CT1's concept image of height included a perpendicular connection to the base (i.e., H-PERP-TO-BASE). In other words, her evocation of height required a prior identification of base.

In Interview Task 2.1a, David(R26) identified the base as the bottom side of the triangle (i.e., B-BOTTOM). Then he said, "this would be the height because it's perpendicular to the base [*touched the base*] and it touches the corner point at the top [*pointed to vertex opposite the base*].” The primary difference, as compared to the previous example, is that I could see David(R26)’s drawing and hand motions as he was discussing the base and height.

Base Depends on Height

I also observed the relationship of base concept images depending on height. Although I identified this relationship in both surveys and interviews, I observed this relationship less often than height depending on base. Of those who demonstrated the relationship of base depending on height, participants described height first without describing base. Then, they always described base as an object perpendicular to the height.

Among my interviewees, I only found evidence of the relationship of base depending on height with Nate(R21). Also, Nate(R21) was the only interviewee who drew height first and base second. In his response to Interview Task 2.1a, he drew the height first. Then regarding the base and said, “because this [the base] is perpendicular and down, I’m gonna use this [*drew the base*] as the base.” Nate(R21) was consistent in his drawing of height first and base second.

Base and Height Interrelated

I found two different ways in which participants demonstrated an interrelated understanding of triangle base and height. First, they placed the initial emphasis on locating the right-angle symbol, then described base and height as being connected to that symbol. Second, and only subtly different than the first, participants described base as perpendicular to height and height as perpendicular to base. In other words, the two types of understanding were nearly identical but only differed in the way that descriptions of base and height included or did not include height and base respectively. Four interviewees discussed a way of thinking about triangle base and height that demonstrated a kind of interrelationship between them.

I also found evidence of an interrelated understanding of base and height within 30 surveys. For example, in Survey Question 2a, R3 described base as, “perpendicular to height,” and in Survey Question 3a, he described height as, “perpendicular with base.” During my earlier phases of analysis, I had thought that these two descriptions were contradictory to each other. The first one implied that R3 had identified height first and base second. The next explanation implied that he had identified base first and height second. It wasn’t until Phase 9 of analysis that I realized that participants with responses like R3 were likely joining base and height together via the idea of perpendicularity and that they may be imagining base and height simultaneously.

David(R26) typically identified a triangle side as the base and drew an auxiliary segment for the triangle height. However, in his response to Survey Question 6 he identified two sides as base and height. So, during Interview Task 1.6, I asked him to clarify what was different about Survey Question 6. He said, “the right-angle symbol makes me know for sure that these are sides [*pointed to both triangle legs*].” Given the context of our discussion and his pointing to the base and height, I knew that the phrase “these are sides” was referring to the base and height. So, for David(R26), it was the right-angle symbol that caused him to see the base and height, perhaps simultaneously, in Survey Question 6.

Relationships Among Base, Height, and Area

In many surveys and interviews, participants wrote out similar versions of the standard triangle area formula, using the parameters *b* and *h*. Then they substituted numerical values for base and height into those parameters to compute the area. I viewed those as examples of participants making a connection, via the formula, from base and height to area. I found two different ways that participants seemed to relate base and height together. First, they related base and height separately to area. Second, sometimes they related base and height together and then related the two to area. In Table 12, I show the frequencies of the two ways that base and height related to area.

Table 12

Frequencies of Base and Height Related to Area from Surveys and Interviews

Connection Type	No. of Participants' Surveys (n = 95)	No. of Interviewees (n = 7)
Base and Height Separately Relate to Area	2	2
Base and Height Jointly Related to Area	19	3

Base and Height Separately Related to Area

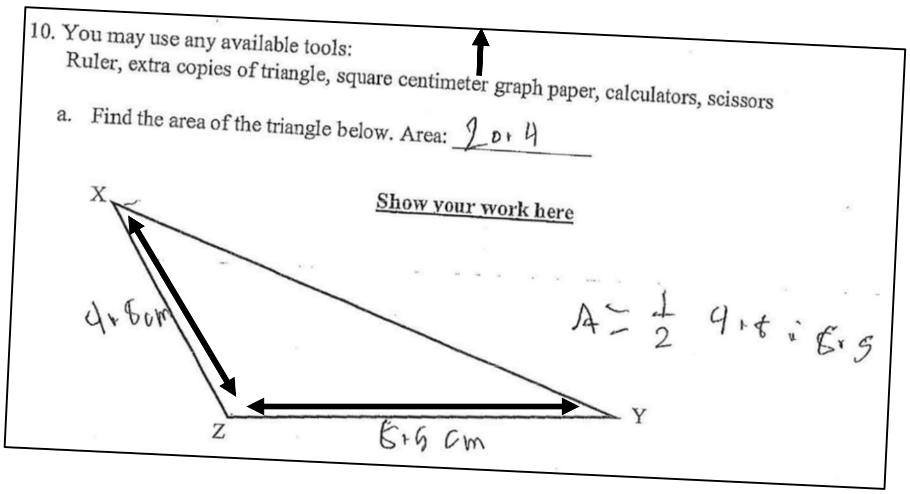
I found that some participants identified a base and drew a height with evoked concept images that seemed separate from each other. For example, the concept images B-BOTTOM does not depend on the height and H-SIDE does not depend on the base. Then, they related both separately to the standard area formula by substituting in the measures of the base and height. This phenomenon was uncommon. I only observed it with two interviewees—Nate(R21) and Phoebe(CT11). I only found written evidence of this once within two surveys as well.

For example, during Interview Task 1.10a, Nate(R21) was discussing Survey Question 10a (see Figure 63). He discussed using one of the triangle sides as the base when he said, "I used the ruler and got [measured] these two [*traced finger along the base-side and then along the height-side*]." He paused slightly between finger tracings. Then he pointed to the standard area formula and said, "This is the formula here, and I got twenty point four." Because of his pausing while describing and tracing over base and height and because he did not discuss any relationship between, I inferred that Nate(R21) related base with the A-STANDARD-

FORMULA concept image and, separately, height with the A-STANDARD-FORMULA concept image.

Figure 63

Nate(R21)'s Task 1.10a: B-BOTTOM, H-SIDE, and A-STANDARD-FORMULA



Note. The bi-directional arrows indicate Nate(R21)'s finger tracing motion—first back and forth along the base and secondly back and forth along the height.

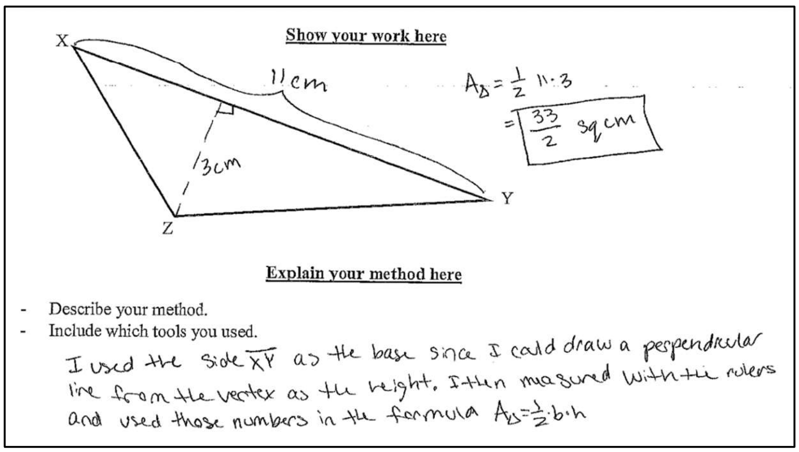
Base and Height Related First Then Connected to Area

I found evidence of participants relating the ideas of base and height, using the three categories described earlier (e.g., height depends on base), and then using those measures in the standard area formula. I found this combination of relationships within 19 surveys and within seven groups of episodes (i.e., base, height, and area episodes) associated with three interviewees.

In her response to Survey Question 10a (see Figure 64), H27 wrote "I used the side XY as the base since I could draw a perpendicular line from the vertex as the height. I then measured with the rulers and used those numbers in the formula *A=1/2bh.*" I infer that H27 was either considering different base height combinations simultaneously and selected the one which enabled her to have an interior height, or that she was imaging height options only, then selected an interior height, and then identified the base as the side that the height intersected perpendicularly.

Figure 64

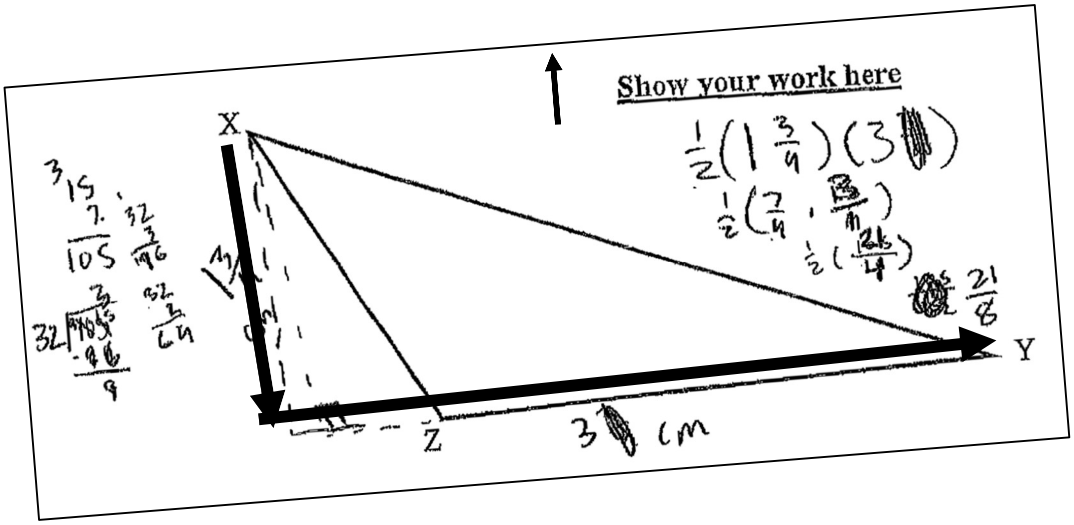
H27's Question 10a: Base Potentially Depends on Height



During Interview Task 1.10a, when Erica(H24) discussed her process with Survey Question 10a, she said, “I tried to draw a line from here [*traced finger downward from vertex opposite the base*], and then like match it up with this base [*traced finger along base*].” I show Erica(H24)’s finger tracing motion in Figure 65.

Figure 65

Erica(H24)s' Task 1.10a: B-BOTTOM, H-EXTERIOR, A-STANDARD-FORMULA



Note. The arrows indicate Erica(H24)'s sweeping motion of her finger as she was discussing the triangle height.

Then, Erica(H24) went on to describe how she used her base and height measures in the standard area formula. Her description was straightforward. She said, "Then, I measured it with a ruler." By, "it," she was referring to the triangle base and height. She then said, "Then I did the same formula," as she pointed to the standard area formula—an indication that she was relating both base and height to the A-STANDARD-FORMULA concept image.

To me, Erica(H24)'s tracing motion indicated that she was relating the triangle height with the base (i.e., connecting them). She traced her finger fully along the height, then along the line containing the base, indicating to me that she was visualizing the height to meet up with an extension of the base. Although she did not speak about the right angle, she had drawn a right-angle symbol in her survey. Erica(H24)'s work serves as a good example of relating base, height, and area together. It also suggests a close connection among the concept images H-ALTITUDE, H-PERP-TO-BASE, and H-EXTERIOR.

Summary of Relationships Among Base, Height, and Area

After analyzing the survey and interview data, I found several examples of participants having related base and height together. Typically, the unifying factor was perpendicularity. In this section, I discussed how some participants viewed height as dependent on base, some viewed base as dependent on height, and some viewed them as interrelated. Recall that almost every participant, in their surveys, and every interviewee used the standard area formula at some point during their work. This means that almost all participants related base and height to the standard area formula in some way. In the next section, I discuss how the influences of triangle orientation, gravity, and triangle type influenced participants concept images of base, height, and area.

Influences of Orientation, Gravity, and Triangle Type

In this section, I discuss relevant data related to Research Question 3. This primarily includes evidence of the influence of triangle orientation, gravity, and triangle type on participants' concept images of triangle base, height, and area.

Influence of Triangle Orientation

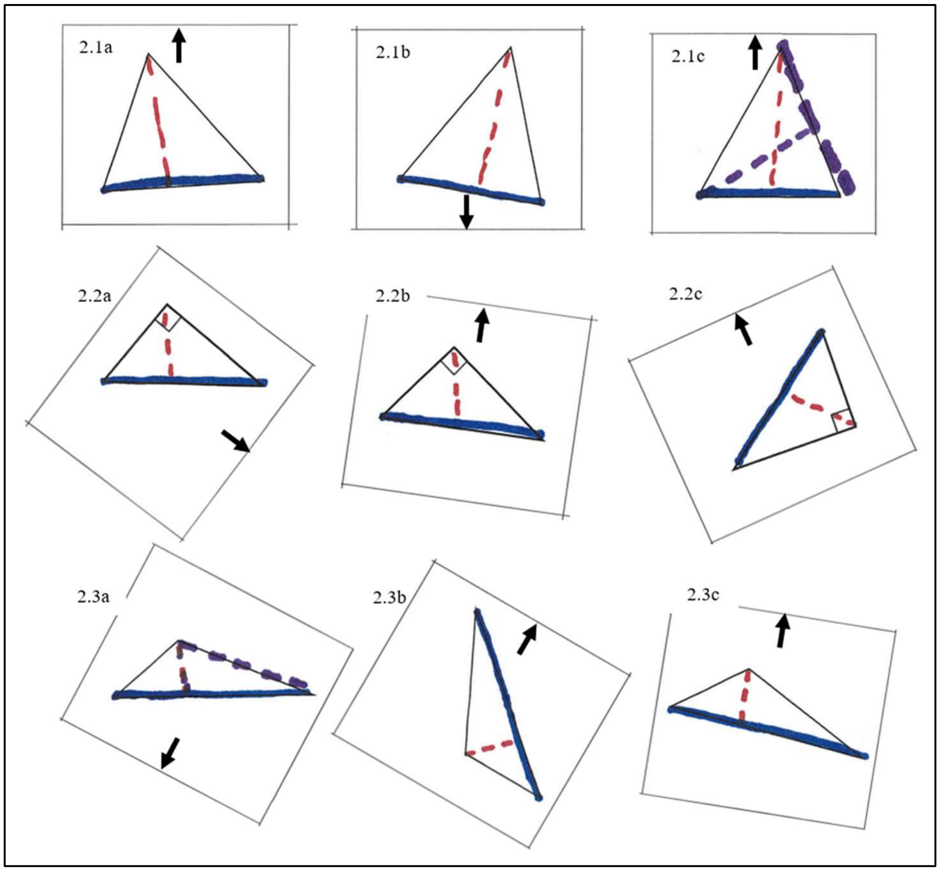
In the analysis of this study, I found evidence that triangle orientation relates to, or influences, some participants' concept images of triangle base. I also found evidence that some participants view triangles as having a "right-side up" orientation. In this section, I present examples related to each of those findings.

Orientation of Triangle for Base

When Phoebe(CT11) was working on Interview Task 2, she rotated the paper almost every time before drawing the triangle base. In Figure 66, I show the collection of nine triangles from Interview Task 2, each as rotated and viewed from Phoebe(CT11)'s perspective.

Figure 66

Phoebe(CT11)'s Task 2: Base-Height-Orientation Connections



Note: The dashed blue lines in Tasks 2.1c and 2.3a were hypothetical bases, which I had asked Phoebe(CT11) to consider.

For the three acute triangles, it appeared that Phoebe(CT11) had rotated the paper so that an edge of the paper was approximately parallel with the edge of the desk. Then, for the right and obtuse triangles, it seemed like the edge of the paper with respect to the desk no longer held the same level of influence.

Although Phoebe(CT11) did not discuss the rotation of the paper, it became clear during this task that she had some underlying preference about the rotation of each triangle, since she rotated each triangle differently. I inferred that sometimes Phoebe(CT11) rotated the paper so that the base was sometimes approximately horizontal, and sometimes rotated the paper for ease of drawing.

Viewing Triangles as Right Side Up

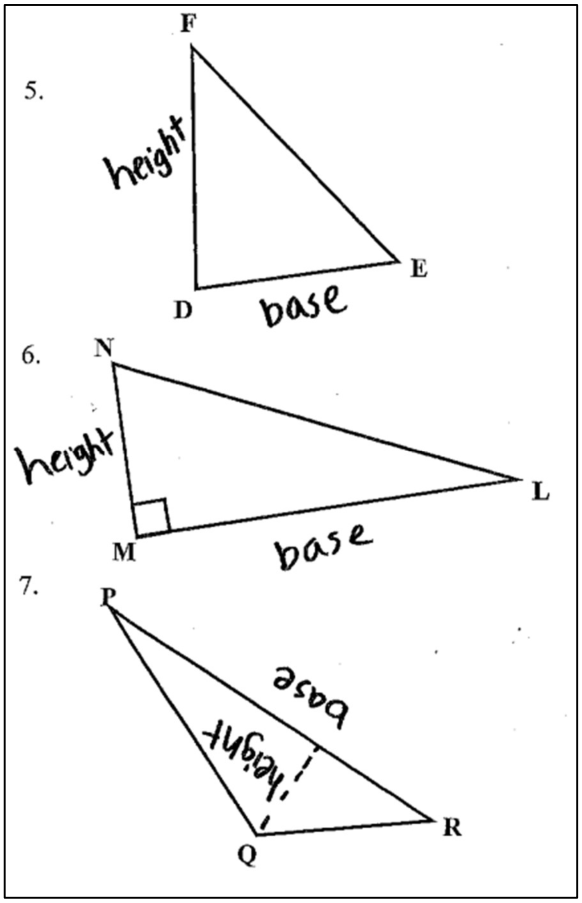
Some participants described triangles as having a top or bottom. In other words, they viewed the triangles as having a correct orientation. In response to Survey Question 9, H5 wrote:

> I used 14 as my base because it has an altitude drawn to it and I used the altitude as the height because that is the height of the triangle from the lowest point to the highest point of the triangle if it was right side up.

Note that it was nearly impossible for me to determine if participants had been influenced by triangle orientation based on examination of their survey responses alone. For example, R35 described base as "the bottom of something," in her response to Survey Question 2b. However, there seemed to be evidence of orientation, based on her work in Survey Questions 5–7 (see Figure 67). I inferred that R35 likely rotated the paper for Question 7 based on her writing of the words base and height. However, I am less certain about rotation having occurred in Questions 5 and 6 because those words could have been written slightly askew just based on the participant's normal writing style.

Figure 67

R35's Questions 5–7: Evidence of Triangle Orientation Based on Written Work



Influence of Gravity

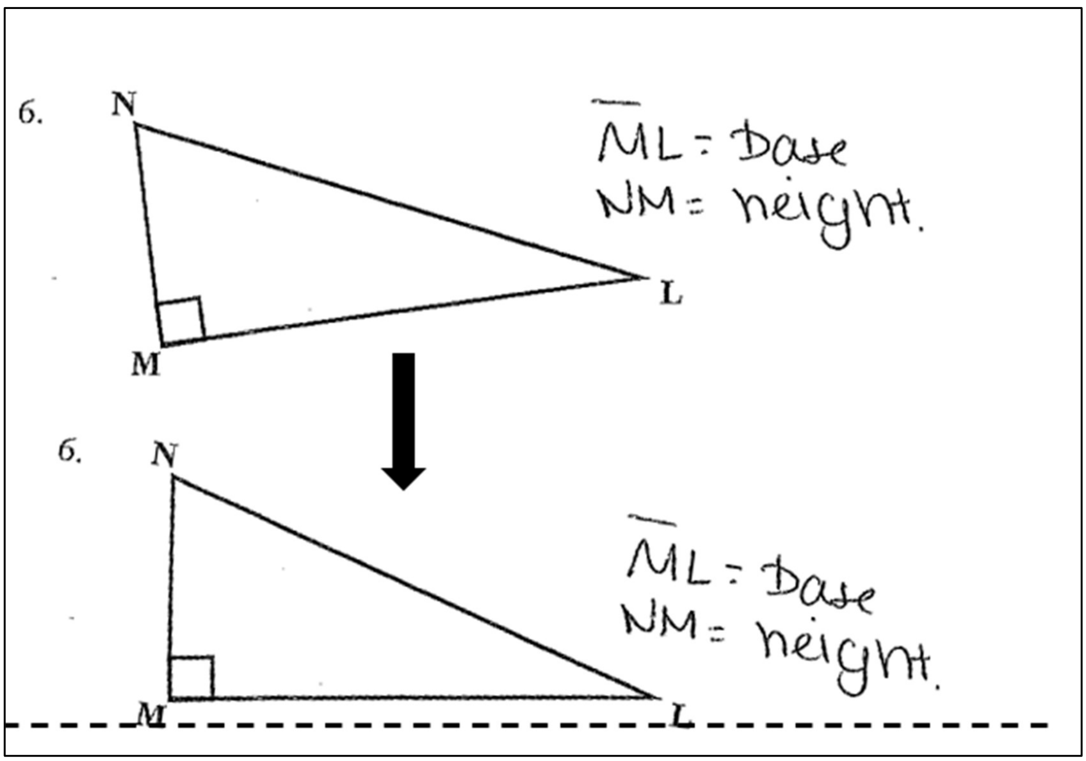
Only a few participants showed evidence of a gravitational factor (Vinner & Hershkowitz, 1980) and the evidence was subtle and limited. However, the limited evidence may reflect my task and question choices rather than an indication of rarity of this factor. In this section, I share the most prominent examples of the gravitational factor and the potential role of gravity in participants' concept images of base and height.

Influence of Gravity on Base

In Survey Question 8, R14 wrote, "The base is the 'bottom' of the triangle or the part you put on the ground to have it stand straight up and the height is the vertical line on one of the sides." Notice R14's usage of the phrases, "put on the ground," and, "stand straight up." The idea of imagining the placement of a triangle on an imagined ground may be connected to the idea of gravity being applied to the triangle. In Figure 68, I show my interpretation of what R14 may have been visualizing, based on their response.

Figure 68

R14's Question 8: Researcher's Interpretation of R14's Visualization



Note. The dashed line represents an imagined horizontal line that the triangle "stands" or "sits" on once gravity is applied to the triangle.

Another participant responded similarly to Survey Question 8. When describing the base and height of the right triangle in the survey, H3 wrote, "The longest of the two legs of the triangle and what leg the triangle appears to be sitting on is the base and the other leg is the height." The phrase "sitting" seemed, to me, connected to the idea of gravity when it was applied to a triangle. During her work on Interview Task 2.1a, Lisa(H16) also used the phrase, "sitting," in reference to her chosen base, which was close to horizontal.

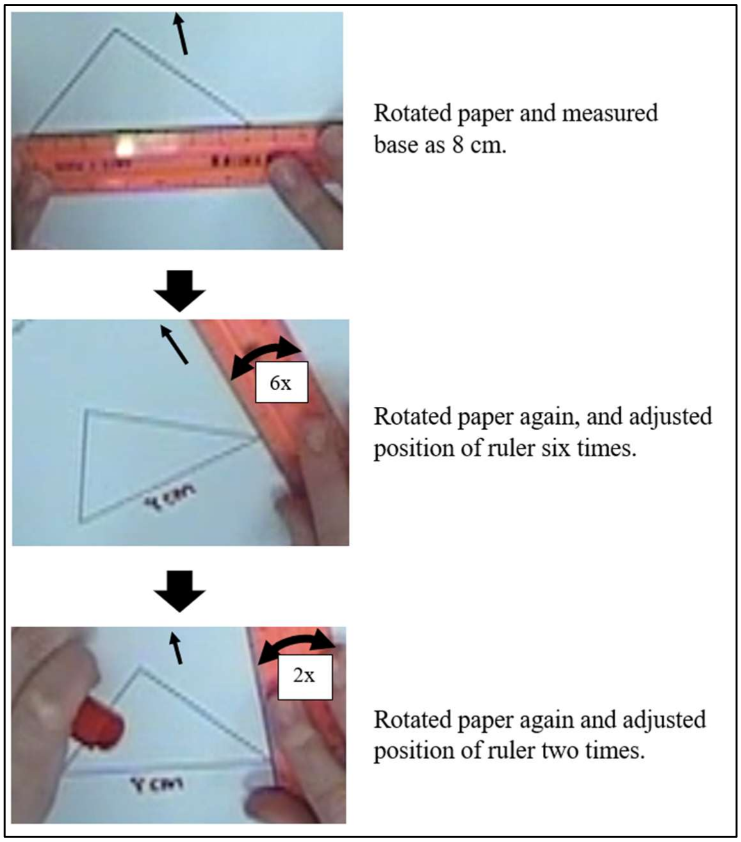
Influence of Gravity on Height

Two of my interviewees produced potential evidence of the gravitational factor. First, I discuss Alyssa(R23)'s work in Interview Task 5, then I discuss Lisa(H16)'s work in Interview Task 1, as she was talking about her survey response to Question 10b.

Alyssa(R23)'s Process of Orienting the Height. Recall that I had introduced Alyssa(R23)'s work in Interview Task 5 in an earlier section called "Height as Vertical." Now, I share the details leading up to her eventual drawing of the height for that task. Alyssa(R23) began the task by measuring the base as 8 cm. Then she adjusted the paper at least twice and position of the ruler at least eight times (see Figure 69). She said, "Just trying to make the lines as straight as possible."

Figure 69

Alyssa(R23)'s Task 5: Struggle to Orient the Height

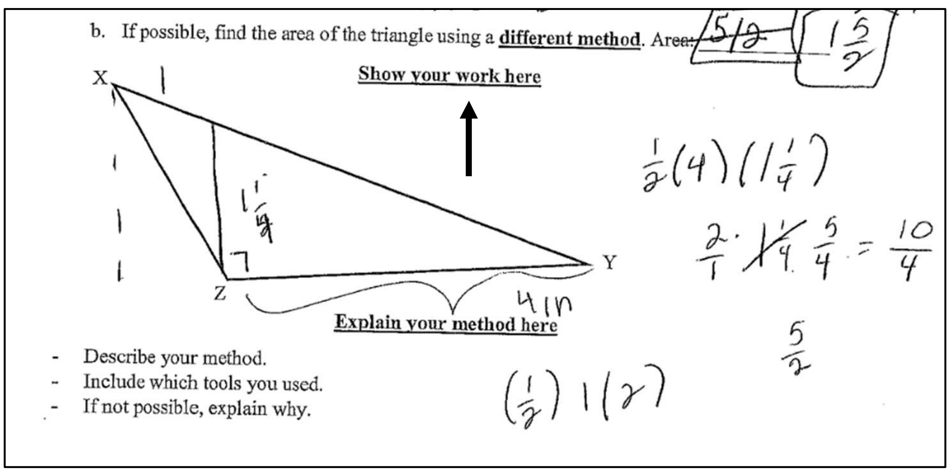


After Alyssa(R23) had adjusted the ruler at to visualize the triangle height, I asked, "How do you determine the right way to hold the ruler?" Alyssa(R23) replied, "Because this [*placed hand along base*] is at an angle, so once I turn it back [*slightly rotated paper*] so it's straight." I believe that during the process of adjusting the ruler and adjusting the paper, Alyssa(R23) was wrestling with multiple visual considerations. First, it seemed like she was thinking about how the base was not horizontal in its original position on the paper. Second, seeing the base "at an angle" may be an indication of the influence of gravity in the that gravity, if applied to the triangle, would cause the triangle to sit "straight" along an imagined horizontal line.

Lisa(H16)'s Relating Two Heights to a Single Base. In this section, I discuss a potential influence of gravity on Lisa(H16)'s concept image of base in the context of her working through discussing Survey Question 10b. When describing her strategy for Survey Question 10b, Lisa(H16) said, "I broke it into two triangles and found the area of the first triangle and the area of the second triangle by using one half base times height and then I added the two areas together." Lisa(H16) had evoked the A-DECOMPOSITION concept image based on her description (see Figure 70).

Figure 70

Lisa(H16)'s Task 1.10b: A-DECOMPOSITION then A-STANDARD-FOMRULA

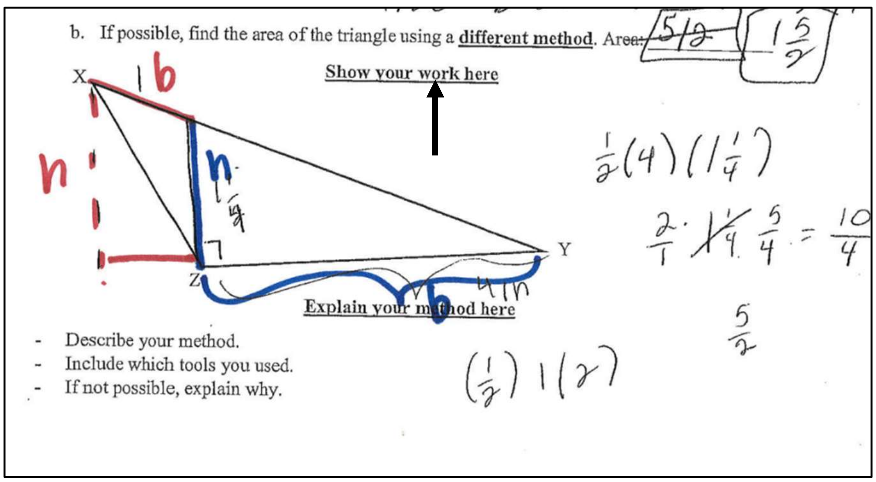


I then asked Lisa(H16) if she could clarify the markings in her work from Survey Question 10b. She talked about the exterior dashed line representing the height of the smaller triangle. Then she described how she created a right triangle by drawing a height perpendicular to a base. At this moment, it seemed that Lisa(H16) had identified one base and one height for each triangle.

I was interested in Lisa(H16)'s thoughts about the exterior height. Therefore, I asked her if she could trace over the base and height of one triangle with red marker and the base and height of the second triangle with blue marker (see Figure 71). Lisa(H16) described the drawing of the red base and the exterior red dashed-line height by saying, "This is the base of the first triangle, and this is the height … going to there." She clarified the meaning of "there" by extending the base outside of the triangle. It seemed to me that Lisa(H16) had made some kind of connection between the red dashed-segment height and the blue base.

Figure 71

Lisa(H16)'s Task 1.10b: Identification of Bases-Height Pairs



Based on Lisa(H16)'s descriptions, she had identified one base and one height for the smaller triangle. However, it seemed like the red dashed-line height was nearly parallel to the blue solid-lined height. This indicated to me that Lisa(H16)'s drawing of both heights was influenced by the blue base of the larger triangle (i.e., with a 4 in. measurement).

Recall that gravitational factor has been used to describe imagining a triangle falling onto an imagined horizontal line and thus causing a triangle side to become horizontal (Vinner &

Herhskowitz, 1980). In contrast, Lisa(H16) did not rotate the triangle when discussing Survey Question 10b. Her concept image of the larger-red height did seem to have a potential connection to the gravitational factor in the way that she drew it nearly vertically.

Influence of Triangle Type

One aspect of Research Question 3 related to a potential relationship between triangle type and students' concept images of triangle base, height, and area. In short, I found evidence that some participants' concept images were related to triangle type. In Figure 72, I show the distribution of the base concept images from Interview Task 2, and in Figure 73, I show the distribution of the height concept images from the same task.

Figure 72

Frequency of Base Concept Images Within Interview Task 2

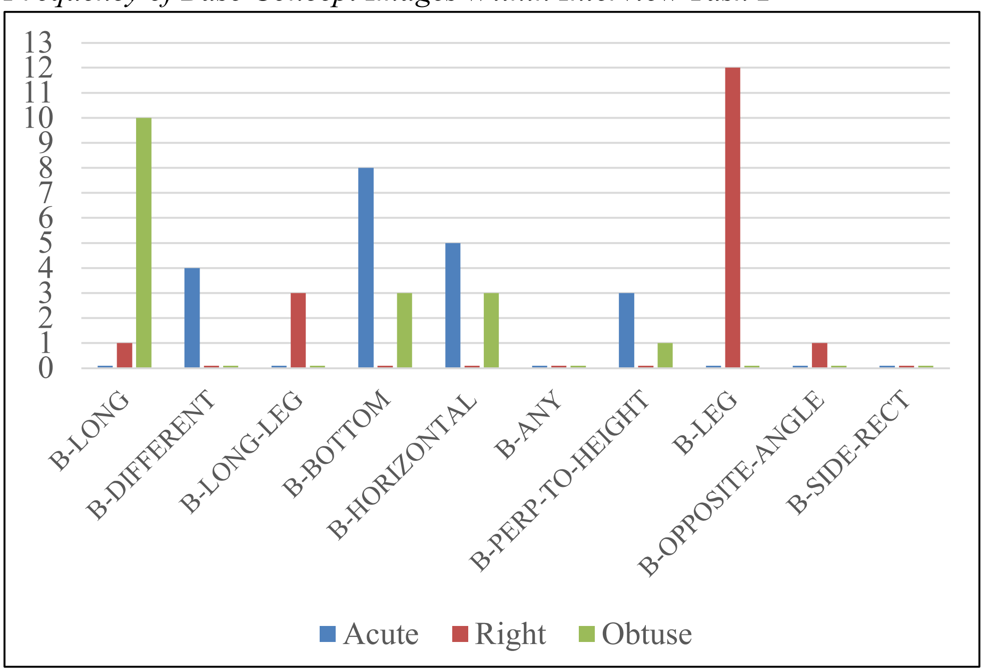
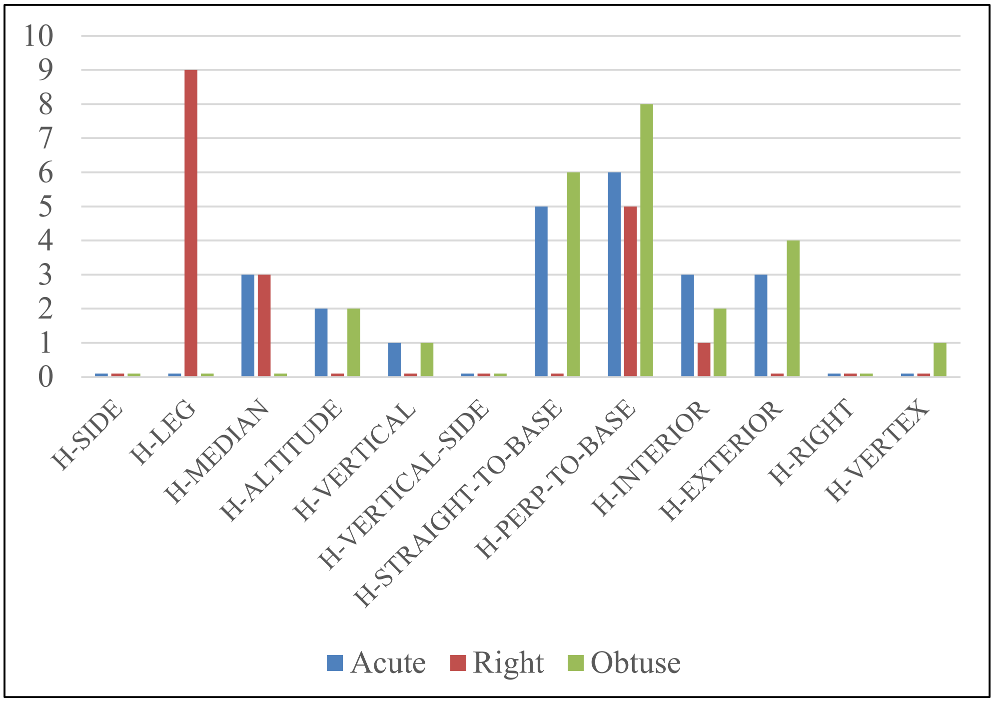


Figure 73

Frequency of Height Concept Images Within Interview Task 2



Within Figure 72 and Figure 73, you can see a broad perspective of the interviewees' concept images as compared to triangle type within Interview Task 2. For example, notice that participants seemed to evoke the H-LEG and B-LEG concept images for right triangles. You could also use Figure 72 and Figure 73 to focus on an individual concept image. For example, notice that there are many instances of the B-LONG concept image for obtuse triangles.

Next, I discuss the same data as shown in Figure 72 and Figure 73 but separated by interviewee. To clarify, in Figure 74 and Figure 75, I show the base and height concept images, respectively, for each interviewee, per triangle within Interview Task 2. During some interviews, I discussed some triangles a second time with some interviewees. Therefore, some triangles contain two concept image codes for either base or height. One of the benefits of displaying the information per interviewee is that you can see the evocations of individuals as they progressed through the tasks.

Figure 74

Evoked Base Concept Images Per Interviewee Across Task 2

		Odessa(CT10)	David(R26)	Lisa(H16)	Nate(R21)	Alyssa(R23)	Erica(H24)	Phoebe(CT11)
Acute	2.1a		B-BOTTOM	B-BOTTOM	B-PERP-TO-HEIGHT	B-BOTTOM	B-DIFFERENT	B-BOTTOM
	2.1b		B-DIFFERENT	B-HORIZONTAL x 2	B-PERP-TO-HEIGHT	B-HORIZONTAL	B-DIFFERENT	B-DIFFERENT
	2.1c		B-BOTTOM	B-BOTTOM	B-PERP-TO-HEIGHT	B-HORIZONTAL	B-DIFFERENT	B-BOTTOM
Right	2.2a	B-LEG	B-LONG-LEG	B-LEG		B-LEG	B-LEG	B-LONG
	2.2b	B-LEG	B-LONG-LEG	B-LEG		B-LEG	B-LEG	B-OPPOSITE-ANGLE
	2.2c	B-LEG	B-LONG-LEG	B-LEG	B-PERP-TO-HEIGHT	B-LEG	B-LEG	
Obtuse	2.3a	B-LONG		B-LONG		B-LONG	B-BOTTOM	B-LONG
	2.3b	B-LONG		B-LONG		B-HORIZONTAL	B-BOTTOM & B-HORIZONTAL	B-LONG
	2.3c	B-LONG		B-LONG	B-PERP-TO-HEGHT	B-HORIZONTAL	B-BOTTOM	B-LONG

Figure 75

Evoked Height Concept Images Per Interviewee Across Task 2

		Odessa(CT10)	David(R26)	Lisa(H16)	Nate(R21)	Alyssa(R23)	Erica(H24)	Phoebe(CT11)
Acute	2.1a	H-PERP-TO-BASE	H-ALTITUDE	H-STRAIGHT-TO-BASE	H-INTERIOR	H-EXTERIOR	H-PERP-TO-BASE	H-MEDIAN
	2.1b	H-PERP-TO-BASE	H-PERP-TO-BASE	H-STRAIGHT-TO-BASE & H-VERTICAL	H-INTERIOR	H-EXTERIOR	H-PERP-TO-BASE	H-MEDIAN
	2.1c		H-ALTITUDE	H-STRAIGHT-TO-BASE x 2	H-INTERIOR	H-EXTERIOR	H-PERP-TO-BASE	H-MEDIAN & H-STRAIGHT-TO-BASE
Right	2.2a	H-PERP-TO-BASE	H-PERP-TO-BASE	H-LEG	H-INTERIOR	H-LEG	H-LEG	H-MEDIAN
	2.2b		H-PERP-TO-BASE	H-LEG	H-INTERIOR	H-LEG	H-LEG	H-MEDIAN
	2.2c		H-PERP-TO-BASE	H-LEG	H-INTERIOR	H-LEG	H-LEG	H-MEDIAN
Obtuse	2.3a	H-STRAIGHT-TO-BASE & H-PERP-TO-BASE	H-ALTITUDE	H-STRAIGHT-TO-BASE	H-INTERIOR	H-EXTERIOR	H-VERTICAL	H-PERP-TO-BASE & H-VERTEX
	2.3b	H-PERP-TO-BASE	H-PERP-TO-BASE	H-STRAIGHT-TO-BASE & H-ALTITUDE	H-INTERIOR	H-EXTERIOR	H-PERP-TO-BASE x 2	H-STRAIGHT-TO-BASE
	2.3c	H-PERP-TO-BASE		H-STRAIGHT-TO-BASE		H-EXTERIOR x 2	H-PERP-TO-BASE	H-STRAIGHT-TO-BASE

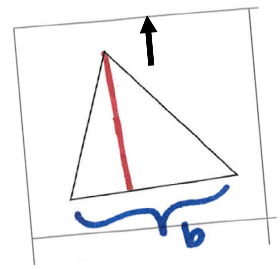
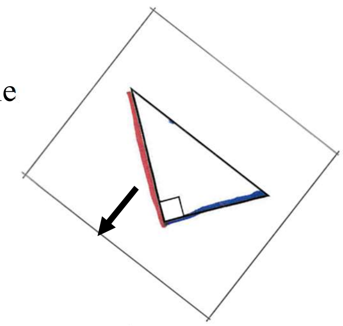
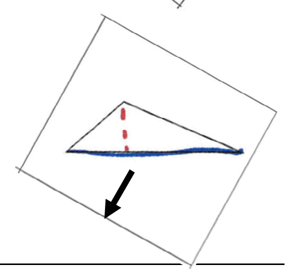
As can be seen in Figure 74 and Figure 75, Nate(R21) was the only interviewee who evoked a consistent base concept image for all triangle types. Both Nate(R21) and Odessa(CT10) seemed to have consistent concept images for height for all triangle types. All other interviewees evoked different concept images as the triangle types changed. These changes are evidence that some interviewees made a connection between triangle type and concept image for triangle base and height. In the following subsections, I share an example of a base-type and a height-type connection.

Example of Base Type Connection with Lisa(H16)

Lisa(H16) demonstrated a relationship between triangle type and concept image of base as she worked on Interview Task 2. In Table 13, I summarize the important moments in Lisa(H16)'s interview that demonstrated her making connections to triangle type.

Table 13

Lisa(H16): Connections Between Base Concept Images and Triangle Type

Task	Representative Transcript Excerpt	Concept Image (Code)	Lisa(H16)'s Work
2.1a	"I'm gonna call this one my base because it's where the … how the triangle is sitting."	Base as Bottom (B-BOTTOM)	
2.2b	"This first leg of the triangle is the base, and then the second leg is the height, and the third is the hypotenuse."	Base as Leg of Right Triangle (B-LEG)	
2.3a	"I'm gonna draw this as base, and then draw in my height inside my triangle, because sometimes it's [the base] easier to see … when it's [the height] inside than when it's outside."	Base as Longest Side (B-LONG)	

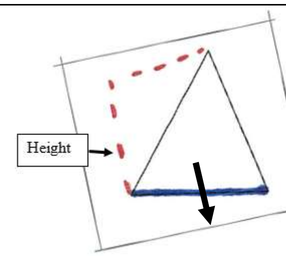
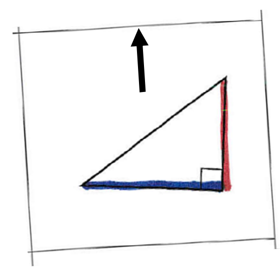
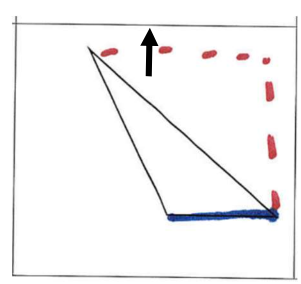
As Lisa(H16) transitioned between the triangle types, from acute to right to obtuse, her evoked concept images of base also changed. Because her concept image of base seemed to change by triangle type, I inferred that Lisa(H16)'s concept image of base was influenced by triangle type. Additionally, in Interview Task 2.2b, she described legs and a hypotenuse. Taken together, these three objects are triangle attributes of only one type of triangle—a right triangle. Unfortunately, I did not inquire about her meaning of her seemingly ordered "first" and "second" legs in Task 2.2b. Lastly, in Task 2.3a, her choice of base was motivated by her desire to have an interior height. This type of distinction between exterior versus interior heights only exists within one type of triangle—an obtuse triangle.

Example of Height Type Connection with Alyssa(R23)

Alyssa(R23)'s concept image of height changed as she progressed through Interview Task 2. In Table 14, I show an example of each type of triangle from Task 2.

Table 14

Alyssa(R23): Connections Between Height Concept Images and Triangle Type

Task	Representative Transcript Excerpt	Concept Image (Code)	Alyssa(R23)'s Work
2.1b	"For height, I'm gonna draw in the height outside the triangle to make it a right triangle."	Height as Vertical (H-EXTERIOR)	
2.2c	"Basically, all the right triangle ones [tasks] are relatively the same to me. Like, these two are like both the legs, and this is the hypotenuse, but like you could choose either one of these [legs] as your base or height [*tapped each leg with marker cap*]."	Height as Leg of a Right Triangle (H-LEG)	
2.3b	"I do it [draw the height] outside the triangle so it's more of a right triangle [*traced an imagined right angle in the exterior triangle with marker cap*] and this [triangle side] is a hypotenuse [*traced longest side of triangle with marker cap*]."	Height Exterior (H-EXTERIOR)	

Alyssa(R23) seemed to evoke H-EXTERIOR of the triangle was acute or obtuse, and H-LEG if the triangle was right. Therefore, I inferred that for Alyssa(R23), she was influenced by the triangle being right or not right—a connection to triangle type. For the acute and obtuse triangles, her process of drawing the exterior height involved the creation of a right triangle outside of the given triangle. Notice how Alyssa(R23) considered, in Task 2.2c, that either leg could be the base or height. This seems to be a different way of thinking as compared to Lisa(H16)'s first and second legs.

Chapter Summary

I organized this chapter by research question. To answer my first research question, I shared details about each of the triangle base, height, and area, concept images. Then, to answer my second research question, I found evidence of participants having related various concept images to one another. Lastly, I offered data towards my third research question, which emphasized connections among triangle orientation, gravity, and triangle type as related to the concept images of triangle base, height, and area.

CHAPTER V: CONCLUSIONS

It is important to examine students' interpretations of concepts relating to triangle area because of the prevalence of the area concept in school curriculum (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2018), the utility of the area concept outside of mathematics classrooms (Cavanagh, 2008; Hirstein et al., 1978), and the way that triangles can be used as a building block for many other area formulas (Lehmann, 2022; Ulusoy, 2020). To examine these important concepts, I addressed the following research questions:

1. What are high school geometry students' concept images of triangle base, height, and area?
2. How are high school geometry students' concept images of triangle base, height, and area related to one another?
3. How are triangle orientation, gravity, and triangle type, related to high school geometry students' concept images of triangle base, height, and area?

This study builds on the work of others who have used the idea of concept image to characterize students' interpretations of base, height, altitude, or area. Specifically, Vinner and Hershkowitz (1980) shed light on students' concept images of triangle attributes and other potential influential factors (i.e., triangle orientation, triangle type, and gravitational factor). Since then, other researchers have examined students' concept images of base (e.g., Herbel-Eisenmann & Otten, 2011; Horzum & Ertekin, 2018), altitude or height (e.g., Gutiérrez & Jaime, 1999; Krajcevski & Sears, 2019), and area (e.g., Tossavainen et al., 2017).

Although researchers have conducted studies related to students' concept images of triangles attributes, I have identified several reasons why it would be valuable for the field to conduct further investigations. Before my study, there had been relatively few studies at the high school level involving a focus on the understanding of triangle base, height, and area (e.g., Herbel-Eisemann & Otten, 2011; Kordaki & Balomenou, 2006). Hence, more research involving high school students was needed. Second, Horzum and Ertekin (2018) examined preservice teachers' evocations of base, whereas Gutiérrez and Jamie (1999) investigated preservice teachers' evocations of altitude. However, Horzum and Ertekin indicated that one of the base concept images that they identified was dependent on height, and Gutiérrez and Jamie described one of the altitude concept images as being related to base. Therefore, research was needed to examine both concepts simultaneously. Finally, researchers hypothesized that several factors (i.e., orientation, gravity, triangle type) may impact students' concept images of base and height (Vinner & Hershkowitz, 1980; Ward, 2004), but the research evidence for such influence was limited. Thus, this study was needed to extend existing research and to address these gaps in the literature.

To address my research questions, I created a survey with 10 questions as well as an interview protocol with five tasks to be used within a task-based interview. I administered the survey to 95 high school geometry students and interviewed seven of those survey participants. Also, I engaged in nine phases of analysis, each of which took me deeper into the survey and interview data.

For this study, I used the theoretical lens of concept image and concept definition (Tall & Vinner, 1981; Vinner & Hershkowitz, 1980) to focus on students' understanding of individual concepts. This lens was appropriate for this study because each concept of interest (i.e., triangle base, height, and area) has different components and because it seems that a variety of contextual factors may be related to students' understanding of those components.

Discussion of Findings

Based on the evidence from this study, I generated findings related to each of the three research questions. In this section, I summarize the major findings, compare the findings to related literature, and discuss the findings from a broad perspective.

Research Question 1: Concept Images of Triangle Base, Height, and Area

In this study, I extended the research about students' concept images of triangle base, height, and area. Some of that evidence was consistent with or extended existing research, but other evidence from the present study was inconsistent with the literature. In this section, I summarize evidence that was consistent with, extended, or was inconsistent with the a priori concept images that I formed for this study. Then, I synthesize the big ideas relating to the concept images of base, height, and area.

Consistent Evidence

Although I found evidence of participants evoking 28 unique concept images in this study, other researchers had previously identified and described 13 of them: three base concept images, six height concept images[12], and four area concept images. My findings relating to B-ANY, B-PERP-TO-HEIGHT, H-ALTITUDE, H-VERTICAL, A-STANDARD-FORMULA, and A-SIZE were consistent with other researchers' findings, as described in Appendix E.

Additional Evidence

The present study also extended existing research. Specifically, there were four extensions for the concept images: B-BOTTOM, H-MEDIAN, H-INTERIOR, and A-SQUARE-UNITS. First, the concept image of B-BOTTOM was described within three studies (i.e., Herbel-Eisenmann & Otten, 2011; Kadarisma et al., 2020; Vinner & Hershkowitz, 1980) with each set

[12] This is not a counting error. There was a seventh a priori concept image, H-PERP-BISECTOR, which I did not find within the present study. This is why I describe the count as six concept images of height that were a priori.

of researchers describing one of four distinct components (i.e., bottom-most side, side closest to person's body, side that is approximately horizontal, or the side that becomes horizontal after rotation). In this study, participants attended to exactly two of those components at a time, either bottom-most side and side closest to person's body or side that is approximately horizontal and the perceived need to rotate a triangle so that a side becomes horizontal. In other words, they seemed to be attending to the bottom side or horizontal (or almost horizontal) side, which caused me to create the concept image of B-HORIZONTAL to account for the latter pairing.

Second, Gutiérrez and Jamie (1999) only found evidence of participants evoking H-MEDIAN for obtuse triangles with exterior altitudes or right triangles with altitudes as legs. In addition to those types of evocations, the participants in my study also evoked H-MEDIAN for acute triangles with interior altitudes.

Third, Gutiérrez and Jamie (1999) and Vinner and Hershkowitz (1980) referred to concept images or interpretations that informed my initial definition of H-INTERIOR. However, Gutiérrez and Jamie (1999) described H-INTERIOR as a segment that connected the base to the vertex opposite the base. In my study, one participant evoked H-INTERIOR without connecting the interior height to the vertex opposite the base.

Fourth, Kamii and Kysh (2006) found that only 6% of Grade 8 students thought about breaking a row of square units, decomposing square units into partial units. However, in the present study, approximately 12% of survey participants demonstrated an understanding of the ability to count or estimate partial squares while evoking A-SQUARE-UNITS. Although not all of the participants in the present study were required to engage with thinking about square units in triangles, the larger proportion of Grade 9–10 students than Grade 8 students may indicate an advancement.

Inconsistent Evidence

For three concept images (i.e., H-SIDE, H-PERP-TO-BASE, and A-ALL-SIDES) the evidence from my study was inconsistent with existing research. For H-SIDE, Guitérrez and Jaime (1999) found only one preservice teacher (out of 190) evoked H-SIDE, but sometimes that participant selected a side to be the altitude that was pre-marked as the triangle base. None of the participants in my study classified a side simultaneously as a triangle base and height. Gutiérrez and Jaime (1999) had preselected a side as the base for the problems in their survey. In contrast, most of the survey questions and interview tasks in the present study (except for Interview Task 3) did not have preselected bases. It is possible that requiring participants to identify both a base and height helped them to interpret base and height as separate objects—thus the inconsistency regarding the H-SIDE concept image. Also, a potential influential factor relating to the inconsistency is that there was a different population of participants in Gutiérrez and Jaime as compared the present study. They worked with preservice elementary teachers whose mathematics class experiences may have differed from those high school geometry students in the present study.

For H-PERP-TO-BASE, Gutiérrez and Jamie (1999) and Krajcevski and Sears (2019) identified two distinct types of evocations of this concept image (see Appendix E). Instead, I found a third type of H-PERP-TO-BASE whereby the height is perpendicular to the base but terminates at a non-base side (i.e., not connected to the vertex opposite the base). Gutiérrez and Jamie's and Krajcevski and Sears' participants were preservice teachers which may account for the differences in our findings.

The evidence of A-ALL-SIDES from the present study was also inconsistent with the evidence in prior research. Specifically, Cavanagh (2008) found evidence of Grade 7 students multiplying all three side lengths to report a measure of area. However, the participants in my study evoked A-ALL-SIDES by multiplying all three side lengths together and then dividing or multiplying by two to report the area of the triangle. Again, sampling from different populations (i.e., middle versus high school students) may account for this inconsistency.

Students' Concept Images of Triangle Base, Height, and Area

The present study also shed light on how high school students perceive base, height, and area. All of the base concept images involved a participant selecting a triangle side, indicating that all of the participants in this study perceived the concept of base to be a side of a triangle. In contrast, only three height concept images, H-SIDE, H-LEG, and H-VERTICAL-SIDE, involved a participant selecting a triangle side as the height. Instead, all of the height concept images involved a participant either implicitly or explicitly identifying a segment with one endpoint on the line containing the base. In other words, all of the participants in this study perceived the concept of height implicitly or explicitly as an object that has a relationship to the line containing the base.

For six of the 12 concept images for height (i.e., H-LEG, H-ALTITUDE, H-EXTERIOR, H-STRAIGHT-TO-BASE, H-PERP-TO-BASE, and H-RIGHT), participants explicitly or implicitly referred to a right angle or perpendicularity. Sometimes that right angle connected the height to the base, as evidenced by drawing a segment that they referred to as the height that was approximately perpendicular to the base—with or without drawing a right-angle symbol. To me, this indicates that participants evoking H-LEG, H-ALTITUDE, H-EXTERIOR, H-STRAIGHT-TO-BASE, H-PERP-TO-BASE, and H-RIGHT may recognize that a right angle or perpendicularity is important, but why or how may not be clear. Although it was possible for participants to evoke the H-MEDIAN, H-VERTICAL, H-VERTICAL-SIDE, or H-INTERIOR while implicitly referring to a right angle or perpendicularity, some participants did not.

As for area, results from the present study indicated that most high school students perceive area to be a numerical value. Participants who evoked five of six area concept images (A-STANDARD-FORMULA, A-ALL-SIDES, A-HERONS, A-DECOMPOSITION, and A-SQUARE-UNITS) reported a numerical value. The area concept image A-SIZE involved an interpretation of area without any relationship to a numerical measurement value. All participants who evoked A-SIZE also evoked one of the other area concept images. This may be an indication that high school geometry students often view area as a numerical value. The area concept image A-DECOMPOSITION always involved the implicit or explicit acknowledgement of how area is conserved followed by an attempt to calculate each decomposed part using the standard area formula. When the area concept image A-SQUARE-UNITS was evoked by participants, it was always preceded by A-STANDARD-FORMULA.

Research Question 2: Dependency and Perpendicularity in Relating Base and Height

In Chapter 4, I reported that some participants considered triangle base to be an object that depends on height, but a greater number of participants considered height to be an object that depends on base. The greater frequency of height depending on base may be caused by the simplicity of the concept definition of base (i.e., any triangle side). In contrast, the concept definition of height requires attending to multiple ideas including a perpendicular connection to the line containing the base. It is also possible that the fact that base comes before height in the standard formula for triangle area influences students' perception that the identification of base precedes the identification of height.

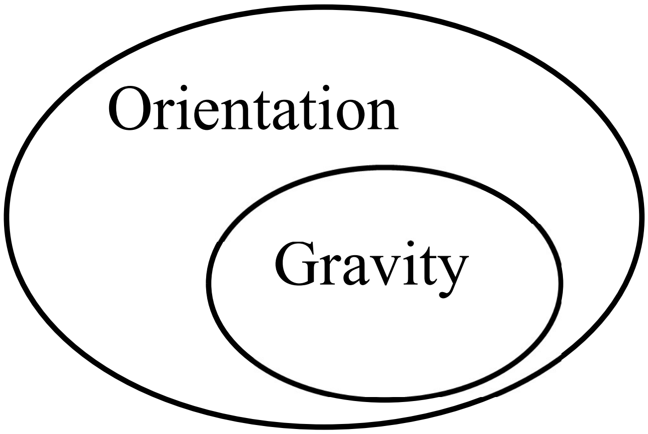
Based on the results of the present study, I infer that perpendicularity seems an important component that students use to relate base with height. First, perpendicularity seems subtly related to the perception of horizontal and vertical. For a triangle side to appear horizontal to a person, it needs to be oriented so that it is perpendicular to a plane containing the body's midline. Second, most participants either drew right angle symbols connecting base and height—with or without the base and height being perpendicular—or they positioned the height to be approximately perpendicular to the base. Drawing a right-angle symbol when a height meets with the line containing the base obliquely may indicate a lack of understanding of right angle. For example, recall from my discussion of the H-MEDIAN concept image, when H31 and Alyssa(R23) drew the median in Survey Question 7 and Interview Task 3a, respectively. They drew right-angle symbols where the median intersected the base, even though the height was visibly oblique. Other researchers who studied preservice elementary (Gutiérrez & Jaime, 1999) and middle school (Krajcevski & Sears, 2019) teachers' concept images also identified perpendicularity as an important component of triangle altitude.

Research Question 3: Influential Factors on Concept Images

Vinner and Hershkowitz (1980) hypothesized that the orientation of angles, gravitational factor, and triangle type may influence students' perception of angles and triangle altitudes. I found examples of all three types of influences relating students' concept images of triangle base and height. Although I do not claim generalizability, all examples of gravity that I found in the present study involved orientation of the triangle. However, not all examples of triangle orientation seemed to involve the gravitational factor, indicating that orientation as an influential factor may include gravity as a subset (see Figure 76).

Figure 76

Nested Relationship of Gravity and Orientation



Besides gravity, students seemed to either rotate triangles for ease of drawing or to place the triangle in a prototypical orientation with a horizontal base from their perspective. To clarify, I am inferring that it is possible to rotate a triangle to make the base approximately horizontal without imagining the triangle falling due to gravity.

In Chapter 4, I reported that most interviewees seemed to make a connection between concept images of base (with Nate(R21) being the exception) and height (with Nate(R21) and Odessa(CT10) being the exceptions) with triangle type. Also, some concept images seemed to be related to triangle type (e.g., B-LEG seemed to relate to right triangles) and some concept images seemed to appear in relatively equal frequency across all three triangle types (e.g., H-PERP-TO-BASE). I infer that some concept images are more attribute-focused and therefore do not tend to be impacted by triangle type. For example, with the H-PERP-TO-BASE concept image, students are focused on the formation of a perpendicular connection between the height and the line containing the base. This formation can occur regardless of the triangle type.

Discussion of Theoretical Lens: Concept Image and Concept Definition

In this section, I discuss some affordances and drawbacks of the theoretical lens of concept image and concept definition based on my experience from this study. Vinner and Hershkowitz (1980) described the lens as beneficial for investigating individual concepts with multiple components. I found this to be the case during my investigation, especially for the concept of height. For example, prior to the present study, I thought that students would either consider the height as perpendicular to the base or not. Because of the focus on the multiple components of height with the lens of concept image and concept definition, I found that when some students are thinking about height, they recognize that identifying a right angle is important, but not necessarily at the intersection of the height and the line containing the base (i.e., H-RIGHT).

Another benefit of the lens was related to the idea that concept images may contain some components of the concept definition (Viholainen, 2008). This helped me in my analysis of students' thinking because I was able to focus on which definition components were present and which components were absent in their evoked concept images. For example, when a student described the height as straight to the base, I was able to recognize that they were connecting the height to the base, but not attending to the perpendicularity between height and base. The base concept definition only had one definitional component—the base is any side. Because all participants selected sides as base, it could be argued that all concept images of base included that component. However, the height concept definition included four definitional components: (a) connection to the vertex opposite the base, (b) connection to the line containing the base, (c) existence of a right angle, but not necessarily joining the height and base; and (d) formation of a right angle at the intersection of the height and line containing the base. Perpendicularity between base and height seemed most prevalent and identifying a right angle not connected to the base seemed least prevalent.

One of the potential drawbacks to the lens is that it does not provide tools for categorizing types of thinking based on the extent to which the concept image is aligned with the concept definition. Although it may seem as simple as determining if students' work is correct or incorrect, I believe that effectiveness rating may require a more robust framework. Perhaps this framework could include consideration of how many components each concept image has compared to its corresponding concept definition or could include consideration of which triangle types would work well with each concept image when the learner intends to use the identified base and height to determine the triangle's area measure.

I found myself wondering if some concept images of triangle base or height might allow students to correctly find area more often than other concept images. For example, I can infer that B-LONG might be more beneficial than B-HORIZONTAL because choosing the base as the longest side will always cause the triangle altitude to be in the triangle's interior. However, B-HORIZONTAL can sometimes create non-interior altitudes for obtuse triangles (i.e., exterior or altitude as a right triangle leg). Notice that this inference being mathematically accurate does not imply that B-LONG is somehow better than B-HORIZONTAL.

Limitations

The present study contained limitations involving the survey and interview instruments. These limitations made it challenging at times to determine participants' concept images of base, height, and area. They also impacted my ability to fully answer Research Questions 2 and 3 regarding relationships among the concept images and other influential factors.

First, sometimes it was challenging to determine participants' concept images of base and height, particularly in Survey Questions 4, 5, and 7. For those questions, participants only had to draw base and height but did not need to explain their work. To conserve time during the survey, I limited the number of survey questions that asked participants to explain their thinking in writing, but written responses provided the richest data for analysis. Furthermore, because students were not recorded while taking the survey, I was unable to determine if participants rotated their papers while completing it. Hence, I was unable to make many claims relating to triangle orientation and gravitational factor on participants' concept images. It was also difficult to determine the impacts of triangle type from the surveys due to the unequal distribution of triangle type (i.e., one acute, one right, and four obtuse). To clarify, I found it easier to make claims about the influence of triangle type in Interview Task 2 where I had three triangles of each type and where participants explained their thinking as they worked.

Second, for Survey Questions 9 and 10, participants often did not describe their interpretations of base, height, and area. Rather, they often just described their selection of base and height in vague terms such as "I used the 12 cm as the base and the 5 cm as the height," and then they used those measures to calculate area. I believe that my direction to "describe your method" was, perhaps, too vague. My original intention was to avoid requiring participants to identify a base and height because it was possible that they may use some other method to find area (e.g., Heron's Formula). I think a different phrasing of instructions could have improved the kinds of responses that I received. For example, phrasing could be changed to, "Describe each aspect of your method," or "If you used an area formula, describe how to identify each component of that formula in the triangle." I would hesitate to directly ask students to describe base and height in the area-finding tasks because that may limit students' thinking to the standard area formula.

During interviews, I sometimes found it challenging to identify interviewees' concept images if they did not verbally describe what they were drawing. At times, I paused them to ask clarifying questions, but I felt like I had to find a balance between allowing the interviewees adequate time to produce a solution that reflected their own thinking without interrupting their train of thought by asking them to pause and share their thinking. This attempted balance was also constrained because I had to limit interviews to 45 minutes, as I had described on the informed consent and assent forms. Often, during Task 2, interviewees would begin drawing and saying something like "This is the base, and this is the height," without describing them any further. I think I could have identified more evocations of concept images if I had paused students more often and encouraged more think-aloud descriptions (e.g., Dursken et al., 2021).

Third, the honors geometry teacher requested that we use a reduced time of 10 minutes for the survey because she had a lot of content to cover on that day. I asked her if she would prefer that I come back another day to conduct the survey and she told me that there was no other possible day to do this. Therefore, I administered the survey with a reduced time of 10 minutes, leaving 23 participants insufficient time to complete all survey questions. Some of those 23 participants did not answer the first three questions, which were more writing intensive, and some did not answer the last few questions. This incomplete information may have affected my determination of new concept images and which concept images occurred most frequently.

Fourth, in the present study, I revised my concept image code descriptions. I met with a researcher who had expertise in geometry, coding, analyzing qualitative data, and measurement-related topics. We discussed a subset of the data to examine and clarify base and height code definitions. Although this code discussion was valuable, we did not examine the reliability of my coding, such as calculating intercoder reliability for an adequate portion of all codable data. In other words, I cannot make claims regarding reliability, including the extent to which other researchers may be able to replicate my findings or conduct a similar study using concept image descriptions from the present study (cf. Goldin, 2000).

Recommended Directions for Future Research

In this section, I present three recommendations for future research. Some of these recommendations come directly from the limitations of the present study.

Recommendation 1: Modifying Sample, Methods, and Instruments

The specific sample for the present study creates an opportunity for researchers to modify the sample in one or more of the following ways: (a) larger sample size, (b) include high school students from different math classes (e.g., Algebra 1), (c) study a group of students both before and after instruction on area, (d) sample students from multiple schools, and (e) specifically seek to include students with learning or other disabilities.

I also recommend modifying the interview process. Due to the richness of data produced by the interviews, I believe it would be beneficial to interview each participant twice. I believe that both interviews could be designed to be at most 30 minutes in length by including Tasks 1, 2, and 3 in the first interview, and then beginning the second interview with any clarifying questions about those tasks followed by Tasks 4 and 5. Interview Task 2 of the present study provided a rich opportunity for discussion of concept images for base and height, and Interview Task 4 provided rich opportunities for students to relate concept images of base and height with their concept image of area.

I also recommend amending the survey instrument to allow more opportunities for students to describe their thoughts in writing. Specifically, I would delete Survey Question 4 and 8 and add an additional requirement of a written explanation of the bases and heights for the remaining drawing items (i.e., Survey Questions 5–7). After deleting Survey Question 4, there would still be one acute, right, and obtuse triangle. These written explanations would negate the need for Survey Question 8.

Recommendation 2: Extending Findings from Present Study

I found evidence that although some students think base depends on height, a greater proportion of students think that height depends on base. Additionally, some students think of base and height simultaneously with both being joined at a right angle. This creates an opportunity for future researchers to create a study that focuses on the idea of dependency between triangle height and base in the context of finding triangle area. Because of the large proportion of students who evoked A-STANDARD-FORMULA, care should be taken if researchers wish to examine linkages beyond the standard area formula. For example, researchers may consider designing tasks with gridded triangles (e.g., Barrett et al., 2017) or tasks that involve filling triangles with square units (e.g., Reynolds & Wheatley, 1994).

Also, researchers may consider examining how evocations of concept images of triangle base and height were associated with correct triangle area calculations compared. It is possible that students are more successful at finding triangle area measures if they evoke some concept images compared to others.

Recommendation 3: Influence of Prior Tasks

Third, although modifying students' concept images was not the focus of the present study, I found evidence that some participants' evoked concept images may have been positively influenced by prior tasks they engaged with. I generally found evidence of this positive influence when interviewees progressed through Tasks 2–4. For example, it seemed that some interviewees drew incorrect altitudes in obtuse triangles of Task 2, correctly identified the exterior altitude in Task 3, and then drew and measured a correct altitude in Task 4. It is possible that the different height options in Task 3c helped the interviewees to evoke different concept images as they progressed through the interview. This type of modification of concept images of triangle base and height has not yet, to my knowledge, been researched.

Likewise, I have not seen research involving the influence of geometry textbooks, school curriculum, or teaching techniques on students' concept images of triangle base, height, and area. Researchers (e.g., Vinner & Hershkowitz, 1980) have hypothesized that geometry textbooks contain several prototypical images of triangles and that these images may influence students' concept images.

Implications for Researchers and Practitioners

In this section, I discuss implications of my study for both researchers and practitioners. I designed this study with the hope that it would have utility both theoretically and practically. I begin by discussing an implication that applies to researchers, then some implications that apply to both researchers and practitioners, and lastly an implication for practitioners.

Implication for Researchers

Because every interviewee evoked A-STANDARD-FORMULA within their interviews, researchers may consider the impacts of the timing of the present study. It is possible that students may have evoked different area concept images if the study were conducted prior to instruction about area. Likewise, it is possible that students from a different class (e.g., Algebra

1) may have evoked different concept images, or evoked concept images with different frequency as compared to the students in the present study.

Implications for Both Researchers and Practitioners

Through the analysis of the survey and interview data, I found that questions and tasks that required explanation tended to improve my ability to identify students' evoked concept images. In contrast, it was challenging, at times, for me to identify evoked concept images when questions only required students to draw a base and height. Gutiérrez and Jaime (1999) identified six concept images of altitude by giving a written test to 190 preservice teachers and generated a category called "other incorrect responses" (p. 267). Although they did not discuss that category, I infer that they would have been able to generate additional concept images from those incorrect responses if they had the ability to ask clarifying questions to the participants or if they had required the participants to explain their ideas in writing.

The benefits of explanations, whether written or spoken, serve as an implication for both researchers and practitioners. Researchers should carefully consider these benefits when creating tasks for future studies—especially those involving the concept image and concept definition lens. Having opportunities for written or spoken explanations can provide rich collections of data for analysis.

My observations about questioning methods may also benefit researchers and practitioners. The survey questions and interview tasks may help generate ideas for classroom examples and assessment items for triangle concepts related to area (e.g., identification of base and height). For example, teachers may consider asking students to identify base and height in a variety of triangles with a variety of orientations, like the nine triangles in Interview Task 2. If giving Interview Task 2 as a written assessment, I recommend that teachers include an opportunity for students to provide written explanations or consider the possibility of assessments involving spoken explanations from students for some or all their base and height drawings.

Another implication for researchers and practitioners is related to a warning from Leder and Gunstone (1990) about the potential negative impact of using the phrase *misconception* to describe students' constructed meaning. I encourage both researchers and practitioners to consider how I focused on participants' evocations of the concepts, as opposed to simply judging their work as correct or incorrect. I hope that this may serve as an example of how to analyze students' interpretations of concepts in one moment of time or as they are developing.

From my personal experience as a high school mathematics teacher, the existence of many concept images that have multiple components has opened my eyes to the possibility that students may only have partial understandings of many mathematical concepts. This has, therefore, influenced the kinds of discussions that I have in class each day about a variety of concepts. Now, during classroom discussions, I am less focused on whether a student's response is right or wrong, and more focused on ascertaining their evoked concept images through a sequence of tasks and questions using a variety of triangle types in different orientations. For example, when a student evokes a response that is missing some elements of a concept definition, I try to uncover students' concept images and help them move closer to the concept definition by asking probing questions that involve a variety of triangle types and orientations. In agreement with Vinner (1991), I recommend that teachers facilitate classroom discussions in such a way as to draw attention to components of students' evoked concept images and allowing time for students to realize which of those components are important for that concept.

Finally, both researchers and practitioners should consider the collection of concept images of triangle base, height, and area when teaching students about triangle area or when studying students' understanding of triangle area or triangle attributes. The concept image collection may impact researchers' task creation for a wide range of studies involving triangle base, height, and area. Specifically, they may design tasks to draw attention to the less common concept images by creating hypothetical student responses and ask participants to analyze those responses, or they may create new versions of Interview Task 3 with new auxiliary lines relating to a wider variety of concept images. Teachers may benefit from this collection of concept images as they pose questions to students during lessons or in assessments—they may be able to improve their ability to identify students' concept image and to help them modify their concept images to become closer to the concept definitions.

Closing Remarks

The present study confirmed and extended existing research regarding students' concept images of triangle base and height. I accomplished this by finding evidence of existing concept images and nearly doubling the concept image collection by identifying new ones. Also, this study is the first of its kind to consider relationships between triangle base, height, and area and to identify student-formed dependencies between base and height. Lastly, the current study extended research that hypothesized the impacts of orientation, gravity, and triangle type. I found gravity to be a subset of orientation, and I found evidence that some students' concept images are impacted by triangle type.

I believe that students' understanding of mathematical concepts goes well beyond correct or incorrect. Additionally, I believe that it is important to focus on identification of the components of students' evocations of concepts and how we might build on their interpretations to help shift their concept images closer to the concept definitions. I will be forever changed as a teacher and researcher because of this dissertation experience.

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APPENDIX A: CONSENT AND ASSENT LETTERS AND FORMS

Geometry Study about Triangles

Date: 11/10/2018

Dear Parents and Guardians:

Your son or daughter is being invited to participate in a very short (10 minute) survey relating to some of the mathematical concepts related to triangles. Some of the students who are surveyed will be invited to be interviewed at a later date. This research is being conducted by Bradley P Heller, who is both a mathematics teacher at the high school in the Midwest Region of the United States
and doctoral student at Illinois State University. The research is being overseen by Dr. Tami Martin, a professor of mathematics education at Illinois State University.

The purpose of this research is to help us gain a better understanding about how students understand and think about mathematical concepts relating to triangles. The survey will take no more than 10 minutes of class time. This survey will be given at the start of a class. Regardless of participating in the study or not, all students will complete the survey as a classroom activity. Of all the students who are surveyed from among all the classes, a total between five and ten will be invited to participate in a follow-up interview.

We do not anticipate any risks more than those minimal risks encountered in everyday class activities. Possible risks include, for example, anxiety over having to perform some calculations by hand, or the possibility of students having me as a teacher in a future mathematics class and feeling intimidated in some way.

In order to minimize these risks, I will encourage students to do their best. Also, I will encourage students by letting them know that I care more about what they think and how they think rather than if they are correct or incorrect. Lastly, I will let students know that I am the only one who will know their individual survey (and interview if applicable) results. I will not share any of their direct results with their teacher. The above risks are minimal, however, there are likely to be benefits.

It is likely that we will gain valuable information about how students think about triangles, and we will then be able to share a summary of this information with the teachers in an anonymous way that protects the identity of each student. Teachers will better understand how students think about triangles, and those teachers may be better able to communicate with the students.

Participation is completely voluntary. If either you or your son/daughter should decide not to participate, or to withdraw from the survey at any time, there will be no penalty or loss of potential benefit to that student. Students may decide to withdraw from the survey at any time.

Please consider granting permission for your son/daughter to participate in the geometry survey. We greatly appreciate your consideration of this matter.

Full confidentiality will be maintained during this research. Teachers will not ever be made aware of individual survey results. If you have any questions, please feel free to contact Brad Heller or Dr. Tami Martin at the emails listed below.

Sincerely,

Dr. Tami Martin
Professor of Mathematics Education
Department of Mathematics
Illinois State University
Normal, IL 61761
tsmartin@ilstu.edu

Brad Heller
Doctoral student at Illinois State University
Department of Mathematics
Illinois State University
Normal, IL 61761
bphelle@ilstu.edu

Parent/Guardian Consent Form

I have read the information presented in the attached letter and have had an opportunity to ask questions and receive answers pertaining to this research project. I am aware that my permission is voluntary, and that I am free to withdraw my permission at any time without any penalties to my child or me.

Do you give permission for Brad Heller and Dr. Tami Martin to …
(please check either yes or no to each)

Yes	No	Permission Request
______	______	Conduct a short survey in my child's geometry class?
______	______	Conduct up to two video-recorded interviews with my child at a time agreeable to both my child and Brad Heller?
______	______	Use short video clips that include my child during conference presentations for scientific purposes only?

Please initial below ONLY if you DO NOT wish to grant permission for your child to participate.

I **do not give permission** for my child to participate in any aspect of the data collection for this study.

Child's Name (print clearly)

Child's Geometry Teacher (print clearly)

Signature of Parent/Guardian

STUDENT ASSENT FORM – MATHEMATICS SURVEY

January 18, 2019

Dear Student:

You are being invited to participate a research project, *Students' Concept Image of Triangles Attributes* (IRB #2018-632). This project serves the purpose of both expanding the knowledge base of the mathematics education research community, and also serves as one of the major parts of a dissertation study for Brad Heller at Illinois State University. The purpose of this research is to help us gain a better understanding about how students understand and think about mathematical concepts relating to triangles.

Sometime soon in class, all students will be competing a classroom assignment. We are hoping to obtain your permission to analyze the results of your assignment. Of all the students who complete written assignment from among all the classes, a total between five and ten will be invited to participate in a follow-up interview. If a student is invited, and then completes the interview, he or she will be compensated with a $10 gift card to a coffee shop in town. My goal is to capture all of the necessary information inside of one interview. On a very rare occasion, I may request a follow-up interview.

Interviews will take approximately 45 minutes, and will take place either before or after school in a classroom or in the library. I will be using two cameras and one audio recording device to serve as a backup for the cameras. One camera will be positioned behind the student and looking over their shoulder at the papers that we are working on. The other camera will be positioned on other side of the student facing their front side, and again aiming down towards the student's paper. It is possible that the cameras and audio recording devices may capture identifiable information such as students' faces, hands, hair, and first names. I may show some of video or audio segments at presentations for educational purposes only (e.g., dissertation defense, or educational conferences). If and when this happens, I will use fake names (i.e., pseudonyms) to help protect the identity of each student. Although the focus of the video segments will be on the mathematical work that we are doing, these segments may contain brief images of students' faces, hands, and hair. Therefore, this will allow for the unlikely possibility someone in the audience recognizing the student being interviewed.

Your mathematics teacher will not be made aware of who is opting in or out of the study, and therefore your choice to participate or not will have no effect on your grade. There will be no anticipated changes to your typical classroom experience as a result of the study.

The risks to participating in this research project are no greater than the risks associated with everyday life. The most likely risk you would be exposed to is the potential loss of confidentiality in the event that a video or sample of your work is shared with other educators. To minimize this risk we will never use your real name or school name during educational presentations or in publications. However, your first name might be heard on a video excerpt.

You will be given a copy of this assent form for your records. Your participation is voluntary. You have the right to withdraw at any time without any penalty or negative consequences to your grade.

If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Research Ethics & Compliance Office at Illinois State University at (309) 438-5527 or via email at rec@ilstu.edu.

You are ineligible to participate if you are currently within the European Economic Area.

Please return the permission request form to the researcher as soon as possible.

If you have any questions, or want additional information, please call or write to any of us. Thank you for your participation and your interest.

Sincerely,

Tami S. Martin
Professor
Department of Mathematics
Illinois State University
Normal, IL 61790-4520
(309) 438-7864

Brad Heller
Doctoral student at Illinois State University Department of Mathematics
bphelle@ilstu.edu

STUDENT ASSENT FORM

__________________________ ________________________ _________________________

Student Name Teacher Name School Name

Agree to Participate
I sign below that I give my **assent (permission)** to participate in the research study. This permission will allow the researchers to make photocopy of my work on this one assignment for analysis. Furthermore, I understand that I may be invited to be interviewed at a later time and can choose to participate in that or not at a later time. ____________________________ ____________________________ Printed Name Email address ____________________________ Mobile phone number ____________________________ ____________________________ Signature Date
By initialing below, I *also give my permission* that IF I am interviewed, to allow video excerpts from the recorded interview to be shown at academic conferences, classes, and for other educational purposes. Every attempt will be made to exclude identifying information, such as my name or school name. However, first names may be heard on video clips. Initial ________ Date __________

APPENDIX B: TRIANGLE SURVEY

Directions: Please write a brief response in a sentence or phrase for each question below.

I am more interested in your thoughts and ideas rather than if you are right or wrong.

1. Think of the word **area.**
   a. How would you find the **area** of a triangle?

   b. What does the word **area** mean outside of the math class?

   c. Draw a triangle and label the **area** of a triangle.

2. Think of the word **base.**
   a. How would you find a **base** of a triangle?

   b. What does the word **base** mean outside of the math class?

   c. Draw a triangle and label a **base** of a triangle.

3. Think of the word **height.**
   a. How would you find a **height** of a triangle?

   b. What does the word **height** mean outside of the math class?

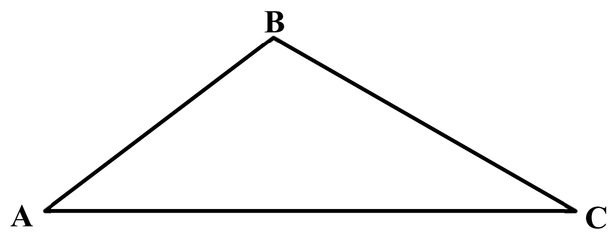
   c. Draw a triangle and label a **height** of a triangle.

I am trying to explain to my students how to find the bases and heights of triangles.

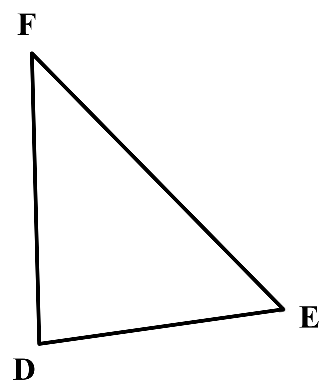
Please identify a base, and a corresponding height in each triangle.

Draw any needed segments and **label** both the base and height.

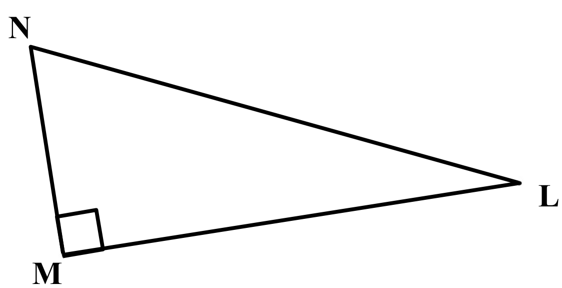
4.



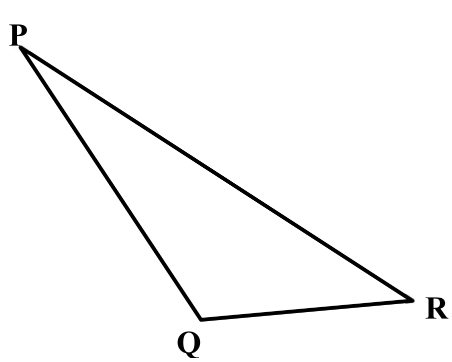
5.



6.



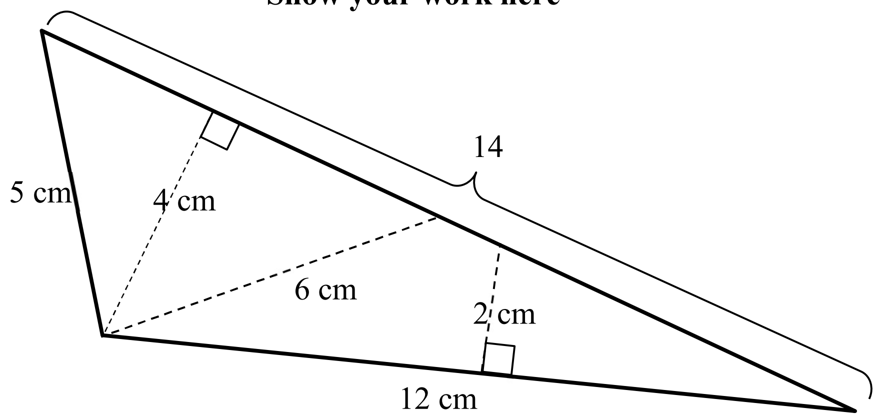
7.



8. Suppose you are helping a friend on the phone with problem 6. What would you say to help them find the base and height?

9. IF POSSIBLE, find the area of the triangle below. Area: _____________

Show your work here



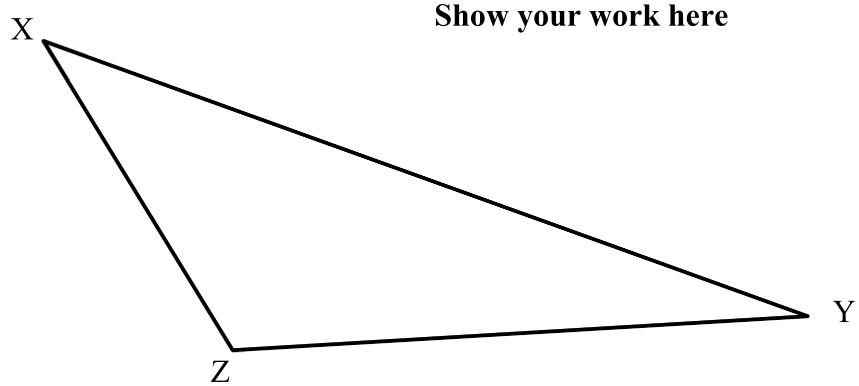
Explain your method here

- Describe your method.
- Which measurements did you use and why?

10. You may use any available tools:
Ruler, extra copies of triangle, square centimeter graph paper, calculators, scissors

a. Find the area of the triangle below. Area: _____________

Show your work here

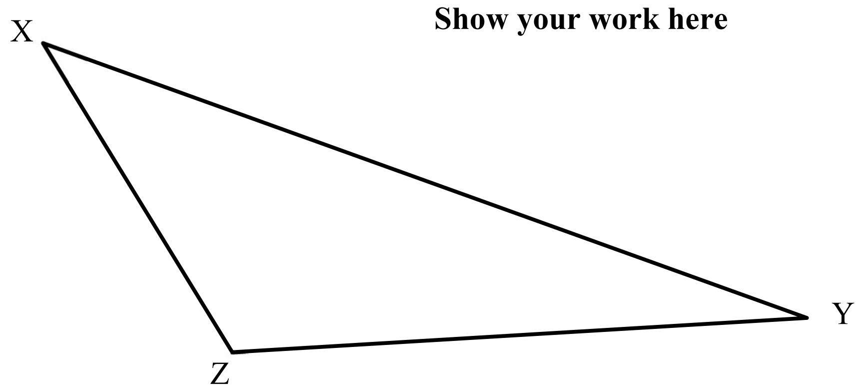


Explain your method here

- Describe your method.
- Include which tools you used.

b. If possible, find the area of the triangle using a **different method**. Area: _____________

Show your work here



Explain your method here

- Describe your method.
- Include which tools you used.
- If not possible, explain why.

APPENDIX C: PRE-INTERVIEW AND STUDENT VERBAL ASSENT PROTOCOL

Pre-Interview Protocol

I began each interview using the following pre-interview protocol:

1. Greetings statement: "Hi, my name is Brad. What's your name?

2. Consent with brief statement about research:
    a. "Today, I would like to talk with you about triangles for about 45 minutes. This is part of a research project designed to help me understand how students think about triangles' attributes."

    b. "Today I am going to ask you a series of questions and record our conversations on this audio device (researcher points to the recording device on the table) and these two cameras (Researcher points to the two video cameras that are set up—one behind the student aiming over his or her shoulder, and one on the other side aiming toward the desk space)."

    c. "The risks involved in this project are no more than what you experience in class on a daily basis. However, I believe that you may benefit by deepening your knowledge about triangles, and triangle area. This is a topic that you will be studying in geometry coming up soon."

    d. "I will keep your name confidential by using a pseudonym, which means fake name, in my writing as well as if I show parts of this conversation at research presentations.

    e. "You have a right to full information regarding the project. If you would like to talk to either me or my dissertation supervisor, Dr. Tami Martin, about this project please feel free to contact either of us. In addition, if you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Research Ethics & Compliance Office at Illinois State University at (309) 438-5527 or via email at rec@ilstu.edu. Here is a paper with our contact information (researcher hands the student a half-sheet of paper with the contact information)"

    f. Your participation is voluntary and you have the right to refuse to participate without any penalty or loss of benefits. You have the right to withdraw at any time without any penalty or negative consequences to your grade."

3. "Okay, let's begin the interview."

APPENDIX D: INTERVIEW PROTOCOL AND TASKS

After completing the pre-interview protocol and obtaining each participant's verbal assent, I began the interview. The following lists contain the interview protocol for the present study. After the interview protocol, I share the interview tasks.

Introduction

- Begin Recording
- Greetings statement: "Hi, my name is Brad. What's your name?
- Demographic questions:
    - "So, what grade are you in this year?"
    - "Are you enjoying your school year so far?"
    - "What are your favorite classes?"
    - "Are you in any sports or clubs?"

- Consent with brief statement about research:
    - "Today, I would like to talk with you about triangles for about 30 to 40 minutes. Is that okay?"
    - "Would it be okay if I recorded our conversation today?"
- General overview
    - "During this interview, I am going to ask you a bunch of questions about triangles and area. Please know that I am very interested in how you think, and less concerned about right or wrong answers."

Task 1: Student Survey

- The main purpose of this task is to clarify and confirm various results from the survey.
- Refer to the student's survey questions
    - Ask if there were any questions that they were unsure about (i.e., guessed at)?
    - If I observed any inconsistencies in descriptions of concepts, ask students to clarify their strategies or ideas.
    - Ask clarifying questions about their survey responses
        - E.g., "Why did you decide to choose this base and this height to calculate area?
        - E.g., "Hey, I was really interested in the way you drew this height here… can you describe to me why you drew it this way and not another [researcher slides finger along another possible path for the height]?

            -

Task 2: Base-Height Creation

- The main purpose of this task is to help me understand students' concept images of triangle attributes for different triangle types and orientations.
- Give the student a red and blue marker and ask them to use red marker to draw a base and blue marker to draw a height in each triangle.
- As the student begins to draw, pause and ask some clarifying questions about why they chose what they chose.

Task 3: Height Identification

- The difference between this and Task 2 is creation versus identification. It could be possible that students can recognize a height when they see one.
- Show the student Task 3, and ask them to identify the height for each chosen base.
- Use this as an opportunity to ask "what if" type questions (e.g., "what if this other side was a base, what would the height be now?").

Task 4: Area of a Triangle

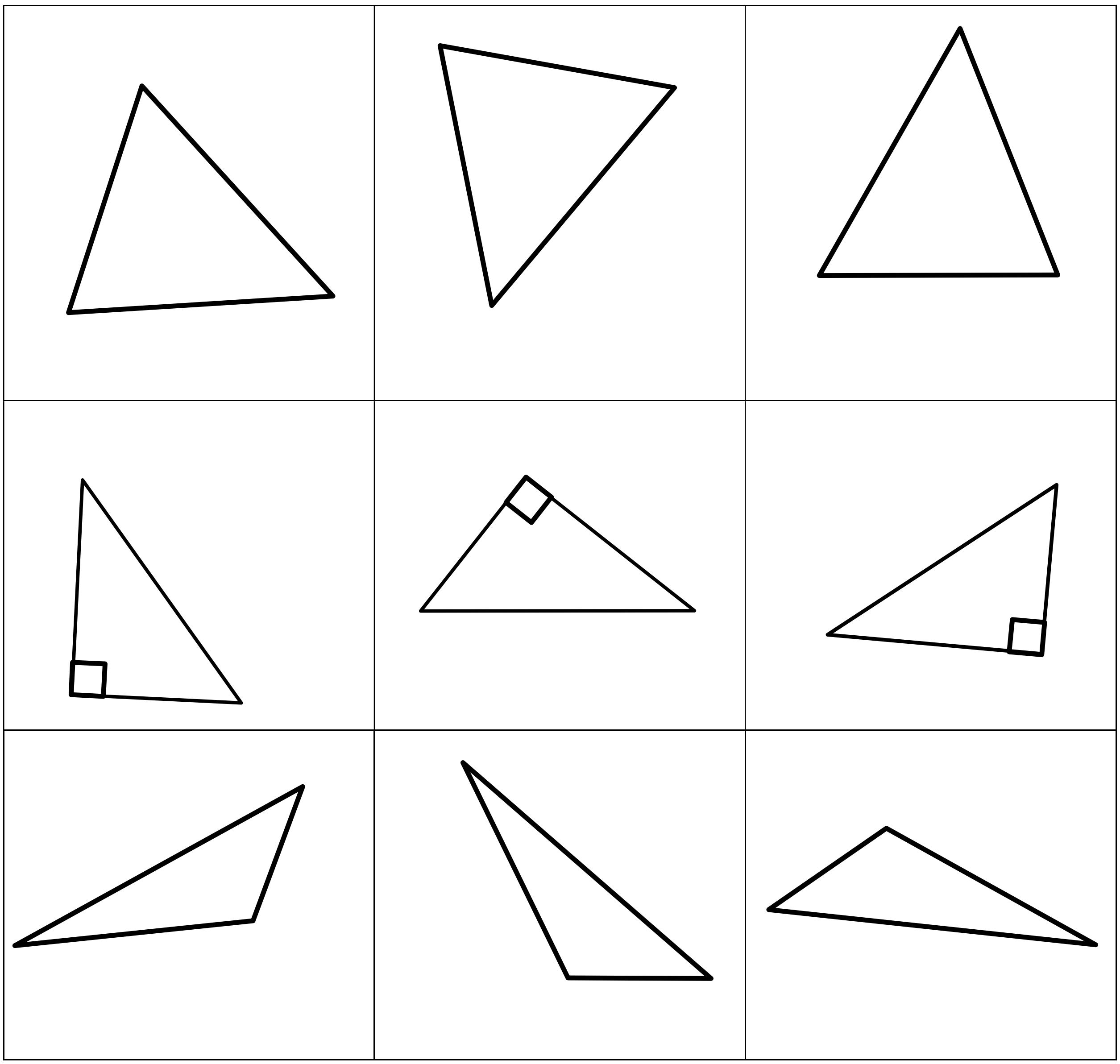
- There are three goals of this task. First, to help identify students' concept images of area. Second, to see students' initial instincts about which base-height combination to work with, or to see if they have some other strategy for finding area other than the standard formula. Lastly, to obtain data about concept images of exterior altitudes for triangles.
- Ask students to find the area of the triangle and offer them a variety of tools without specifying what each tool is such as scissors, metric ruler, extra copies of the triangle that are already cut out, a square centimeter unit.
- Then, the researcher chooses another base, and asks the student where the height would be for that base (note, that because the triangle is obtuse, either the student's chosen base or the researcher's chosen base will yield an exterior altitude).

Task 5: Changing a Triangle into a Rectangle

- This activity will yield more data about students' strategies for finding area, concept images of area, and has the potential to help students make sense of the standard triangle area formula.

Task 2: Base-Height Creation

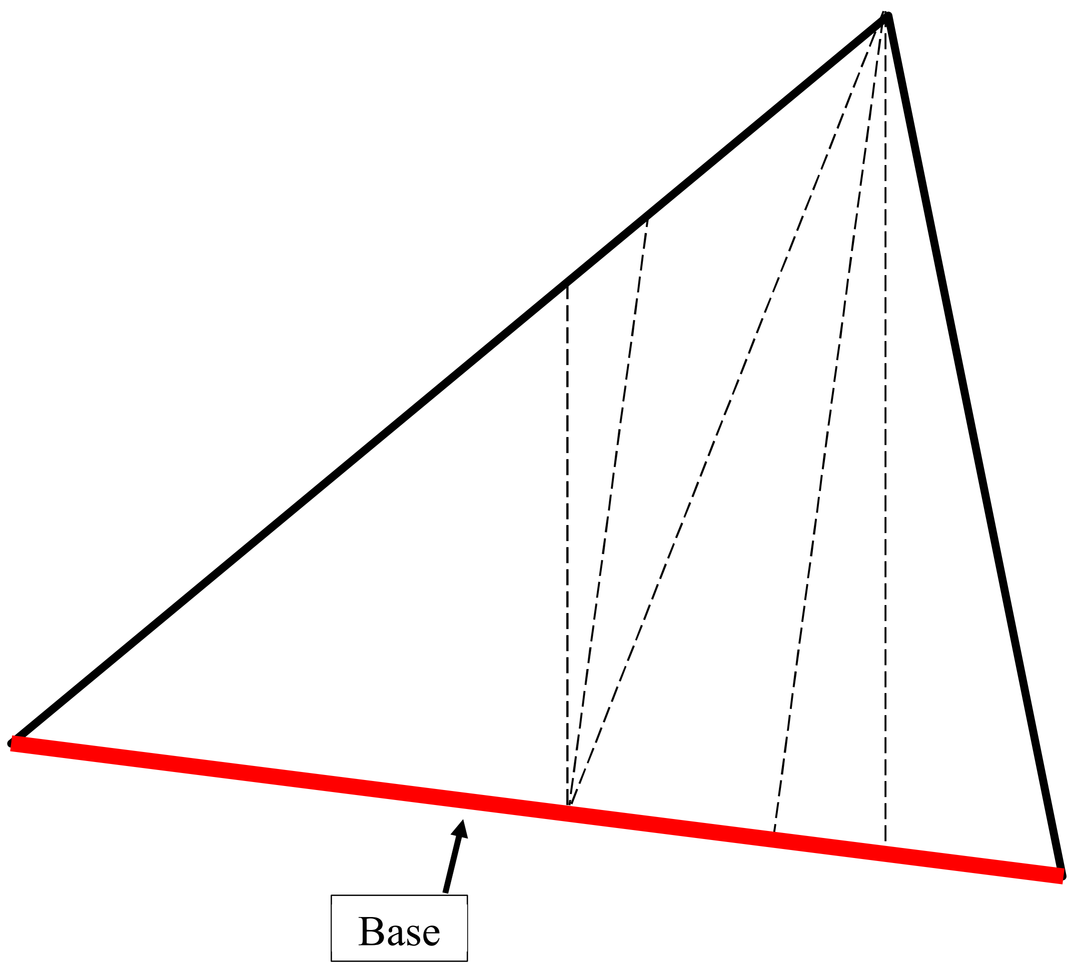
Use BLUE marker to draw a base and RED marker to draw a height in each triangle.



Task 3: Height Identification

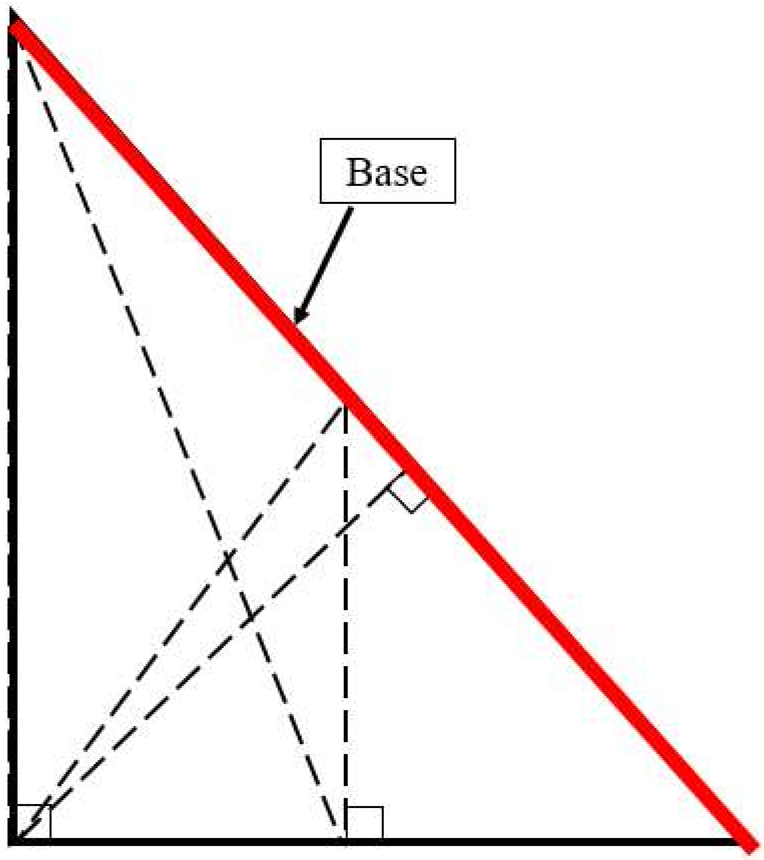
(a)

This time, I will point out a base, and you pick the corresponding height.



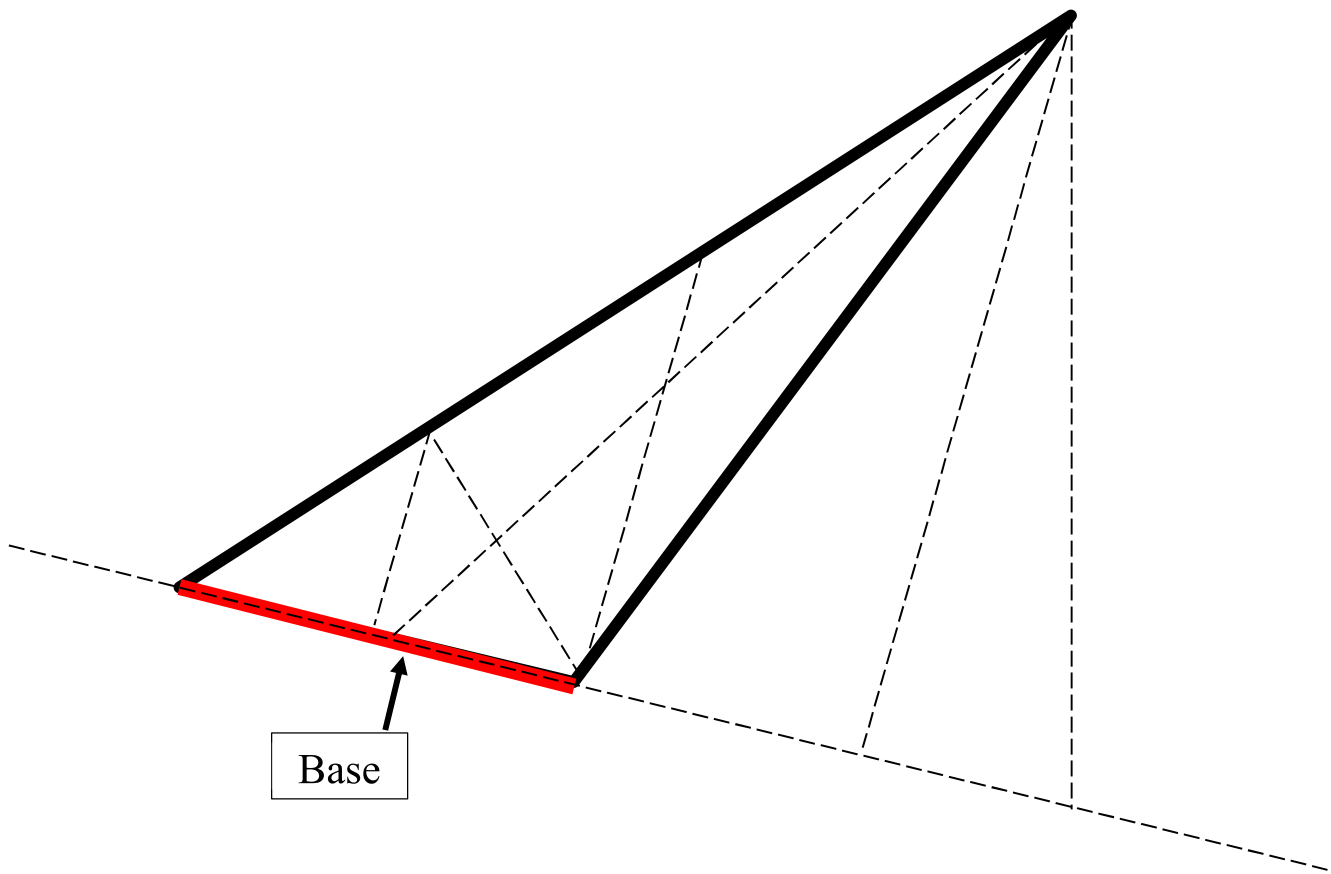
(b)

This time, I will point out a base, and you pick the corresponding height.



(c)

This time, I will point out a base, and you pick the corresponding height.

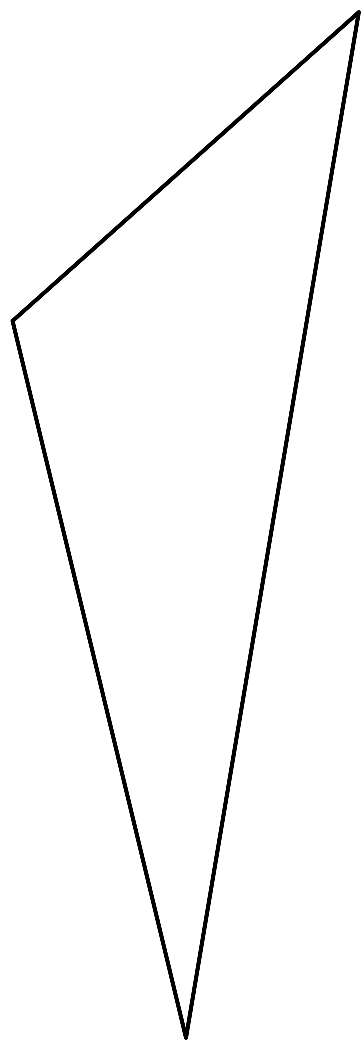


Task 4: Area of a Triangle

Determine the area of the triangle.

Optional materials available:

- Duplicate copies of the triangle
- Square centimeter cut-out
- Square centimeter graph paper
- Scissors
- Calculator
- Ruler

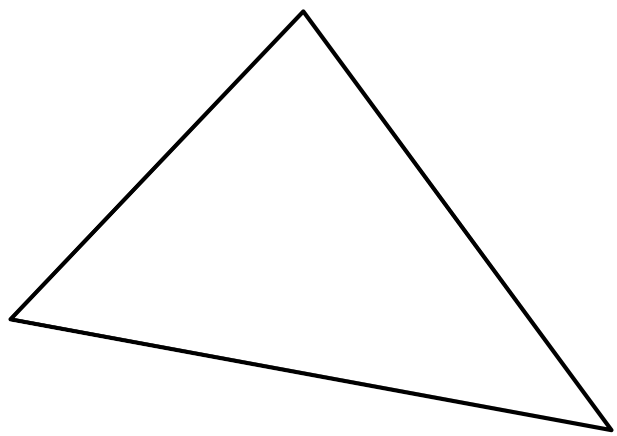


Task 5: Changing a Triangle into a Rectangle

Using whatever method you wish, create a rectangle with **double the area** of the given triangle.

Optional materials available:

- Duplicate copies of the triangle
- Square centimeter cut-out
- Square centimeter graph paper
- Scissors
- Calculator
- Ruler



APPENDIX E: THE A PRIORI CODES FOR ANALYSIS

In this appendix, I describe the a priori concept image codes that I used for survey and interview analysis. These descriptions include the relevant literature which I used to develop the a priori concept images.

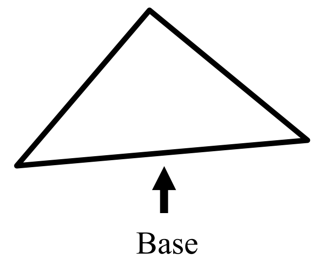
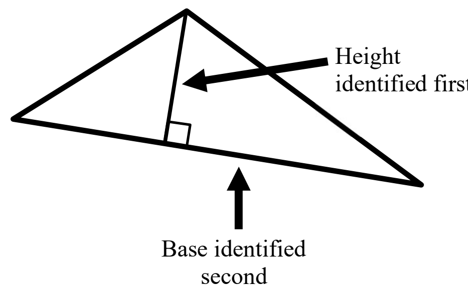
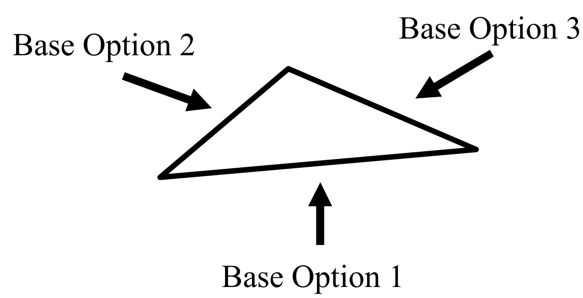
Initial Base Concept Images Used

Although some researchers did not use the interpretive lens of concept image and concept definition (e.g., Herbel-Eisenmann & Otten, 2011), in this section, I describe concept images for base, inferred from existing literature. Additionally, some researchers did use the interpretive lens of concept image and concept definition to examine students' understanding of base (Horzum & Ertekin, 2018). Using relevant research literature, I created a priori concept images for base which I used in my analysis processes.

In Figure B1, I share the list of a priori concept images that I used for this study. I created the list based on existing research literature discussed in Chapter 2 (e.g., Herbel-Eisenmann & Otten, 2011; Horzum & Ertekin, 2018; Vinner & Hershkowitz, 1980). Note that the descriptions in Figure B1 are only intended to describe each concept image, and therefore are not intended to describe how a teacher or researcher might detect the concept images. Throughout my study, I had refined the descriptions. Chapter 4 contains my revised descriptions after my phases of analysis. For clarification, the descriptions in Figure E1 reflect my initial descriptions, prior to my survey and interview analysis, based on existing literature.

Figure E1

A Priori Concept Images of Base

Name (Abbreviation): Description	Sample Figure
Base as Bottom (B-BOTTOM): Side which is approximately horizontal or viewed as closest to the person viewing the triangle.	
Base as Perpendicular to Height (B-PERP-HEIGHT): The side which is perpendicular with the height. Note the way that height is identified first, and base second.	
Base as Any Side (B-ANY-SIDE): Any triangle side can be the base. This concept image is equivalent to the concept definition for triangle base.	

Note: I created the names and abbreviations for these base concept images and used them as my a priori concept images of base for my study.

Base as Bottom

In this section, I describe the a priori concept image which I formed called B-BOTTOM. For clarification, I inferred the components of this concept image from existing research literature, some of which used the lens of concept image and definition (e.g., Vinner & Hershkowitz, 1980), and many of which did not (e.g., Herbel-Eisenmann & Otten, 2011;

Kadarisma et al., 2020). First, and perhaps most obvious, is that some students view the triangle base as the bottom-most side (Herbel-Eisenmann & Otten, 2011).

Second, this concept image contains the idea that an approximately horizontal side is the base (Vinner & Hershkowitz, 1980). This idea of a base being horizontal may be influenced by prototypical textbook images (Cunningham & Roberts, 2010; Kadarisma et al., 2020; Vinner & Hershkowitz, 1980). Third, if the triangle does not contain a horizontal side, a student may consider a triangle side to be a base if it is approximately horizontal, as described by a gravitational factor (Vinner & Hershkowitz, 1980), or they may rotate the triangle so that a side appears to be horizontal (Ward, 2004), or they may even consider the side which is closest to their body as the base (Horzum & Ertekin, 2018).

Base as Perpendicular to Height

I created the a priori concept image B-PERP-TO-HEIGHT to describe a situation whereby the student identifies the height first, and then describes the base as the geometric object perpendicular to the height (Horzum & Ertekin, 2018). Using the concept image and concept definition framework, Horzum and Ertekin (2018) described phenomena this as a height-dependent base and found this to be the least common in their study, with only 4 of 139 participants having evoked this concept image. Their primary example of this occurred when a preservice teacher was describing the base of a frustrum (i.e., circular cone with the top removed, creating a circle parallel with the original base) as "The [region] dependent on the height of a solid" (p. 191).

Regarding triangles, Horzum and Ertekin (2018) shared how one preservice teacher claimed that any side can be the base. Then, without rotation, they sketched three different base-height combinations. Although Horzum and Ertekin did not specify if the preservice teacher mentioned perpendicularity in this example, in many other examples, the researchers shared that perpendicularity between base and height was a key component in this base concept image. Note that Horzum and Ertekin had created their concept image to describe students' concept images of both 2- and 3-dimensional figures. In contrast, the present study only involved triangles, and therefore I created a new name that included the same emphasis on perpendicularity and height-dependency.

Base as Any Side

I created the a priori concept image, B-ANY, to include the idea that any triangle side is eligible to be a base. For example, Horzum and Ertekin (2018) shared a student's description of the base concept for a 2-dimensional figure. After drawing three triangles, the student explained that "each side would be a base and the base would change according to the point of view of individuals" (p. 191). Based on this example, it seems that a person can hold this concept image while still being influenced by other visual factors. I infer that the student was accepting all sides of the triangle as potential bases, and that they also were influenced by the orientation of the triangle because of how the student mentioned the point of view of the individuals—as if someone sitting on a different side of a table might choose a different base.

Section Summary

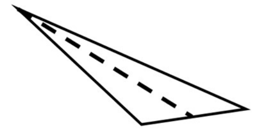
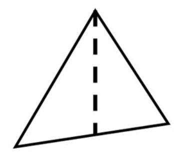
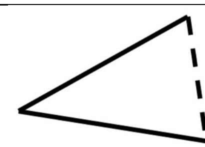
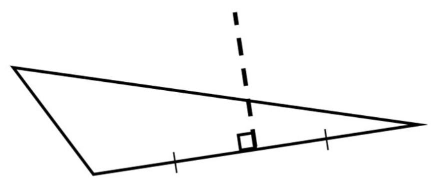
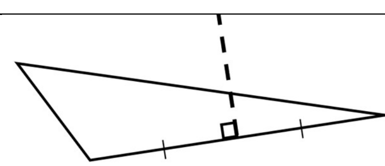
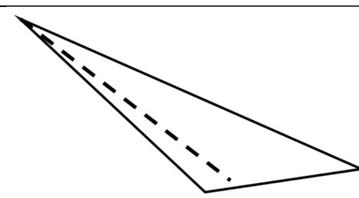
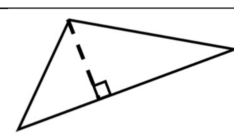
In this section, I identified attributes of three a priori base concept images by referring to research that either specifically used the lens of concept image and concept definition (e.g., Horzum & Ertekin, 2018; Vinner & Hershkowitz, 1980) or that did not use the lens, but did include discussion of students' understanding of base as part of their research (e.g., Herbel-Eisenmann & Otten, 2011). Note that one of the concept images—base as perpendicular to height—seemed related to a concept image of height because of the way that a student first identifies a height and then a base. In the next section, I will discuss the concept images for triangle height that I used as a priori for my study.

Initial Height Concept Images Used

In this section, I describe the collection of concept images of triangle height that I used in the beginning of my study as a priori. I formed this collection by looking both at studies that employed the interpretive lens of concept image and definition (e.g., Gutiérrez & Jaime, 1999; Krajcevski & Sears, 2019; Vinner & Hershkowitz, 1980) and by inferring concept images from studies which used other lenses to examine students' understanding of height as an aspect of their study (e.g., Baturo & Nason, 1996; Herbel-Eisenmann & Otten, 2011; Hong & Runnalls, 2020). In Figure B2, I summarize the a priori concept images for height that I used at the beginning of my study. As a reminder, like the a priori base concept images, the descriptions in Figure E2 represent my initial descriptions of the concept images. I share the final descriptions as part of my results in Chapter 4.

Figure E2

A Priori Concept Images of Height

Name (Abbreviation): Description	Sample Figure
Height as Median (H-MEDIAN): The segment that is approximately a median, regardless of the shape of the triangle.	
Height as Vertical (H-VERTICAL): A vertical segment, regardless of the orientation of the triangle. This segment has one endpoint as the vertex opposite the base.	
Height as Side (H-SIDE): A segment that is a triangle side, regardless of the type of the triangle.	
Height as Perpendicular Bisector (H-PERP-BISECTOR): A segment that is concurrent with the perpendicular bisector of the base, and may extend outside the triangle.	
Height as Perpendicular with Base (H-PERP-TO-BASE): A segment that contains the triangle altitude, but is longer than the altitude.	
Height as Interior to the Triangle (H-INTERIOR): The segment must be interior, but does not demonstrate perpendicularity and is also not close to being a median.	
Height as Altitude (H-ALTITUDE): The segment that is an altitude of the triangle.	

Note: The "heights" are indicated with a dashed line.

Height as Median

The concept image, H-MEDIAN, is characterized by a confusion between altitude and median. Using the concept image and definition framework, Gutiérrez and Jaime (1999) found this to be the most common concept image of altitude. Interestingly, they also found that many of the students who coded with this concept image correctly drew altitudes when they were internal but drew medians when the altitude was either external or coincided with a side. This finding is possibly explainable by considering Vinner's (1991) reminder about evoked concept images: "In a specific cognitive task we deal only with one's evoked concept image. We do not claim that under different circumstances the same image will be evoked again" (p. 73). Note that the H-MEDIAN concept image contains two of the three important attributes of altitude. Namely, endpoints at both the vertex opposite the base and on the line containing the base. It is missing the requirement of perpendicularity with the base.

Height as Vertical

I infer the a priori concept image, H-VERTICAL, based on evidence from a few studies (Barrett et al., 2012; Blanco, 2001; Gürefe & Gültekin, 2016; Kadarisma et al., 2020). For this concept image, the vertical height is a segment in the plane that bisects the body, with the line containing the segment passing through the midline (e.g., Pizzamiglio et al., 2000). This segment has endpoints as a triangle vertex and opposite side. Blanco (2001) described this as an error that may be caused because typical textbook representations are limited to those in which the triangle's height is vertical with respect to the page. Note that this concept image also does not contain any requirement of perpendicularity with the base.

Height as Side

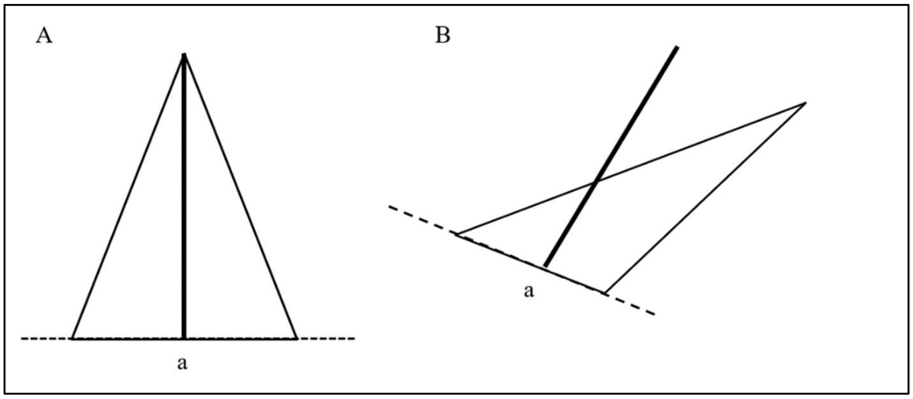
Various researchers have found evidence to suggest the formation of a concept image that I have called H-SIDE (e.g., Cavanagh, 2008; Gutiérrez & Jaime, 1999; Kospentaris et al., 2011; Krajcevski & Sears, 2019). For this concept image, students view the height as a side of the triangle. In this case, students are not attending to the perpendicularity of the altitude. Although some researchers have found this concept image to be a rarely occurring one (Gutiérrez & Jaime, 1999), this type of thinking has been observed when students worked towards finding the area of triangles (e.g., Kospentaris et al., 2011) and parallelograms (e.g., Cavanagh, 2008). Note that this concept image also does not contain the requirement of perpendicularity with the base.

Height as Perpendicular Bisector

The concept image, H-PERP-BISECTOR, which I used as a priori was observed by Gutiérrez and Jaime (1999), who described it as a confusion among the concepts associated with altitude and perpendicular bisector. They documented this as a rarely occurring concept image. The preservice teachers in their study who evoked this concept image showed an understanding of a perpendicular connection with their "altitude" and the triangle base but did not necessarily place a strong importance on a connection with the vertex opposite the base. Interestingly, the preservice teachers who evoked this concept image drew a segment that was approximately the same length as the correct altitude. An example of this is in a figure in Chapter 2 and shown again here in Figure E3.

Figure E3

Examples of the Altitude as Perpendicular Bisector Concept Image



Note. Images adapted from "Preservice primary teachers' understanding of the concept of altitude of a triangle," by A. Gutiérrez and A. Jaime, 1999, *Journal of Mathematics Teacher Education*, *2*, 253–275 (https://doi.org/10.1023/A:1009900719800). The lower-case *a* in each panel represents the base that was pre-selected by the researchers. Panel A: Example of an altitude drawn in an isosceles triangle. Panel B: Example of an 'altitude' drawn in an obtuse triangle that is approximately the same length as the triangle height to the given base.

Height as Perpendicular to Base

The a priori concept image, H-PERP-TO-BASE, may seem somewhat similar, based on the name, to the previous concept image. However, with this concept image, the height is not the bisector of the base. Gutiérrez and Jaime (1999) described this concept image as containing the important altitude characteristics of perpendicularity with the base. In their examples, they showed a variety of triangle types in a variety of orientations, with a segment that was longer than the altitude, but contained the altitude. In every example shown, one endpoint of the altitude was on the triangle base, and the other endpoint extended beyond the vertex opposite the base.

Krajcevski and Sears (2019) found similar results of incorrect altitude lengths, but with altitudes being orientated differently compared to Gutiérrez and Jaime (1999). Krajcevski and Sears reported altitudes that students drew with one endpoint as the vertex opposite the base and the other endpoint extending beyond the base. Additionally, they conjectured that this "concept image of a height is one that is represented by a segment drawn perpendicularly to 'the base' of the triangle given horizontally" (p. 96). In other words, they suspected that in addition to students not attending to the correct altitude length, they hypothesized an orientation-based influence.

Therefore, I created this a priori concept image to contain the altitude attribute of perpendicularity, but to not necessarily include the altitude endpoints on the base and vertex opposite the base. Note that, in contrast with the base concept image B-PERP-TO-HEIGHT, the existing literature relating to this concept image did not specify or imply any type of order that students identified the base and height.

Height as Interior to the Triangle

I formulated the a priori concept image, H-INTERIOR, based on existing research results, which happened to use the concept image and concept definition lens (Gutiérrez & Jaime, 1999; Vinner & Hershkowitz, 1980), and based on a study that used a different lens (Şengün & Yılmaz, 2021). This concept image includes the idea that heights must be interior to the triangle, regardless of what angle they form with the base.

Note that Vinner and Hershkowitz (1980) found that only 8% of their participants, Grades 7–9, correctly drew an exterior altitude in an obtuse triangle. They hypothesized that this "can be a result of a (sometimes implicit) common belief that an altitude should always fall inside the triangle" (p. 182). Gutiérrez and Jaime (1999) found similar results with rates of participants being classified with this concept image ranging from 3.1–9.8% with participants having drawn a segment from the vertex to the base at an oblique angle. However, the rate of participants—preservice teachers—being classified with this concept image with right and acute triangles dropped, ranging from 0–5%. Şengün and Yılmaz, (2021) found this type of thinking among 8.75% of their Grade 7 participants. To clarify, this concept image does not contain the perpendicularity component of the concept definition of altitude, but it does include the altitude endpoints at the base and vertex opposite the base.

Height as Altitude

I created the a priori concept image, H-ALTITUDE, to be aligned with the concept definition of altitude. Gutiérrez and Jaime (1999) asked preservice teachers to draw altitudes in triangles to pre-selected sides marked as a base. They found that some of the preservice teachers correctly drew the altitudes to the pre-selected base, and some drew correct altitudes to a base other than the pre-selected base.[13] Vinner and Hershkowitz (1980) had a similar task, given to their participants (i.e., students in Grades 7–9), and found varying levels of success based on triangle type, ranging from 8% for exterior altitudes to 41.6% with interior altitudes.

Section Summary

In this section, I described seven a priori concept images which I created based on a review of existing research literature. Many of the concept images were missing one or two components of the concept definition for altitude. I used the seven concept images of height as a starting collection of concept images of height for my study.

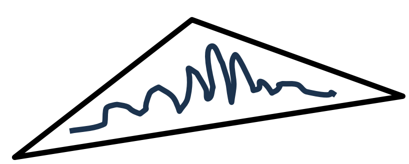
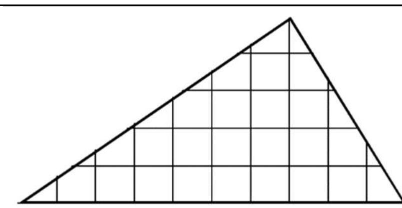
[13] In their article, Gutiérrez and Jaime (1999) provided a percentage breakdown for all concept images other than this one, and also did not provide statistics for the *no answer* category. This is why I use the description of *some*, rather than offering any specific percentages.

Initial Area Concept Images Used

Although more studies may exist, Tossavainen et al. (2017) was the only one that I could find that explicitly used the concept image and concept definition framework to describe preservice teachers' concept definitions of area. They identified concept images having to do with interpretations of area based on the idea of size and having a focus on formulas. Other researchers have also documented students' emphasis of formula usage in conjunction with area (e.g., Baturo & Nason, 1996; Tierney et al., 1990). In Figure E4, I share the a priori concept images which I used for area.

Figure E4

A Priori Concept Images of Area

Name (Abbreviation): Description	Sample Figure
Area as Standard Formula (A-STANDARD-FORMULA): Using the formula *A=1/2bh* to find the area of a triangle.	$A = (1/2)bh$ $A=(1/2)*5*6$ $A=15$
Area as Perimeter (A-ALL-SIDES): Finding the perimeter by adding or multiplying the sides together when trying to find area or making a judgment about the area by reasoning about perimeter.	$A = s_1+s_2+s_3$
Area as Size (A-SIZE): A general understanding that area involves a region in a closed figure.	
Area as Square Units (A-SQUARE-UNITS): the understanding of area as a collection of squares or partial squares which are used to fill a closed region.	

Area as Standard Formula

Tossavainen et al. (2017) found that 27% of the preservice teachers in their study viewed area as the result of a computation, based on a standard formula. Therefore, I created the a priori concept image H-STANDARD-FORMULA to depict a concept image that includes an understanding of area with a strong emphasis on formula usage. In other words, students with this concept image have a confusion between the concept of area and the idea of area formulas.

Area as All Sides

Researchers found that some young children (Yuzawa et al., 2000) and pre-service elementary teachers (Tierney et al., 1990) compared side lengths of figures to make judgments about area. For example, Tierney et al. (1990) found that pre-service elementary teachers judged a square and parallelogram to have different areas because the parallelogram had longer sides—actually, the two figures had the same area. Researchers also found evidence of students in Grade 7 finding attempting to find the area of a triangle and parallelogram by adding the three sides and by multiplying the three sides together (Cavanagh, 2008).

Area as Size

Tossavainen et al. (2017) also found three distinct concept images for area that all included an emphasis on size: (a) having explicitly discussed 2-dimensionality, (b) having explicitly discussed figures being bounded, and (c) describing size of figures but with vague terminology. Because all the figures in my study were triangles, and therefore did not include any un-bounded shapes or any 3-dimensional figures, I decided to condense the three concept images from Tossavainen et al. down to one a priori concept image which I called A-SIZE.

Area as Square Units

Based on various studies involving students' interpretations of unit concepts (e.g., Cullen, Eames, et al.,2018; Lamon, 1996, Outhred & Mitchelmore, 2000), I created the a priori concept image A-SQUARE-UNITS to include the notion that area involves either structuring, filling, or iterating square units. As discussed in Chapter 2, Kamii and Kysh (2006) found that many students think that the square unit cannot be decomposed, and therefore considered it impossible to create an 18 square centimeter rectangle having one dimension set to four squares. In contrast, some students allow for partial square units either when structuring a triangle (e.g., Reynolds & Wheatley, 1994) or when examining a gridded triangle (Barrett et al., 2017). All these types of interpretations could be considered as potential components of the concept image A-SQUARE-UNITS.