2023

Hyperplane Arrangements over the Ring of Integers Modulo N.

Ehiareshan James Obeahon
Illinois State University, obeahonjames@yahoo.com

HYPERPLANE ARRANGEMENTS OVER THE

RING OF INTEGERS

MODULO n.

EHIARESHAN J. OBEAHON

49 Pages

Let V be a finite vector space of dimension n over the field K. A hyperplane in V is an $n-1$ dimensional subspace of V defined by an equation of the form $\sum_{i=1}^{n} a_i x_i = 0,\ a_i \in K, (x_1, ..., x_n) \in V$. A hyperplane arrangement is any finite collection of hyperplanes. This work focuses on some hyperplanes arrangements such as the *braid and the graphical arrangement* $\{x_i - x_j = 0 : 1 \leq i < j \leq n\}$, the *shi arrangement* $\{x_i - x_j = 1 : 1 \leq i < j \leq n\}$, the *threshold arrangement* $\{x_i + x_j = 0 : 1 \leq i < j \leq n\}$, and the *complete arrangement* $\{\sum_{i \in S} x_i = 0 : S \subseteq [n]\}$. The graphical arrangement is a subarrangement of the threshold arrangement. Chapter 1 gives an introductory overview of the entire work, the significance of hyperplane arrangements, and their interconnections. Chapter 2 presents some of the important definitions and the general theory underlying hyperplane arrangements. Chapter 2 also introduces what we would call the most important concept in the study of hyperplane arrangement, which is the *characteristic polynomial.* Every hyperplane arrangement(or simply arrangement) can be identified by a unique characteristic polynomial. Chapter 3 covers the braid, graphical, and shi arrangement, and application to graph coloring, while Chapter 4 explores the threshold arrangement, and how it relates to threshold graphs. There is a bijection between the set of threshold graphs on n vertices and the set of regions of the *boxed threshold arrangement.* Finally, chapter 5 gives an introduction to the complete arrangement.

KEYWORDS: Threshold Arrangement, Characteristic Polynomial, Finite Field Method.

HYPERPLANE ARRANGEMENTS OVER THE RING OF INTEGERS MODULO n.

EHIARESHAN J. OBEAHON

A Thesis Submitted in Partial
Fulfillment of the Requirements
for the Degree of

MASTER OF SCIENCE

Department of Mathematics

ILLINOIS STATE UNIVERSITY

2023

HYPERPLANE ARRANGEMENTS OVER THE
RING OF INTEGERS
MODULO n.

EHIARESHAN J. OBEAHON

COMMITTEE MEMBERS:
Sunil K. Chebolu, Chair
Papa Sissokho

ACKNOWLEDGMENTS

The writer wishes to thank the supervisors, Dr. Sunil Chebolu and Dr. Papa Sissokho, for their support, corrections, and important comments throughout the work.

E. J. O.

CONTENTS

FIGURES

CHAPTER I: INTRODUCTION

Hyperplane arrangements are a fascinating and essential topic in mathematics with profound connections to various areas of pure and applied mathematics. They represent a captivating landscape where geometry, combinatorics, and algebra intertwine to reveal intricate structures and patterns. Hyperplane arrangements can be thought of as a geometric puzzle: a collection of hyperplanes that intersect in a particular way, partitioning the underlying space into a finite number of regions. These regions have a rich combinatorial structure, which has captivated mathematicians for decades. Hyperplane arrangements offer a broad spectrum of complexity and beauty.

I.1 Why Study Hyperplane Arrangements

The study of hyperplane arrangements is not just an abstract pursuit; it has profound implications in various mathematical and scientific fields. These arrangements can be found at the heart of algebraic geometry, combinatorics, representation theory, and topology. They play a fundamental role in computer science, statistical physics, and the study of complex systems. Understanding hyperplane arrangements provides insight into the geometry of spaces and the structure of problems, making them indispensable tools in many mathematical and scientific endeavors.

I.2 Overview

In this work, we aim to shed light on four specific hyperplane arrangements: the Braid arrangement, the Shi arrangement, the threshold arrangement, and the complete arrangement. These arrangements represent diverse aspects of hyperplane arrangement theory and have their unique stories to tell. As part of my contribution, I constructed a finite sequence and gave its general proof in **Theorem IV.0.7** and **Theorem IV.0.8** in terms of its recurrence relation. The terms of this sequence are the the coefficients of a polynomial not in standard form but equal to the threshold characteristic polynomial.

With my finite sequence, we can almost, without thinking, write down the threshold characteristic polynomial of any dimension. Whereas the common expression of this polynomial involves Stirling's number of the second kind and some product notation under the summation symbol, my finite sequence gives some computational ease. This means we have a new simplified way of computing the Stirling's number of the second kind. Also, as part of my contribution, I derived the exponential generating function of the threshold arrangement in **Theorem IV.0.6** by leveraging the work from another source which is properly referenced but did not even attempt to give the generating function of the threshold arrangement. I do not claim any originality here because finding the exponential generating function for the threshold arrangement is a problem listed in Richard Stanley's note as an unsolved exercise, not an unsolved problem. I decided to supply proof of my own. The only proof I have seen so far is different from mine.

I.2.1 **The Braid Arrangement** Inspired by the intricate patterns of braids and their connections to permutation groups, this arrangement reveals the elegance of combinatorics and topology. The *graphical arrangement* is a subarrangement of the braid arrangement in that every hyperplane in the graphical arrangement also belongs to the set of hyperplanes of the braid arrangement of the same vector space.

I.2.2 **The Shi Arrangement** Named after the founder Jiaqun Shi, who introduced them in a paper published in 1986, this arrangement connects to algebraic structures, such as matroid theory and Coxeter groups. It can be used to study the algebraic and combinatorial properties of these structures. Both braid and shi arrangement belong to a larger class known as *deformation of the braid arrangement* $\{x_i - x_j = a : 1 \leq i < j \leq n, a \in \mathbb{R}\}$. Other notable arrangements that belong to this broader class are the *Catalan arrangements* $\{x_i - x_j = -1, 0, 1 : 1 \leq i < j \leq n\}$ and the *Linial arrangement* $\{x_i - x_j = 1 : 1 \leq i < j \leq n\}$ but are not considered in this work

I.2.3 **The Complete Arrangement** . This straightforward yet powerful arrangement encompasses all possible hyperplanes in a given space, and its study is foundational for understanding the theory of hyperplane arrangements.

I.2.4 **The Threshold Arrangement** The threshold arrangement is related to threshold graphs.

I.3 Exploring Hyperplane Arrangements

In the following chapters, we will delve into each of these arrangements, exploring their properties, relevance, and applications. We will employ a blend of geometric intuition, algebraic techniques, and combinatorial insights to unravel these mathematical constructs.

I.4 The characteristic Polynomial

Every hyperplane arrangement has its unique characteristic polynomial that retains virtually all the important information we want concerning the arrangement under consideration. Evaluating the characteristic polynomial at -1 gives us the number of connected regions created by the hyperplanes of the arrangement. Again, when we evaluate the characteristic polynomial at 1 we get the number of bounded regions. So, obtaining this polynomial is something we strive to achieve for every hyperplane. In a case when this is not available, then we must be content with the generating function of the polynomial if we have it. However, when closed forms formula for characteristic polynomials and the generating function prove elusive, we must be content with bounds and asymptotics. This is precisely the case with the complete arrangement.

I.5 The Role of Exclusion-Inclusion Principle

The significance of the exclusion-inclusion principle cannot be exaggerated in combinatorics, generally. Because we are interested in the complement of the union of the hyperplanes, the method of exclusion-inclusion principle plays a huge role in starting and establishing the two ways of constructing the characteristic polynomial of an arrangement. However, this method quickly proves impractical as the dimension of the vector space increases. A more sophisticated way is to construct the *Poset*(partially ordered set) of an arrangement, and then with the help of the Möbius function (see definition II), the polynomial. Again, this approach becomes cumbersome as the number of interactions (intersections) increases due to the nature of the hyperplane and the high dimension. This leads to another important method for constructing the characteristic polynomial of an arrangement - the finite field(see section II.4) method.

I.6 The finite Field

When we say hyperplanes over the ring of integers, the finite field method is precisely what we have in mind. The main result (Theorem II.4.2) is implicit in the work of Crapo and Rota[1]. It was first developed into a systematic tool for computing characteristic polynomial by Athanasiadis, after a closely related but not as general technique was presented by Blass and Sagan[1]. It is called the *finite field method*. This method avoids working over the field $\mathbb{R}$, which makes evaluating the characteristic polynomial a difficult task. Instead, it works on some prime field or, more generally, some finite rings. Therefore, it converts the original problem into that which requires just counting points to get the characteristic polynomial. All these are later illustrated in the chapters that follow.

CHAPTER II: GENERAL FRAMEWORK FOR HYPERPLANE ARRANGEMENTS

II.1 Basic Definitions

A finite *hyperplane arrangements* $\mathcal{A}$ is a finite set of hyperplanes in some finite dimensional vector space $V \cong K^n$, K is a field. A linear hyperplane is an $n-1$ dimensional subspace of H of V. That is

$$H = \{v \in V : \alpha.v = 0\},$$

where α is a fixed nonzero vector in V and $\alpha.v$ is the usual dot product (assuming V is an inner product space) defined by

$$\alpha.v = \sum_i \alpha_i \; v_i$$

An affine hyperplane is a translate J of a linear hyperplane defined by

$$J = \{v \in V : \alpha.v = a\},$$

where α is a fixed nonzero vector in V and $a \in K$. Let $\mathcal{A}$ be an arrangement in a vector space V. The dimension $\dim(\mathcal{A})$ of $\mathcal{A}$ is defined to be $\dim(V) = n$, while the $\operatorname{rank}(\mathcal{A})$ of $\mathcal{A}$ is the dimension of the space spanned by the normals to the hyperplanes in $\mathcal{A}$. We say that $\mathcal{A}$ is *essential* if $\operatorname{rank}(\mathcal{A}) = r$, and take $V = K^n$. Now let $K = \mathbb{R}$, a *region* of an arrangement $\mathcal{A}$ is a connected component (with respect to the subpace topology) of the complement X of the hyperplanes:

$$X = \mathbb{R}^n - \bigcap_{H \in \mathcal{A}} H.$$

Let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$, and let

$$r(\mathcal{A}) = \#\mathcal{R}(\mathcal{A}),$$

the number of regions.

If we assume $K = \mathbb{R}$, then it is easy to see that every region $R \in \mathcal{R}(\mathcal{A})$ is open and convex. If W is the subspace of V spanned by the normals to the hyperplanes in $\mathcal{A}$, then $R \in \mathcal{R}(\mathcal{A})$ if and only if $R \cap W \in \mathcal{R}(\mathcal{A}_{\mathcal{W}})$.

We say that a region $R \in \mathcal{R}(\mathcal{A})$ is *relatively bounded* if $R \cap W$ is bounded. If $\mathcal{A}$ is essential, then relatively bounded is the same as bounded. We write $b(\mathcal{A})$ for the number of relatively bounded regions of $\mathcal{A}$

A closed *half-space* is a set $\{x \in \mathbb{R}^n : x \,.\, \alpha \geq c\}$ for some $\alpha \in \mathbb{R}^n, c \in \mathbb{R}$. If H is a hyperplane in $\mathbb{R}^n$, then the complement $\mathbb{R}^n - H$ has two components whose closures are half-spaces. It follows that the closure $\bar{R}$ of a region R of $\mathcal{A}$ is a finite intersection of half-spaces; that is, a convex polyhedron of dimension n. A bounded polyhedron is called a convex polytope.

An arrangement $\mathcal{A}$ is in *general position* if

$$\{H_1, H_2, ..., H_p\} \subseteq \mathcal{A}, p \leq n \implies dim(H_1 \cap ... \cap H_p) = n - p, \text{ and}$$

$$\{H_1, H_2, ..., H_p\} \subseteq \mathcal{A}, p > n \implies H_1 \cap ... \cap H_p = \emptyset.$$

For instance, if $n = 2$, a set of lines is in general position if no two are parallel and no three meet at a point.

Example 2.1 Let $\mathcal{A}$ consist of m lines in general position in $\mathbb{R}^2$, then $r(\mathcal{A}_m)$ can be computed using the following method. Initially $\mathcal{A}_k = \emptyset$, then add a line L to $\mathcal{A}_k$ such that $\mathcal{A}_k \cup L$ is in general position. When we travel along L from one end at infinity to the other, every time we intersect a line in $\mathcal{A}_k$, we create a new region at the end. Before we add any lines, we have one region, the whole of $\mathbb{R}^2$. Therefore,

$$r(\mathcal{A}_m) = \#intersections + \#lines + 1 = \binom{m}{2} + m + 1$$

II.2 The Intersection Poset

Recall that a **partially ordered set** called a *poset*, denoted as $(S, \preceq)$, is a set S together with a partial order relation $\preceq$, where $\preceq$ on S is called a partial ordering if it is

- **reflexive**: $a \preceq a \; \forall \; a \in S$
- **antisymmetric**: $a \preceq b \; and \; b \preceq a \Longrightarrow a = b$
- **transitive**: $a \preceq b \; and \; b \preceq c \Longrightarrow a \preceq c$

A partial order denoted by $x \preceq y$ means x precedes y, while $x \prec y$ means x strictly precedes y. For instance, the inclusion relation $\subseteq$ is a partial ordering on the power set $P(S)$ of a set S; that is, $(P(S), \subseteq)$ is a poset. Similarly, the relation $\leq$ is a partial ordering on the set of integers $\mathbb{Z}$; that is $(\mathbb{Z}, \leq)$ is a poset. If $x \preceq y$ in S, then the *closed interval* $[x, y]$ is defined by

$$[x, y] = \{z \epsilon S : x \leq z \leq y\}$$

Here, the empty set is not a closed interval.

Definition 1. Let $\mathcal{A}$ be an arrangement in V, and let $L(\mathcal{A})$ be the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including V itself as the intersection over the empty set. Define $x \preceq y$ in $L(\mathcal{A})$ if $x \supseteq y$ (as a subset of V). In other words, $L(\mathcal{A})$ is partially ordered by reverse inclusion. We define the *intersection poset* of $\mathcal{A}$ as $L(\mathcal{A})$. The element $V \in L(\mathcal{A})$ satisfies $V \preceq x$ for every $x \in L(\mathcal{A})$. We say that y *covers* x in a poset S, if $x \prec y$ and no $z \in S$ satisfies $x \prec z \prec y$.

Every finite poset is determined by its cover relations. The Hasse diagram of a finite poset is obtained by drawing elements of S as dots, with x drawn lower than y if $x \prec y$, and with an edge between x and y if xy.

A chain of length k in a poset S is a set $x_0 \prec x_1 \prec \cdots \prec x_k$ of elements of S.

The chain is saturated if $x_1 x_2 ... x_k$. We say that S is graded of rank n if every maximal chain of S has length n. In this case, S has a rank function $rk : P \longrightarrow \mathbb{N}$ defined by

(a) $rk(x) = 0$ if x is a minimal element of S

(b) $rk(y) = rk(x) + 1$ if xy in P.

If $x \prec y$ in a graded poset S, then we write $rk(x, y) = rk(y) - rk(x)$, which is equal to the length of the interval $[x, y]$.

Proposition II.2.1. *let $\mathcal{A}$ be an arrangement in a vector space $V \cong K^n$. The intersection poset $L(\mathcal{A})$ is graded of rank equal to rank($\mathcal{A}$). The rank function of $L(\mathcal{A})$ is given by*

$$rk(x) = codim(x) = n - dim(x)$$

where $dim(x)$ is the dimension of x as an affine subspace of V.

Proof. Since $L(\mathcal{A})$ has a unique minimal element V, the whole space, it suffices to show that (a) if xy in $L(\mathcal{A})$ then $dim(x) - dim(y) = 1$, and (b) all maximal elements of $L(\mathcal{A})$ have dimension $n - rank(\mathcal{A})$. By linear algebra, if H is a hyperplane and x an affine subspace, then $H \cap x = x$ or $dim(x) - dim(H \cap x) = 1$, so (a) follows. Now suppose that x has the largest codimension of any element of $L(\mathcal{A})$, say $codim(x) = d$. Thus x is an intersection of d linearly independent hyperplanes $H_1, H_2, ..., H_d$ in $\mathcal{A}$. Let $y \in L(\mathcal{A})$ with $e = codim(y) < d$. Thus y is an intersection of e hyperplanes, so some $H_i (1 \leq i \leq d)$ is linearly independent of them. Then $y \cap H_i \neq \emptyset$ and $codim(y \cap H_i) > codim(y)$. Hence, y is not a maximal element of $L(\mathcal{A})$, proving (b). ∎

II.3 The Characteristic Polynomial

A poset S is locally finite if every interval $[x, y]$ is finite. Let $Int(S)$ denote the set of all closed intervals of S. For a function $f : Int(S) \longrightarrow \mathbb{Z}$, write $f(x, y)$ for $f([x, y])$. We now come to a fundamental invariant of locally finite posets.

Definition 2. Let S be a locally finite poset. Define a function $\mu = \mu_S : Int(S) \longrightarrow \mathbb{Z}$, called the *Möbius function* of S by the conditions:

$$\begin{aligned} \mu(x,x) &= 1, \forall x \in S \\ \mu(x,y) &= -\sum_{x \leq z \leq y} \mu(x,z), \text{for all } x \prec y \text{ in } S \end{aligned}$$

This second condition can also be written as

$$\sum_{x \leq z \leq y} \mu(x,z) = 0, \text{ for all } x \prec y \text{ in } S$$

If S has a $\hat{0}$, then we write $\mu(x) = \mu(\hat{o}, x)$. An important application of the Möbius function is the *Möbius inversion formula.* Let $\mathcal{J}(S) = \mathcal{J}(S, K)$ denote the vector space of all functions $f : Int(S) \longrightarrow K$. Write $f(x,y)$ for $f([x,y])$. For $f, g \in \mathcal{J}(S)$, define the product $fg \in \mathcal{J}(S)$ by

$$fg(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$$

This product makes $\mathcal{J}(P)$ an associative $\mathbb{Q}$ algebra, with multiplicative identity given by the Kronecker delta function δ where

$$\delta = \begin{cases} 1, \text{if } x = y, \\ 0, \text{if } x < y. \end{cases}$$

Define the *zeta function* $\zeta \epsilon \mathcal{J}(P)$ of P by $\zeta(x,y) = 1$ for all $x \leq y$ in P. Note that the Möbius function μ is an element of $\mathcal{J}(P)$. The previous definition of μ is equivalent to the relation $\mu\zeta = \delta$ in $\mathcal{J}(P)$. In any finite-dimensional algebra over a field, one-sided inverses are two-sided inverses, and so $\mu = \zeta^{-1}$ in $\mathcal{J}(P)$.

Theorem II.3.1. *[1] Let P be a finite poset with möbius function μ, and let*

$f, g : P \longrightarrow K$. Then the following two conditions are equivalent:

$$\begin{aligned} f(x) &= \sum_{y \geq x} g(y), \forall x \in P \\ g(x) &= \sum_{y \geq x} \mu(x,y) f(y), \forall x \in P \end{aligned}$$

Proof. The set K^P of all functions $P \longrightarrow K$ forms a vector space on which $\mathcal{J}(P)$ acts on the left as an algebra of linear transformations by

$$(\xi f)(x) = \sum_{y \geq x} \xi(x,y) f(y)$$

where $f \epsilon K$ and $\xi \epsilon \mathcal{J}(P)$. The möbius formula is then nothing but the statement

$$\zeta f = g \Longleftrightarrow f = \mu g$$

■

Definition 3. The characteristics polynomial

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{dim(x)}$$

Proposition II.3.2. *Let $\mathcal{A}$ be given by $\mathcal{Q}_{\mathcal{A}}(x) = x_1 x_2 ... x_n$, then*

$$\chi_{\mathcal{A}}(t) = (t-1)^n$$

Theorem II.3.3. ***Whitney's Theorem:*** *Let $\mathcal{B} \subseteq \mathcal{A}$ be central; meaning $\bigcap_{H \,\epsilon\, \mathcal{B}} H \neq \emptyset$, then*

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{dim(x)} = \sum_{\mathcal{B} \subseteq \mathcal{A}\ ,\ \mathcal{B}\ is\ central} (-1)^{\#\mathcal{B}} t^{n - rank(\mathrm{B})}$$

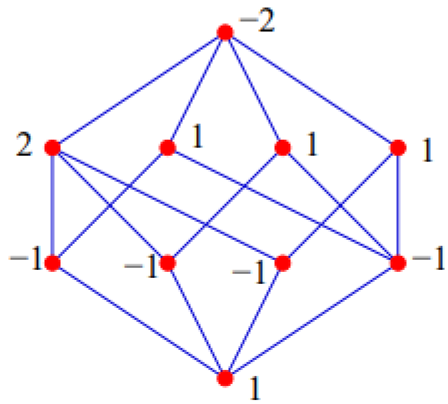

Figure 1: Poset of hyperplanes

Theorem II.3.4. ***Zaslavsky's Theorem:(1975)***

$$
\begin{aligned}
r(\mathcal{A}) &= (-1)^n \chi_{\mathcal{A}}(-1) \\
b(\mathcal{A}) &= (-1)^{rank(\mathcal{A})} \chi_{\mathcal{A}}(1)
\end{aligned}
$$

Example II.3.5. Let $\mathcal{A}$ be the arrangement of Figure 1 above, then

$$\chi_{\mathcal{A}}(t) = t^3 - 4t^2 + 5t - 2 = (t-1)^2(t-2)$$

This is because, from the definition of $\chi_{\mathcal{A}}(t)$ where $\mathcal{A}$ is in K^n, we have

$$\chi_{\mathcal{A}}(t) = t^n - (\#\mathcal{A})t^{n-1} + ...$$

II.3.1 Arrangement in General Position

Proposition II.3.6. *Let $\mathcal{A}$ be an $n-$dimensional arrangement of m hyperplanes in general position, then*

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^i \binom{m}{n} t^{n-i}$$

In particular, if $\mathcal{A}$ is a real arrangement, then

$$
\begin{aligned}
r(\mathcal{A}) &= \sum_{i=0}^{n} \binom{m}{i} \\
b(\mathcal{A}) &= (-1)^n \sum_{i=0}^{n} (-1)^i \binom{m}{i} = \binom{m-1}{n}
\end{aligned}
$$

Proof. Every $\mathcal{B} \subseteq \mathcal{A}$ with $\#\mathcal{B} \leq n$ defines an element $x_{\mathcal{B}} = \bigcap_{H \in \mathcal{B}} H$ of $L(\mathcal{A})$. Hence $L(\mathcal{A})$ is a truncated boolean algebra:

$$L(\mathcal{A}) \cong \{S \subseteq [m] : \#S \leq n\}$$

ordered by inclusion. If $x \in L(\mathcal{A})$ and $rk(x) = k$, then $[\,0, x] \cong B_k$, which is a boolean algebra of rank k. By a previous result, we have $\mu(x) = (1)^k$. Hence

$$
\begin{aligned}
\chi_{\mathcal{A}}(t) &= \sum_{S \subseteq [m], \#S \leq n} (-1)^{\#S} t^{n-\#S} \\
&= \sum_{i=0}^{n} (-1)^i \binom{m}{n} t^{n-i}
\end{aligned}
$$

■

Example II.3.7. Consider the coordinate hyperplane $\mathcal{A}$ with defining polynomial $\mathcal{Q}_{\mathcal{A}}(x) = x_1 x_2 ... x_n$. Every subset of the hyperplanes in $\mathcal{A}$ has a different nonempty intersection, so $L(\mathcal{A})$ is isomorphic to the boolean algebra B_n of all subsets of $[n]$, ordered by inclusion. We claim that

$$\chi_{\mathcal{A}}(t) = (t-1)^n$$

To show this, let $y \in L(\mathcal{A}), rk(y) = k$. We claim that

$$\mu(y) = (-1)^k$$

The claim is clearly true for $rk(y) = 0$ when $y = \hat{0}$. Now let $y > \hat{0}$. We need to show that

$$\sum_{x \le y} (-1)^{rk(x)} = 0$$

The number of x such that $x \le y$ and $rk(x) = i$ is $\binom{k}{i}$ so that the last equation is equivalent to the well-known identity $\sum_{i=0}^{k} (-1)^i \binom{k}{i} = 0$ for $k > 0$. This completes the proof of the claim.

II.4 The Finite Field Method.

This is an interesting method [5] for computing the characteristics polynomial of an arrangement defined over $\mathbb{Q}$. If an arrangement $\mathcal{A}$ is defined over $\mathbb{Q}$, then we may assume that it is defined over $\mathbb{Z}$ as well because multiplying the equations of hyperplanes in $\mathcal{A}$ by a suitable integer gives an arrangement over $\mathbb{Z}$. In that case, we can take coefficients modulo a prime p and get an arrangement $\mathcal{A}_q$ defined over the finite field $\mathbb{F}_q, q = p^r$. We say that $\mathcal{A}$ has a *good reduction* mod p (or over $\mathbb{F}_q$) if $L(\mathcal{A}) = L(\mathcal{A}_q)$.

Proposition II.4.1. *Let $\mathcal{A}$ be an arrangement defined over $\mathbb{Z}$. Then $\mathcal{A}$ has a good reduction for all but finitely many primes p.*

Proof. Let $H_1, H_2, ..., H_j$ be affine hyperplanes, where $H_i, 1 \le i \le j$, is given by the equation $v_i \, . \, a_i (v_i, a_i \in \mathbb{Z})$ By linear algebra, we have $H_1 \cap ... \cap H_j \ne \emptyset$ if and only if

$$rank \begin{pmatrix} v_1 & a_1 \\ \vdots & \vdots \\ v_j & a_j \end{pmatrix} = rank \begin{pmatrix} v_1 \\ \vdots \\ v_j \end{pmatrix}$$

Moreover, if this equation holds, then

$$\dim(H_1 \cap ... \cap H_j) = n - rank \begin{pmatrix} v_1 \\ \vdots \\ v_j \end{pmatrix}$$

Now, for any $r \times s$ matrix A, we have rank$(A) \geq t$ if and only if some $t \times t$ submatrix B satisfies $\det(B) \neq 0$. It follows that $L(\mathcal{A}) \ncong L(\mathcal{A}_p)$ if and only if at least one member S of a certain finite collection $\mathcal{S}$ of subsets of integer matrices B satisfies the following condition:

$$(\forall B \in S), \ \det(B) \neq 0 \text{ but } \det(B) \equiv 0 (\text{mod} p)$$

This can only happen for finitely many p; that is, for certain B we must have p divides $\det(B)$, so that $L(\mathcal{A}) = L(\mathcal{A}_p)$ when p is sufficiently large. ■

Theorem II.4.2. *[1] Let $\mathcal{A}$ be an arrangement in $\mathbb{Q}^n$, and suppose that $L(\mathcal{A}) = L(\mathcal{A}_q)$ for some prime power q. Then*

$$\begin{aligned} \chi_{\mathcal{A}}(q) &= \#(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H) \\ &= q^n - \#(\bigcup_{H \in \mathcal{A}_q} H) \end{aligned}$$

Proof. Let $x \in L(\mathcal{A}_q)$ so $\#x = q^{\dim}(x)$. Here, $\dim(x)$ can be computed over $\mathbb{Q}$ or $\mathbb{F}_q$. Define two functions $f, g : L(\mathcal{A}_q) \longrightarrow \mathbb{Z}$ by

$$\begin{aligned} f(x) &= \#x \\ g(x) &= \#(x - \bigcup_{y > x} y) \end{aligned}$$

In particular,

$$g(\hat{0}) = g(\mathbb{F}_q^n) = \#(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H)$$

Clearly,

$$f(x) = \sum_{y \geq x} g(y)$$

Let μ denote the Möbius function of $L(\mathcal{A}) = L(\mathcal{A}_q)$. By the Möbius inversion theorem, we have

$$\begin{aligned} g(x) &= \sum_{y \geq x} \mu(x, y) f(y) \\ &= \sum_{y \geq x} \mu(x, y) q^{\dim(x)} \end{aligned}$$

Put $x = \hat{0}$ to get

$$g(\hat{0}) = \sum_{y \geq x} \mu(y) q^{\dim(y)} = \chi_{\mathcal{A}}(q)$$

■

CHAPTER III: THE BRAID, GRAPHICAL AND SHI ARRANGEMENTS

III.1 Stirling's Numbers

Before we proceed to talk about the hyperplane arrangements of this chapter, it is a good idea to review some interesting combinatorial objects that will be used in the later work. These are the Stirling's numbers of the first and second kinds.

III.1.1 Stirling's Numbers of the second kind.

By a *partition* of a set S we mean a collection of nonempty, pairwise disjoint sets whose union is S. Another name for a partition of S is an *equivalence relation* on S.

Definition 4. Let $S(n, k)$ be the number of partitions of $[n]$ into k subsets. We call the sequence $S(n, k)$ *Stirling's number of the second kind.*[9]

These numbers are of great combinatorial significance as they naturally feature when we enumerate other interesting mathematical objects, as we will later see. Without much work, we can see that these numbers satisfy the recurrence relation

$$S(n, k) = S(n-1, k-1) + kS(n-1, k); S(0, 0) = 1$$

where $S(n, k) = 0$ for $k > n$ or $n < 0$ or $k < 0$ and $S(n, 0) = 0$ for $n \neq 0$. Hence the recurrence is valid for all n, k except at $(0, 0)$ at which nit equals 1. Let the generating function of $S(n, k)$ be defined as

$$B_k(x) = \sum_{n \geq 0} S(n, k)x^n$$

Taking the generating function of the recurrence and simplifying gives

$$B_k(x) = \frac{x}{1 - kx} B_{k-1}(x) \ \ (k \geq 1; B_0(x) = 1)$$

Solving this first order recurrence relation gives

$$B_k(x) = \sum_{n\geq 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)...(1-kx)}, \quad k \geq 0$$

This last equation means that

$$\begin{aligned} S(n,k) &= [x^n]B_k(x) \\ &= [x^n]\Big(\frac{x^k}{(1-x)(1-2x)...(1-kx)}\Big) \\ &= [x^{n-k}]\Big(\frac{1}{(1-x)(1-2x)...(1-kx)}\Big) \\ &= \sum_{r=1}^{k}(-1)^{k-r}\frac{r^n}{r!(k-r)!} = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}r^n, \quad (n,k\geq 0) \end{aligned}$$

As a side note, the number of onto functions from a set of size n to a set of size k is equal to the number of ordered partitions of a set of size n into k subsets, which is equal to

$$k!S(n,k) = \sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}r^n \quad (n,k\geq 0)$$

III.1.2 **Stirling's Numbers of the first kind** The sequence $s(n,k)$ which represents the number of permutations of $[n]$ with k cycles is called the *Stirling's number of the first kind*. It satisfies the recurrence relation

$$s(n+1,k) = ns(n,k) + s(n,k-1);\ s(0,0) = 1, s(n,0) = s(0,n) = 0\ ; n \geq k \geq 0$$

Let

$$B_n(y) = \sum_{k\geq 0} s(n,k)y^k$$

Taking the generating function of both sides of the recurrence relation with respect to k, simplifying and solving it, we have

$$B_n(y) = \sum_{k\geq 0} s(n,k)y^k = y(y+1)...(y+n-1)$$

This implies that

$$s(n,k) = [y^k]\Big(y(y+1)...(y+n-1)\Big)$$

A closely related sequence to $s(n,k)$ is the *signed Stirling's number of the first kind* $s^*(n,k)$ which is defined as

$$s^*(n,k) = (-1)^{n-k}s(n,k)$$

Multiplying both sides of the last equation by $(-1)^{n-k}$ and then replacing $-y$ by x, we see that

$$s^*(n,k) = [x^k]\Big(x(x-1)...(x-n+1)\Big)$$

One interesting relationship between Stirling's numbers is the following equation called *Stirling reciprocity* whose proof will be given as an application of the **Braid Arrangement**.

$$\sum_{k=j}^{n} s^*(n,k)S(k,j) = \delta_{jn}$$

III.2 The Braid Arrangement

Recall that the characteristics polynomial $\chi_{\mathcal{A}_n}(p)$ of the hyperplane arrangement $\mathcal{A}_n = \{H_1, ..., H_l\}$ is defined as

$$\chi_{\mathcal{A}_n}(q) = \sum_{X\in L(\mathcal{A}_n)} \mu(x,\bar{1})p^{\dim X} = \#(\mathbf{F}_p^n - \cap_{i=1}^{l} H_i)$$

Definition 5. [2] The Braid arrangement $\mathcal{B}_n$ is defined as

$$\mathcal{B}_n = \{H_{ij}\}_{1\leq i<j\leq n}$$

Where

$$H_{ij} = \{x \in \mathbf{F}_p^n : x_i = x_j\}$$

From the above definition we see that

$$\chi_{\mathcal{A}}(p) = \#\{v \in \mathbf{F}_p^n : x_i \neq x_j\}$$

We can choose x_1 in p ways, x_2 in $p-1$ ways, since it must be distinct from x_1.

Continuing this way, we find that

$$\chi_{\mathcal{A}}(p) = p(p-1)(p-2)...(p-n+1)$$

Recall that the signed Stirling's number of the first kind $s^*(n,k)$ is given by

$$s^*(n,k) = [p^k](p(p-1)...(p-n+1).$$

Therefore, we have

$$\chi_{\mathcal{A}}(p) = \sum_{k=1}^{n} s^*(n,k)p^k.$$

We know that Stirling's number of the first kind counts Stirling's number of the second kind; that is, partitions. Knowing that the braid arrangements are also closely related to partitions, it turns out that the key to this relation is *Stirling's reciprocity*. Recall also that Stirling's number of the second kind $S(k,j)$ is defined as

$$S(k,j) = \#\{\text{partitions of } [k] \text{ into } j \text{ nonempty parts}\} = \#\{\lambda \in \Pi_k : |\lambda| = j\}.$$

where Π_k = the set of all partitions of $[k]$. If we consider some fixed $\lambda \in \Pi_k$ where $|\pi| = k$, we can then write

$$S(k,j) = \#\{\lambda \in \Pi_k : |\lambda| = j, \lambda \geq \pi\}$$

For signed Stirling's number of the first kind, we know that

$$\chi_{\mathcal{A}}(p) = \sum_{k=1}^{n} s^*(n,k)p^k = \sum_{X \in L(\mathcal{B}_n)} \mu(X, \hat{1})p^{dimX}$$

where $\hat{1} = \bigcap_{X \in \mathcal{B}_n} X$. This implies that

$$\begin{aligned} s^*(n,k) &= \sum_{X \in L(\mathcal{B}_n), dimX = k} \mu(X, \hat{1}) \\ &= \sum_{\pi \in \Pi_n, |\pi| = k} \mu(\hat{0}, \pi) \end{aligned}$$

Putting these together gives a (not so common) proof of Stirling's reciprocity in the following theorem.

Theorem III.2.1.

$$\sum_{k=j}^{n} s^*(n,k)S(k,j) = \delta_{jn}$$

where δ_{jn} is the Kronecker delta notation.

Proof. Using the definitions of $s^*(n,k)$ and $S(k,j)$, we have

$$\begin{aligned} \sum_{k=j}^{n} s^*(n,k)S(k,j) &= \sum_{k=j}^{n} \Big(\sum_{\pi \in \Pi_n, |\pi| = k} \mu(\bar{0}, \pi)).(\#\{\lambda \in \Pi_k : |\lambda| = j, \lambda \geq \pi\} \Big) \\ &= \sum_{\lambda \in \Pi_n, |\lambda| = j} \Big(\sum_{\pi \leq \lambda} \mu(\bar{0}, \pi)\zeta(\pi, \lambda) \Big) \\ &= \sum_{\lambda \in \Pi_n, |\lambda| = j} \mu\zeta(\bar{0}, \lambda) = \delta_{jn} \end{aligned}$$

■

III.3 The Graphical Arrangement

There are close connections between certain invariants of a graph G and an associated arrangement $\mathcal{A}_G$. Let G be a simple graph on the vertex set $[n]$, and let $E(G)$ denote the set of edges of G, regarded as two-element subsets of $[n]$. Write ij for the edge $\{i, j\}$.

Definition 6. The graphical arrangement $\mathcal{A}_G$ in K^n is the arrangement defined by the equations:

$$x_i - x_j = 0, \; ij \in E(G).$$

Thus, a graphical arrangement is simply a subarrangement of the braid arrangement $\mathcal{B}_n$.[1] If $G = K^n$, the complete graph on $[n]$ vertices, then

$$\mathcal{A}_{K^n} = \mathcal{B}_n$$

Definition 7. A coloring of a graph G on $[n]$ is a map

$$\kappa : [n] \longrightarrow \mathbf{P}$$

The coloring κ is *proper* if $\kappa(i) \neq \kappa(j)$ whenever $ij \in E(G)$. If $q \in \mathbf{P}$ then let $\chi_G(q)$ denote the number of proper colorings $\kappa : [n] \longrightarrow [q]$ obtained by choosing a vertex, say 1, and coloring it in q ways. Then choose another vertex, say 2, and color it in $q-1$ ways, and so on. This gives

$$\chi_{K_n}(q) = q(q-1)(q-2)...(q-n+1)$$

Let $e_i(G)$ denote the number of surjective proper colorings $\kappa : [n] \longrightarrow [i]$ of G. We can choose an arbitrary proper coloring $\kappa : [n] \longrightarrow [q]$ by first choosing the size $i = \#\kappa([n])$ of its image in $\binom{q}{i}$ ways, and then choose κ in e_i ways. Hence

$$\chi_G(q) = \sum_{i=0}^{n} e_i \binom{q}{i}$$

It is clear that $\chi_G(q)$ is a polynomial in q. Since every surjection (also a bijection) $\kappa : [n] \longrightarrow [n]$ is proper, then $e_n = n!$. This means that $\mathcal{X}_G(q)$ is a monic polynomial of degree n.

Theorem III.3.1. *For any graph G, we have*

$$\chi_{\mathcal{A}_G}(t) = \chi_G(t)$$

Proof. Let $\pi \epsilon L_G$. For $q \epsilon \mathbf{P}$, define $\chi_\pi(q)$ to be the number of colorings $\kappa : [n] \longrightarrow [q]$ of G satisfying:

(a)If i, j are in the same block of π, then $\kappa(i) = \kappa(j)$

(b) If i, j are in different blocks of π and $ij \epsilon E(G)$, then $\kappa(i) \neq \kappa(j)$

Given any $\kappa : [n] \longrightarrow [q]$, there is a unique $\sigma \epsilon L_G$ such that κ is enumerated by $\chi_\sigma(q)$. Moreover, κ will be constant on the blocks of some $\pi \epsilon L_G$ if and only if $\sigma \geq \pi$ in L_G. Hence

$$q^{|\pi|} = \sum_{\sigma \geq \pi} \chi_\sigma(q) \ \ \forall \pi \epsilon L_G$$

where $|\pi|$ denote the number of blocks of π. By möbius inversion, we have

$$\chi_\pi(q) = \sum_{\sigma \geq \pi} q^{|\sigma|} \mu(\pi, \sigma)$$

Letting $\pi = \ 0$, we get

$$\chi_G(q) = \sum_{\sigma \in L_G} q^{|\sigma|} \mu(\sigma)$$

Since $|\sigma| = dim X_\sigma$, so comparing the form of $\chi_G(q)$ and the definition of $\chi_{\mathcal{A}}$ earlier given, the result follows. ■

Corollary III.3.2. The Characteristic polynomial of the Braid arrangement $\mathcal{B}_n$ is given by

$$\chi_{\mathcal{B}_n}(t) = t(t-1)...(t-n+1).$$

Proof. Since $\mathcal{B}_n = \mathcal{A}_{K_n}$, then from theorem 4.2, we have

$$\chi_{\mathcal{B}_n}(t) = \chi_{\mathcal{A}_{K_n}}(t)$$

the result follows from earlier result. ■

III.4 The Shi Arrangement

The *shi arrangement* is a deformation of the braid arrangement. Let $\mathcal{S}_n$ represent the shi arrangement, then

$$\mathcal{S}_n = \{(x_1, ..., x_n) \in \mathbf{F}_p^n : x_i + x_j = 0, 1,\ 1 \leq i < j \leq n\}$$

Thus, $\mathcal{S}_n$ has $n(n-1)$ hyperplanes and $\text{rank}(\mathcal{S}_n) = n - 1$

Theorem III.4.1. *[3] The characteristics polynomial of $\mathcal{S}_n$ is given by*

$$\mathcal{S}_n = t(t-n)^n.$$

Proof. Let p be a large prime. By Theorem 3.8, we have

$$\chi \mathcal{S}_n(p) = \#\{(x_1, ..., x_n) \in \mathbf{F}_p^n : x_i + x_j = 0, 1,\ i < j \implies x_i + x_j \neq 0, 1\}$$

Choose a weak (empty set is allowed) ordered partition $\pi = (B_1...., B_{p-n})$ of $[n]$ into $p - n$ blocks, meaning that $B_i \cap B_j = \emptyset$ if $i \neq j$ such that $1 \in B_1$. For $2 \leq i \leq n$ there are $p - n$ choices for j such that $i \in B_j$, so $(p-n)^{n-1}$ choices in all. There is a bijection from the $(p-n)^{n-1}$ weak ordered partitions $\pi = (B_1, ..., B_n)$ of $[n]$ into $p - n$ blocks such that $1 \in B$, together with the choice of $x_1 \in \mathbf{F}_p$ to the set $\mathbf{F}_p^n - \cup_{H \in (S_n)_p} H$. There are $(p-n)^{n-1}$ choices for π and p choices for x_1, so it follows from theorem 3.8 that $\chi\mathcal{S}_n(t) = t(t-1)^{n-1}$ ■

Corollary III.4.2.

$$r(\mathcal{S}_n) = (n+1)^{n-1},\ b(\mathcal{S}_n) = (n-1)^{n-1}.$$

CHAPTER IV: THE THRESHOLD HYPERPLANE ARRANGEMENT

A hyperplane arrangement $\mathcal{T}_n$ in $\mathbb{R}^n$ is called *threshold arrangement* if

$$\mathcal{T}_n = \{x_i + x_j = 0 \ : 1 \leq i < j \leq n\}$$

The finite field method earlier described, developed by Athanasiadis, converts the computation of the characteristics polynomial to a point counting problem. A combination of the results of Athanasiadis and Zaslavsky allows for the computation of the number of regions of several arrangements of interests.

Theorem IV.0.1. *Let n be a positive integer, then there exists a monic polynomial $f_n(x)$ of degree n with integer coefficients such that for any sufficiently large prime $p, \chi_{\mathcal{T}_n}(p) = f_n(p)$.*

Proof. Consider the $\binom{n}{2}$ hyperplanes defined by the equations $x_i + x_j = 0, 1 \leq i < j \leq n$; that is, the threshold arrangement on K^n. For every non-empty subset P_i of these hyperplanes, let A_i be a matrix whose rows are the coefficients of hyperplanes contained in P_i. Then A_i defines the linear map

$$A_i : F_p^n \to F_p^{m_i}$$

where $m_i = \#$ of hyperplanes in P_i. Then $Null(A_i)$ is the intersection of the hyperplanes in P_i. Therefore, we have

$$\begin{aligned}
\chi_{\mathcal{T}_n}(p) &= p^n - |\text{set of all points in the } \tbinom{n}{2} \text{ hyperplanes}| \\
&= p^n - |\text{set of all points in the union of the } \tbinom{n}{2} \text{ hyperplanes}| \\
&= p^n - |\{(x_1, x_2, \cdot, x_n) : x_i + x_j = 0, 1 \leq i < j \leq n\}|
\end{aligned}$$

Using the exclusion-inclusion principle, we have

$$\chi_{\mathcal{T}_n}(p) = p^n - \sum_{A_i \subseteq S} \pm |Null(A_i)|$$

where S is the collection of all the $(2^{\binom{n}{2}} - 1)$ sets of hyperplanes. By the rank-nullity theorem,

$$\begin{aligned} dim(Null(A_i)) &= dim(F_p^n) - dim(Range(A_i)) \\ &= n - rank(A_i) \\ \Longrightarrow Null(A_i)) &= p^{n-rank(A_i)} \end{aligned}$$

Therefore,

$$\chi_{\mathcal{T}_n}(p) = p^n - \sum_{A_i \subseteq S} \pm p^{n-rank(A_i)} = f_n(p)\ ;\ \ p > v_n$$

We know that $rank(A_i)$ is equal to the maximal order of a non-zero minor of A_i. But the set of all minors of A_i is finite. Therefore, if v_n is the maximum prime divisor of one of these minors of A_i, then for every $p > v_n$, we have $rank(A_i)$ independent of p. Also, $rank(A_i) \geq 1$ because the entries of A_i are not all zeros. Therefore,

$$\chi_{\mathcal{T}_n}(p) = O(p^n) = f_n(p)$$

∎

Theorem IV.0.2. *[4] The Characteristic polynomial of the threshold arrangement $\mathcal{T}_n$ is given by*

$$\chi_{\mathcal{T}_n}(p) = \sum_{k=1}^{n} (S(n,k) + nS(n-1,k)) \prod_{k=1}^{k} (p - (2i-1)).$$

Here $S(n,k)$ are Stirlings number of the second kind.

Proof. By the previous results of the finite field method, the characteristics polynomial of

$\mathcal{T}_n$ satisfies, for large odd values of p,

$$\chi_{\mathcal{T}_n}(p) = \#(\mathbb{Z}_p^n \setminus \bigcup_i H_i), H_i \in \mathcal{T}_n$$

This is equivalent to

$$\chi_{\mathcal{T}_n}(p) = |\{(x_1, ..., x_n) \in \mathbb{Z}_p^n : x_i + x_j \neq 0 \text{ for all } 1 \leq i < j \leq n\}|$$

This implies that we need to count the functions $f : [n] \longrightarrow \mathbb{Z}_p$ such that

(1)$|f^{-1}(0)| \leq 1$.

(2)f can take at most one value from each of the sets $\{1, -1\}, \{2, -2\}, ..., \{\frac{p-1}{2}, \frac{-(p-1)}{2}\}$.

We split the count into two cases. If 0 is not attained by f, then all values must be from

$$\{1, -1\} \cup \{2, -2\} \cup ... \cup \{\frac{p-1}{2}, \frac{-(p-1)}{2}\}.$$

with at most one value attained in each set. So, there are

$$\binom{\frac{p-1}{2}}{k} 2^k k! S(n, k)$$

ways for f to attain values from exactly k of these sets. This is because we have $\binom{\frac{p-1}{2}}{k} 2^k$ ways to choose k sets and which elements f should attain and $k!S(n, k)$ to choose the images of the elements of $[n]$ after making this choice. So, the total number of f such that 0 is not attained is

$$\sum_{k=1}^{n} \binom{\frac{p-1}{2}}{k} 2^k.k!.S(n, k)$$

When 0 is attained there are n ways to choose which element of $[n]$ gets mapped to 0, and using a similar logic for choosing the images of other elements, we get that the total number of f where 0 is attained is

$$n.\sum_{k=1}^{n}\binom{\frac{p-1}{2}}{k}2^k.k!.S(n-1,k)$$

So we get that for large p,

$$\begin{aligned}\chi\mathcal{T}_n(p) &= n.\sum_{k=1}^{n}\binom{\frac{p-1}{2}}{k}2^k.k!.S(n,k)+n.\sum_{k=1}^{n}\binom{\frac{p-1}{2}}{k}2^k.k!.S(n-1,k)\\ &= \sum_{k=1}^{n}(S(n,k)+nS(n-1,k))\prod_{k=1}^{k}(p-(2i-1))\end{aligned}$$

Since $\chi\mathcal{T}_n$ is a polynomial, we get the required result. ■

IV.0.1 **Threshold Graphs** Labeled Threshold graphs on n vertices are inductively constructed starting from the empty graph. Vertices labeled $1, ..., n$ are added in a specific order. At each step, the vertex added is either dominant or recessive. A dominant vertex is one that is adjacent to all vertices added before it and a recessive vertex is one that is isolated from all added before it. Figure 2 below shows a construction of a labeled threshold graph on five vertices. Any construction of a labeled threshold graph can be expressed as a *signed permutation* (a permutation whose symbols or numbers may carry a tilde). The construction associated to a signed permutation is the one where the vertices are added in the order given by the permutation such that any number assigned a tilde sign is dominant while others are recessive. For example, the signed permutation associated to the construction of Figure 2 assuming that vertex 2 is added as dominant is $\tilde{2}3\tilde{1}\tilde{4}5$. Clearly, there may be more than one way to construct a particular labeled threshold graph. For example, it can be checked that the construction obtained from $32\tilde{4}\tilde{1}5$ also yields the same graph as that in figure 2. Hence after making a canonical choice of construction for each graph, we can represent labeled threshold graphs by certain signed permutations. Using this logic, we can show that labeled threshold graphs are in bijection

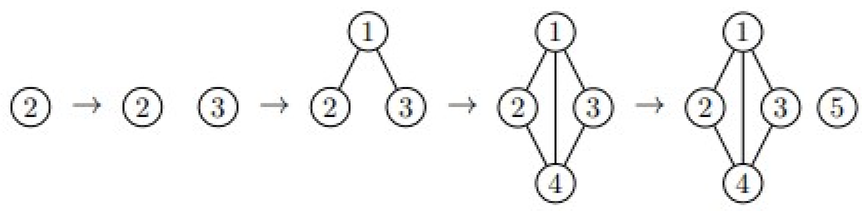

Figure 2: Construction of a labeled threshold graph on 5 vertices

with with certain signed permutations called *threshold pairs in standard form*

Definition 8. A threshold graph is defined recursively as follows

(1) The empty graph is a threshold graph.

(2) A graph obtained by adding an isolated vertex to a threshold graph is a threshold graph. Such a vertex will be called recessive vertex of the graph.

(3)A graph obtained by adding a vertex adjacent to all vertices of a threshold graph is a threshold graph. Such a vertex will be called a dominant vertex of the graph.

Definition 9. A threshold pair in standard form of size n is a signed permutation on $[n]$ that satisfies the following conditions:

(1) The signs of the first two numbers are the same.

(2) If the sign of any two consecutive numbers is the same, the first number is smaller.

Example IV.0.3. $235\tilde{1}\tilde{4}6$ is a threshold pair in standard form whereas $\tilde{2}35\tilde{4}\tilde{1}6$ is not. The labeled threshold graphs on n vertices are in bijection with threshold pairs in standard form of size n. A threshold pair in standard form (π, w) corresponds to a labeled threshold graph obtained by adding vertices in the order given by the permutation such that:

(1) The vertex π_i is added as a dominant vertex if π_i carries a tilde, meaning that $w_i = +$

(2) The vertex π_i is added as a recessive vertex if π_i does not carry a tilde, meaning that $w_i = -$.

We will use this threshold pairs in standard form to show that the regions of the threshold arrangement are in bijection with the labeled threshold graphs.

Theorem IV.0.4. *[4] The regions of the threshold hyperplane arrangement $\mathcal{T}_n$ are in bijection with threshold pairs in the standard form of size n.*

Proof. Given a threshold pair in standard form (π, w), we associate the region of $\mathcal{T}_n$ where for any $1 \leq i < j \leq n$, we have:

(1)$x_{\pi_i} + x_{\pi_j} > 0, w_j = +$.

(2)$x_{\pi_i} + x_{\pi_j} > 0, w_j = -$. This does correspond to a region since a point $(a_1, ..., a_n)$ in $\mathbb{R}^n$ with a_i nonzero, sign of a_i being w_i,and $|a_{\pi_i}| < |a_{\pi_{i+1}}|$ for all $i \in [n-1]$ satisfies the inequality of the threshold region. It can be checked that different pairs in standard form are associated with different threshold regions.

Conversely, given a threshold region R, it is possible to find a point $(a_1, ..., a_n)$ in the region with all coordinate nonzero and having distinct absolute values. Let b_i denote the absolute value of a_i and $w_i^{'}$ be the sign of a_i for all $i \in [n]$. Let $\pi_1^{'}\pi_2^{'}...\pi_n^{'}$ be the permutation on $[n]$ such that $0 < b_{\pi_1^{'}} < b_{\pi_2^{'}} < ... < b_{\pi_n^{'}}$. Let $\pi^{'}$ be the signed permutation of $\pi_1^{'}\pi_2^{'}...\pi_n^{'}$ where $\pi_i^{'}$ is assigned the sign $w_{\pi_i^{'}}$. In fact, we can assume that $w_1^{'} = w_2^{'}$. If $w_1^{'} \neq w_2^{'}$, consider the point $(c_1, ..., c_n)$ where $c_i = a_i$ if $i \neq \pi_i^{'}$ and $c_{\pi_1^{'}} = a_{\pi_1^{'}}$. We claim that this point is also in R by showing that $a_i + a_j > 0$ if and only if $c_i + c_j > 0$ for any $1 \leq i < j \leq n$. This is clear if neither i nor j is $\pi_1^{'}$. The number $a_{\pi_1^{'}}$ has the least absolute value, hence for any $j \neq \pi_1^{'}, a_j$ does not lie between $-a_{\pi_1^{'}}$ and $a_{\pi_1^{'}}$; that is, between $c_{\pi_1^{'}}$ and $a_{\pi_1^{'}}$. Consequently, for any $j \neq \pi_1^{'}, a_j > -a_{\pi_1^{'}}$ if and only if $c_j > -c_{\pi_1^{'}}$. So $(c_1, ..., c_n)$ lies in R and the signed permutation constructed as described above has the first two signs equal. The threshold pair in the standard form obtained by reordering all maximal strings of numbers of the same sign in $(\pi^{'}, w^{'})$ to ascending order is the one associated with this region. Let (π, w) be this threshold pair in standard form associated to R.

Let $1 \leq i < j \leq n$. If $w_j = +$, this means that $a_{\pi_j} > 0$ and hence $b_{\pi_j} = a_{\pi_j}$. If $w_i = +$; that is $a_{\pi_i} > 0$, we automatically get $a_{\pi_j} > -a_{\pi_i}$. If $w_i = -$; that is, $a_{\pi_i} < 0$, we must have $b_{\pi_j} > b_{\pi_i} <$ since π_i and π_j have different signs and hence have the same relative positions in π as they did in π'. In any case, we have $a_{\pi_i} + a_{\pi_j} = 0$. This means that R should lie in the half-space $x_{\pi_i} + x_{\pi_j} < 0$. This shows that R is the same region as the one associated to (π, w) as described earlier. Hence the two maps described above are inverses. ■

Corollary IV.0.5. There is a bijection between regions of $\mathcal{T}_n$ and labeled threshold graphs on n vertices.

Proof. Combining the bijections described above gives us a bijection between regions of $\mathcal{T}_n$ and labeled threshold graphs on n vertices where, for any $1 \leq i < j \leq n$, there is an edge between the vertices of i and j in the labeled threshold graph if and only if $x_i + x_j > 0$ in the corresponding region. ■

Proposition IV.0.6. *[8] Let p be an odd integer. Then exponential generating function of $\mathcal{T}_n$ is given by*

$$\sum_{n \geq 0} \chi_{\mathcal{T}_n}(p) \frac{x^n}{n!} = (1+x)(2e^x - 1)^{\frac{p-1}{2}}$$

Proof. We know that

$$\chi_{\mathcal{T}_n}(p) = \sum_{k=1}^{n} (S(n,k) + nS(n-1,k)) \prod_{i=1}^{k} (p - (2i-1))$$

Therefore,

$$\begin{aligned} \sum_{n \geq 0} \chi_{\mathcal{T}_n}(p) \frac{x^n}{n!} &= \sum_{n \geq 0} \sum_{k=1}^{n} (S(n,k) + nS(n-1,k)) \prod_{i=1}^{k} (p - (2i-1)) \frac{x^n}{n!} \\ &= \sum_{k=1}^{n} \sum_{n \geq 0} f(k) S(n,k) \frac{x^n}{n!} + \sum_{k=1}^{n} \sum_{n \geq 1} f(k) nS(n-1,k) \frac{x^n}{n!} \end{aligned}$$

where

$$f(k) = \prod_{i=1}^{k} (p - (2i-1))$$

Now,

$$
\begin{aligned}
\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} &= \sum_{n\geq 0}\sum_{i\geq 0}\frac{1}{k!}(-1)^{k-i}\binom{k}{i} i^n \frac{x^n}{n!} \\
&= \sum_{i\geq 0}\frac{1}{k!}\sum_{n\geq 0}(-1)^{k-i}\binom{k}{i}\frac{(xi)^n}{n!} \\
&= \sum_{i\geq 0}\sum_{i\geq 0}\frac{1}{k!}(-1)^{k-i}\binom{k}{i} e^{xi} \\
&= \frac{(-1)^k}{k!}\sum_{i\geq 0}\binom{k}{i}(-e^x)^i \\
&= \frac{(-1)^k}{k!}(1-e^x)^k
\end{aligned}
$$

Also,

$$
\begin{aligned}
\sum_{n\geq 1} nS(n-1,k)\frac{x^n}{n!} &= x\sum_{n\geq 1} S(n-1,k)\frac{x^{n-1}}{(n-1)!} \\
&= x\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} \\
&= \frac{x(-1)^k}{k!}(1-e^x)^k
\end{aligned}
$$

Since p is odd, let $p = 2r-1, r = 1, 2, ...,$ then

$$
\begin{aligned}
f(k) &= \prod_{i=1}^{k}(p-(2i-1)) = \prod_{i=1}^{k} 2(r-i) \\
&= 2^k \prod_{i=1}^{k}(r-i) \\
&= 2^k \frac{1}{r}\binom{r}{k} k!(r-k)
\end{aligned}
$$

Putting these together, we have

$$\begin{aligned}
\sum_{n\geq 0}\chi_{\mathcal{T}_n}(p)\frac{x^n}{n!} &= \sum_{k\geq 1}\frac{(-1)^k}{k!}(1-e^x)^k 2^k \frac{1}{r}\binom{r}{k}k!(r-k) \\
&+ \sum_{k\geq 1}\frac{x(-1)^k}{k!}(1-e^x)^k 2^k \frac{1}{r}\binom{r}{k}k!(r-k) \\
&= (1+x)\sum_{k\geq 1}(2e^x-2)^k\frac{1}{r}\binom{r}{k}(r-k) \\
&= (1+x)\sum_{k\geq 1}(2e^x-2)^k\binom{r-1}{k} \\
&= (1+x)(2e^x-1)^{r-1} \\
&= (1+x)(2e^x-1)^{\frac{p+1}{2}-1} = (1+x)(2e^x-1)^{\frac{p-1}{2}}
\end{aligned}$$

■

Let $[x^n]f(x)$ be read as the coefficient of x^n in $f(x)$, then from proposition 5.6, we see that

$$\begin{aligned}
\chi_{\mathcal{T}_n}(p) &= [\frac{x^n}{n!}]\ (1+x)(2e^x-1)^{\frac{p-1}{2}} \\
\implies \chi_{\mathcal{T}_n}(p) &= [\frac{x^n}{n!}]\ (2e^x-1)^{\frac{p-1}{2}} + [\frac{x^n}{n!}]\ x(2e^x-1)^{\frac{p-1}{2}} \\
&= [\frac{x^n}{n!}]\ \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ (2e^x)^r\binom{\frac{p-1}{2}}{r} + [\frac{x^{n-1}}{n!}]\ \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ (2e^x)^r\binom{\frac{p-1}{2}}{r} \\
&= \sum_{r\geq 0}(-1)^{\frac{n-1}{2}-r}\ [\frac{x^n}{n!}]\ (2e^x)^r\binom{\frac{p-1}{2}}{r} + \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ [\frac{x^{n-1}}{n!}]\ (2e^x)^r\binom{\frac{p-1}{2}}{r} \\
&= \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ 2^r r^n\binom{\frac{p-1}{2}}{r} + \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ [\frac{x^{n-1}}{n!}]\ 2^n d\ r^{n-1}\binom{\frac{p-1}{2}}{r}
\end{aligned}$$

Therefore,

$$\chi_{\mathcal{T}_n}(p) = \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ 2^r r^{n-1}\binom{\frac{p-1}{2}}{r}(r+n)$$

The last equation is a polynomial in p of degree n.

Special Cases:

For $n = 3 : (1, 6, 4)$, we have

$$\chi_{\mathcal{T}_3}(p) = \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\, 2^r \binom{\frac{p-1}{2}}{r}(r^3 + 3r^2)$$

Let $(f(x))^{(k)} = \frac{d^k}{dx^k}(f(x))$, and $[f(x)]_{x=k} = f(k)$, then

$$\begin{aligned}
\chi_{\mathcal{T}_3}(p) &= \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\, 2^r \binom{\frac{p-1}{2}}{r}[(x^r)^{(3)} + 6(x^r)^{(2)} + 4(x^r)^{(1)}]_{x=1} \\
&= [\frac{d^3}{dx^3}\sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\, 2^r x^r \binom{\frac{p-1}{2}}{r} + 6\frac{d^2}{dx^2}\sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\, 2^r x^r \binom{\frac{p-1}{2}}{r} \\
&\quad +4\sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\, 2^r x^r \binom{\frac{p-1}{2}}{r}]_{x=1} \\
&= [((2x-1)^{\frac{p-1}{2}})^{(3)} + 6((2x-1)^{\frac{p-1}{2}})^{(2)} + 4((2x-1)^{\frac{p-1}{2}})^{(1)}]_{x=1} \\
&= (2^3)\ \frac{p-1}{2}\ \frac{p-3}{2}\ \frac{p-5}{2} + 6\ (2^2)\ \frac{p-1}{2}\ \frac{p-3}{2} + 4\ (2)\ \frac{p-1}{2} \\
&= (p-1)(p-3)(p-5) + 6(p-1)(p-3) + 4(p-1) \\
&= (p-1)^3
\end{aligned}$$

For $n = 4 : (1, 10, 19, 5)$, we have

$$\begin{aligned}
\chi_{\mathcal{T}_4}(p) &= \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ 2^r\binom{\frac{p-1}{2}}{r}(r^4+4r^3) \\
\chi_{\mathcal{T}_4}(p) &= \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ 2^r\binom{\frac{p-1}{2}}{r}[(x^r)^{(4)}+10(x^r)^{(3)}+19(x^r)^{(2)}+5(x^r)^{(1)}]_{x=1} \\
&= [((2x-1)^{\frac{p-1}{2}})^{(4)}+10((2x-1)^{\frac{p-1}{2}})^{(3)}+19((2x-1)^{\frac{p-1}{2}})^{(2)} \\
&+ 5((2x-1)^{\frac{p-1}{2}})^{(1)}]_{x=1} \\
&= (p-1)(p-3)(p-5)(p-7)+10(p-1)(p-3)(p-5) \\
&+ 19(p-1)(p-3)+5(p-1) \\
&= p^4-6p^3+15p^2-17p+7
\end{aligned}$$

Theorem IV.0.7. *Let the finite sequence* $\{\alpha_k\}_{k=0}^{n-1}$ *be such that* $r^n + nr^{n-1} = [\sum_{k=0}^{n-1}\alpha_k(x^r)^{(n-k)}]_{x=1},\ 0\leq k\leq n-1,\ n-k\leq r,$ *then*

$$\chi_{\mathcal{T}_n}(p) = \sum_{k=0}^{n-1}\alpha_k(\prod_{r=1}^{n-k}(p-(2r-1)).$$

Proof.

$$\begin{aligned}
\chi_{\mathcal{T}_n}(p) &= \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ 2^r\binom{\frac{p-1}{2}}{r}(r^n+nr^{n-1}) \\
&= \sum_{r\geq 0}(-1)^{\frac{p-1}{2}-r}\ 2^r\binom{\frac{p-1}{2}}{r}[\sum_{k=0}^{n-1}\alpha_k(x^r)^{(n-k)}]_{x=1} \\
&= [(\sum_{k=0}^{n-1}\sum_{r\geq 0}\alpha_k(-1)^{\frac{p-1}{2}-r}\ \binom{\frac{p-1}{2}}{r}(2x)^r)^{(n-k)}]_{x=1} \\
&= [(\sum_{k=0}^{n-1}\alpha_k(2x-1)^{\frac{p-1}{2}})^{(n-k)}]_{x=1} \\
&= \sum_{k=0}^{n-1}\alpha_k[((2x-1)^{\frac{p-1}{2}})^{(n-k)}]_{x=1} \\
&= \sum_{k=0}^{n-1}\alpha_k\ 2^{n-k}\ \frac{p-1}{2}\ \frac{p-3}{2}\ \frac{p-5}{2}\cdots\frac{p-2n+2k+1}{2} \\
&= \sum_{k=0}^{n-1}\alpha_k 2^{n-k}\prod_{r=1}^{n-k}\frac{p-(2r-1)}{2} \\
&= \sum_{k=0}^{n-1}\alpha_k(\prod_{r=1}^{n-k}(p-(2r-1))
\end{aligned}$$

■

For $n=3, \alpha_k = 1, 6, 4$, for $n=4, \alpha_k = 1, 10, 19, 5$. Therefore, computing $\chi_{\mathcal{T}_n}(p)$ for all odd p is a problem reduced to computing the finite sequence α_k which are the coefficients of $(x^r)^{(n-k)}$ such that $r^n + nr^{n-1} = [\sum_{k=0}^{n-1}\alpha_k(x^r)^{(n-k)}]_{x=1}$.

Theorem IV.0.8. *In general, for $n = n$, let $\alpha_{m,n}$ be such that*

$$r^n + nr^{n-1} = [\sum_{m=0}^{n-1}\alpha_{m,n}(x^r)^{(n-m)}]_{x=1},\ 0\leq n-m\leq r$$

Then,

$$\alpha_{k,n} = -\sum_{m=0}^{k-1}\alpha_{m,n}(-1)^{k-m}S_{k-m,n-m-1},\ 2\leq k\leq n$$

where

$$
\begin{aligned}
S_{p,q} &= \sum_{0\le a_1<a_2<\cdots<a_p\le q} a_1a_2\cdots a_p \\
&= \sum_{1\le a_1<a_2<\cdots<a_p\le q} a_1a_2\cdots a_p \\
S_{0,q} &= 1
\end{aligned}
$$

Proof.

$$
[r^k][\sum_{m=0}^{n-1}\alpha_{m,n}(x^r)^{(n-m)}]_{x=1} = \sum_{m=0}^{n-k}\alpha_{m,n}[[r^k](x^r)^{(n-m)}]_{x=1},\ 0\le k\le n-m\le r.
$$

$$
\begin{aligned}
&= \sum_{m=0}^{n-k}\alpha_{m,n}[[r^{n-m-(n-m-k)}](x^r)^{(n-m)}]_{x=1} \\
&= \sum_{m=0}^{n-k}\alpha_{m,n}(-1)^{n-m-k}S_{n-m-k,n-m-1}.
\end{aligned}
$$

In particular, since

$$
[r^n](r^n+nr^{n-1})=1,\ [r^{n-1}](r^n+nr^{n-1})=n.
$$

we have

$$
\alpha_{0,n}=1,\ \alpha_{1,n}=n+S_{1,n-1}=d+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}=S_{1,n}
$$

Also, for $k=0,1,2,\cdots,n-2$, we have the recurrence relation

$$
\sum_{m=0}^{n-k}\alpha_{m,n}(-1)^{n-m-k}S_{n-m-k,n-m-1}=0.
$$

This gives

$$\alpha_{n-k,n} = -\sum_{m=0}^{n-k-1} \alpha_{m,n}(-1)^{n-m-k} S_{n-m-k,n-m-1}.$$

Replacing k by $n-k$ on both sides gives

$$\alpha_{k,n} = -\sum_{m=0}^{k-1} \alpha_{m,n}(-1)^{k-m} S_{k-m,n-m-1},\ 2 \leq k \leq n$$

■

For instance, for $n = 3$, we have

$$\begin{aligned} \alpha_{2,3} &= -\alpha_{0,3}S_{2,2} + \alpha_{1,3}S_{1,1} = -2 + \frac{(3)(4)}{2}.1 = 4,\ \alpha_{1,3} = \frac{(3)(4)}{2} = 6,\ \alpha_{0,3} = 1 \\ \Longrightarrow \chi_{\mathcal{T}_3}(p) &= (p-1)(p-3)(p-5) + 6(p-1)(p-3) + 4(p-1) \end{aligned}$$

For $n = 4$, we have

$$\begin{aligned} \alpha_{0,4} = 1, \alpha_{1,4} &= 10,\ \alpha_{2,4} = -(\alpha_{0,4}S_{2,3} - \alpha_{1,4}S_{1,2}),\ S_{2,3} = (1)(2) + (1)(3) + (2)(3) = 11 \\ \Longrightarrow \alpha_{2,4} &= -(11 - 10(3)) = 19 \\ \alpha_{3,4} &= -(-\alpha_{0,4}S_{3,3} + \alpha_{1,4}S_{2,2} - \alpha_{2,4}S_{1,1}) \\ &= -(-(1)(2)(3) + 10(1)(2) - 19(1)) = 5 \\ \Longrightarrow \chi_{\mathcal{T}_4}(p) &= (p-1)(p-3)(p-5)(p-7) + 10(p-1)(p-3)(p-5) \\ &+ 19(p-1)(p-3) + 5(p-1) \end{aligned}$$

[4]

n	$\chi_{\mathcal{T}_n}(t)$	$r(\mathcal{T}_n)$
2	$t^2 - t$	2
3	$t^3 - 3t^2 + 3t - 1$	8
4	$t^4 - 6t^3 + 15t^2 - 17t + 7$	46
5	$t^5 - 10t^4 + 45t^3 - 105t^2 + 120t - 51$	332
6	$t^6 - 15t^5 + 105t^4 - 410t^3 + 900t^2 - 1012t + 431$	2874
7	$t^7 - 21t^6 + 210t^5 - 1225t^4 + 4340t^3 - 9058t^2 + 9961t - 4208$	29024
8	$t^8 - 28t^7 + 378t^6 - 3066t^5 + 15855t^4 - 52234t^3 + 104433t^2 - 112163t + 46824$	334982
9	$t^9 - 36t^8 + 630t^7 - 6762t^6 + 47817t^5 - 226380t^4 + 703815t^3 - 1355427t^2 + 1422483t - 586141$	4349492
10	$t^{10} - 45t^9 + 990t^8 - 13560t^7 + 125265t^6 - 801507t^5 + 3541125t^4 - 10491450t^3 + 19546335t^2 - 20068391t + 8161237$	62749906

Figure 3: Characteristic polynomial and the number of regions of $\mathcal{T}_n$ for $n \leq 10$

For $n = 5$, we have

$$\begin{aligned}
\alpha_{0,5} &= 1, \alpha_{1,5} = 15, \alpha_{2,5} = -\alpha_{0,5}S_{2,4} + \alpha_{1,5}S_{1,3} \\
&= -((1)(2) + (1)(3) + (1)(4) + (2)(3) + (2)(4) + (3)(4)) + (15)(6) = 55 \\
\alpha_{3,5} &= \alpha_{0,5}S_{3,4} - \alpha_{1,5}S_{2,3} + \alpha_{2,5}S_{1,2} \\
&= ((1)(2)(3) + (1)(2)(4) + (1)(3)(4) + (2)(3)(4)) - 15((1)(2) \\
&+ (1)(3) + (2)(3)) + 55(3) = 50 \\
\alpha_{4,5} &= -\alpha_{0,5}S_{4,4} + \alpha_{1,5}S_{3,3} - \alpha_{2,5}S_{2,2} + \alpha_{3,5}S_{1,1} \\
&= -4! + 15(3!) - 55(2!) + 50(1!) = 6 \\
\chi_{\mathcal{T}_5}(p) &= (p-1)(p-3)(p-5)(p-7)(p-9) + 15(p-1)(p-3)(p-5)(p-7) \\
&+ 55(p-1)(p-3)(p-5) + 50(p-1)(p-3) + 6(p-1)
\end{aligned}$$

CHAPTER V: THE COMPLETE HYPERPLANE ARRANGEMENT

V.1 Chapter Five: The Complete Arrangement

The complete (aka resonance) arrangement $\mathcal{A}_n$ is the arrangement of hyperplanes in $\mathbb{R}^n$ given by all hyperplanes of the form $\sum_{i \,\epsilon\, I} x_i = 0$ where I is a nonempty subset of $[n]$. We consider the characteristics polynomial $\chi_{\mathcal{A}_n}(t)$ of the resonance arrangement whose whose value at -1 is of particular interest as given by Zaslavsky theorem. No formula is known for either the characteristics polynomial or its evaluation at -1. The coefficients of the characteristics polynomial are also equal to the *Betti numbers* of the complexified hyperplane arrangement; that is the coefficients of t^{n-i} is denoted by the Betti number $b_i(\mathcal{A}_n)$. Explicit formulas are known for the Betti numbers up to $b_3(\mathcal{A}_n)$.

Definition 10. The resonance arrangement $\mathcal{A}_n$, for $n > 0$, is the hyperplane arrangement in $\mathbb{R}^n$ defined by the equations

$$\mathcal{A}_n = \{\sum_{i \,\epsilon\, I} x_i = 0 | I \subseteq [n]\}$$

Equivalently, it is the collection of hyperplanes whose normal vectors are nonzero with entries on $\{0, 1\}$. Moreover, let R_n be the total number of regions created by the hyperplanes in $\mathcal{A}_n$ which is equal to the number of connected regions in the set obtained from $\mathbb{R}^n$ by removing the solution space of all the equations of $\mathcal{A}_n$. Recall that the characteristics polynomial of a hyperplane arrangement $\mathcal{A}$ over a field is given by

$$\chi_{\mathcal{A}}(t) = \sum_{T \subseteq \mathcal{A}} (-1)^{|T|} t^{n-r(T)},\ r(T) = \text{rank of } T$$

By Zaslavsky's theorem, $\chi_{\mathcal{A}}(-1) = R_n$.

V.1.1 Resonance Hyperplane Arrangement Over the ring $\mathbb{Z}_n$

[10] We speak of hyperplane arrangement over a field, not a ring. Therefore, we can talk about hyperplane arrangement over the finite field $\mathbb{Z}_p$ for a prime p, and more generally for $\mathbb{Z}_n$, even though

$\mathbb{Z}_n$ is not a field if n is not prime. This is because any hyperplane arrangement over $\mathbb{Q}$ can be viewed as one over $\mathbb{Z}_n$ by multiplying the equations of the hyperplanes by a sufficiently large integer. This gives a set of hyperplanes over the ring $\mathbb{Z}_n$. Therefore, this work will routinely refer to hyperplane arrangement over $\mathbb{Z}_n$ without any confusion arising.

Definition 11. Let $\mathbb{Z}_n$ denote the ring of integers modulo n. A vector $(v_1, v_2, ..., v_d)$ in $\mathbb{Z}_n^d$ is said to be a zero-sum free $d-$tuple if there is no nonempty subset of its components whose sum is zero in $\mathbb{Z}_n$. Let the collection of all such be denoted as $\mathcal{G}(n, d)$, and let $\alpha(n, d)$ denote the cardinality of $\mathcal{G}(n, d)$.

Theorem V.1.1. *For any fixed* $d, \alpha(p, d) = \chi(\mathcal{A}_d; p)$ *for all* p *such that no 0-1 matrix of size* d *has determinant equal to a nonzero multiple of* p.

Proof. For each subset $S \subset \{0, 1\}^d$, let $f(S)$ denote the number of $d-$tuples v which satisfies $v.x = 0$ for each vector $x \epsilon S$; that is, each designated subset sum is zero. We want to find $f(\subset \{0, 1\}^d \setminus \{(0, ..., 0)\})$. By the principle of inclusion-Exclusion, this is the sum over all subsets of k hyperplanes of $(-1)^k$ times the number of points in the intersection of the k hyperplanes., over all $k \leq d$. The number of points in the intersection of the hyperplanes is given by n^{d-r}, where r is the rank over $\mathbb{F}_p$ of the 0-1 matrix A with rows given by the coordinate vectors of the hyperplanes. We claim that r is also the rank of A over $\mathbb{Q}$. Indeed, since all determinants of minors are 0 mod p if and only if they are zero in $\mathbb{Z}$, the rank must be the same. Thus, they are p^{d-r} solutions, and therefore, $\alpha(p, d)$ is a polynomial in p given by

$$\alpha(p, d) = \sum_{S \subseteq \mathbb{A}_d} p^{d-r(S)},$$

where $r(S) =$ rank of S. This is exactly the same as the definition of $\chi(\mathcal{A}_d; p)$. Therefore,

$$\alpha(p, d) = \chi(\mathcal{A}_d; p)$$

■

Theorem V.1.2. *Let d be a positive integer, then there exists a monic polynomial $f_d(x)$ of degree d with integer coefficients such that for any sufficiently large prime $p, \alpha(p,d) = f_d(p)$.*

Proof. Consider the $2^d - 1$ hyperplanes defined by the equations $\sum_{i \,\epsilon\, I} x_i = 0$ where I is a nonempty subset of $[n]$; that is, the complete arrangement on K^d. For every non-empty subset P_i of these hyperplanes, let A_i be a matrix whose rows are the coefficients of hyperplanes contained in P_i. Then A_i defines the linear map

$$A_i : F_p^d \to F_p^{m_i}$$

where $m_i = \#$ of hyperplanes in P_i. Then $Null(A_i)$ is the intersection of the hyperplanes in P_i. Therefore, we have

$$\begin{aligned} \alpha(p,d) &= p^d - |\text{set of all points in the } 2^d - 1 \text{ hyperplanes}| \\ &= p^d - |\text{set of all points in the union of the } 2^d - 1 \text{ hyperplanes}| \\ &= p^d - |\{(x_1, x_2, \cdot, x_d) : \sum_{i \,\epsilon\, I} x_i = 0, 1 \leq i < j \leq d\ , I \subseteq [n]\}| \end{aligned}$$

Using the exclusion-inclusion principle, we have

$$\alpha(p,d) = p^d - \sum_{A_i \subseteq S} \pm\, |Null(A_i)|$$

where S is the collection of all the $(2^{2^d - 1} - 1)$ nonempty intersections of hyperplanes. By the rank-nullity theorem,

$$\begin{aligned} dim(Null(A_i)) &= dim(F_p^d) - dim(Range(A_i)) \\ &= d - rank(A_i) \\ \implies Null(A_i)) &= p^{d - rank(A_i)} \end{aligned}$$

Therefore,

$$\alpha(p,d) = p^d - \sum_{A_i \subseteq S} \pm\, p^{d-rank(A_i)} = f_d(p)\ ;\ \ p > v_d$$

We know that $rank(A_i)$ is equal to the maximal order of a non-zero minor of A_i. But the set of all minors of A_i is finite. Therefore, if v_d is the maximum prime divisor of one of these minors of A_i, then for every $p > v_d$, we have $rank(A_i)$ independent of p. Also, $rank(A_i) \geq 1$ because the entries of A_i are not all zeros. Therefore,

$$\alpha(p,d) = O(p^d) = f_d(p)$$

■

We know already that for sufficiently large prime p, we have $\alpha(p,d) = \chi(\mathcal{A}_d; p)$; also,since $\alpha(p,d) = f_d(p)$ for sufficiently large prime p, and both $\chi(\mathcal{A}_d; p)$ and $f_d(p)$ are polynomials of degree d, then we have

$$\alpha(p,d) = \chi(\mathcal{A}_d; p) = f_d(p)$$

By the finite field formula, we have

$$\chi(\mathcal{A}_d; p) = \alpha(p,d) = n^d - |\mathbb{Z}_n^d \setminus \bigcup_{i}^{2^d-1} H_i|$$

Therefore, we have

$$\alpha(p,d) = n^d - |\mathbb{Z}_n^d \setminus \bigcup_{i}^{2^d-1} H_i|$$

From our proof above, we refer to a matrix A_i whose rows correspond to the coefficients of the hyperplanes of the set $P_i \subseteq S$. The full 0-1 matrix corresponding to S is the 0-1 $d \times (2^d - 1)$ matrix H_d whose columns are all non-zero binary vectors in $\mathbb{F}_p^d$. For instance,

$$H_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

From the proof of the theorem above, the coefficients of $x^i (0 \leq i \leq d-1)$ of our polynomial $f_d(x)$ is obtained from all possible null spaces of dimension i in the alternating sum giving by the exclusion-inclusion principle. This gives the following proposition.

Proposition V.1.3. *The coefficient of $x^i (0 \leq i \leq d-1)$ of the polynomial $f_d(x)$ is given by $\sum_{j=1}^{2^d-1} (-1)^j m(j,i)$, where $m(j,i)$ is the number of subsets of j columns of H_d that span a $d-$dimensional subspace of $\mathbb{F}_p^d$.*

A closed formula for these coefficients is hard to obtain with the exception of one case. The coefficient of x^{d-1} is given by $\sum_{j=1}^{2^d-1} (-1)^j m(j, d-1)$ But $m(1, d-1) = 2^d - 1$ because every column is nonzero and there are $2^d - 1$ columns, and $m(i, d-1) = 0$ for all $i > 1$ because a subset of size more than 1 cannot span a 1-dimensional subspace when working with binary vectors. Thus, the coefficients of x^{d-1} is $1 - 2^d$. We can also deduce from this formula that the constant term in $\alpha(p,d)$ is $\sum_{j=1}^{2^d-1} (-1)^j m(j, 0)$, where $m(j,0)$ is the number of subsets of j columns that span the whole space $\mathbb{F}_p^d$. Part of what makes a closed formula for $\alpha(p,d)$ is the fact that it is not easy to get an exact value for threshold v_d mentioned in the last proof. Therefore, the best we can hope for is a bound. This leads to the following proposition.

Proposition V.1.4. *[10] Let d be a positive integer. Then, $\alpha(p,d) = f_d(p)$ for all $p > d^{d/2}$. In other words, $v_d \leq d^{d/2}$*

Proof. Recall that if the prime p exceeds the value of all possible minors obtained from H_d, then $\alpha(p,d) = H_d(p)$ Now, note that even when computing all possible minors of H_d, it is enough to consider all $d \times d$ submatrices because the minors from $j \times j, j \leq d$ submatrices will be picked by those of $d \times d$ submatrices as can be seen by simply adding appropriate 0s and 1s. The maximum possible minor of a $d \times d$ submatrix of H_d can be bounded by Hadamard's inequality which states that for any matrix A, $det(A) \leq \prod_{i=1}^{d} ||_i||$, where $||A_i||$ is the Eucleadian norm of the ith column of A. Applying this inequality to a $d \times d$ submatrix of M of H_d gives

$$det(M) \leq \prod_{i} ||M_i|| \leq \prod_{i} \sqrt{d} = d^{d/2}$$

This means that for all $p > d^{d/2}$ we have $\alpha(p, d) = f_d(p)$. ∎

The following theorem offers a generalization to Theorem 2.2.

Theorem V.1.5. *[10] For every positive integer d there exists a monic polynomial $f_d(x)$ of degree d with integer coefficients such that $\alpha(n, d) = f_d(n)$ for all n that are relatively prime to the determinant of any $d \times d$ binary matrix.*

Proof. The proof is much like the proof of Theorem 2.2, but this time we work with $\mathbb{Z}_n$. Following a similar pattern, we have

$$\alpha(n, d) = |\mathbb{Z}_n^d \setminus \bigcup_{i}^{2^d-1} H_i|$$

where $H_i(1 \leq i \leq 2^d - 1)$ are all the possible nonempty intersections of all the hyperplanes A_i in the complete arrangement $\mathcal{A}$. Using the exclusion-inclusion principle, that gives

$$\alpha(n, d) = n^d - \sum_{i} \pm |Ker(A_i)|$$

Since we are no longer working over a finite field, we look for the cardinality of the kernel of the group homomorphisim

$$A_i : \mathbb{Z}_n^d \longrightarrow \mathbb{Z}_n^{m_i} \; m_i \leq d$$

Since n is relatively prime to the determinant of any $d \times d$ binary matrix, in particular, it will also be relatively prime to any minor of A_i. These minors are exactly the collections of all the scalars that we multiply with in the process of converting $A_i's$ into their RRE forms over $\mathbb{Z}_n$. This will show that under the given condition on n, $|Ker(A_i)| = n^{d-rank(A_i)}$, where rank is computed over the reals. Therefore, we have a polynomial function $f_d(x) = x^d - \sum_i \pm x^{d-rank(A_i)}$ such that $\alpha(n,d) = f_d(n)$ for all n that are relatively prime to the determinant of any $d \times d$ binary matrix. ■

Corollary V.1.6. For every positive integer d, we have $\alpha(n,d) = f_d(n)$ if $gcd(n, \lceil d^{d/2} \rceil) = 1$. In particular, $\alpha(p,d) = f_d(p)$ for all sufficiently large primes p.

Proof. Since the determinant of any $d \times d$ matrix is at most $d^{d/2}$, if we choose n such that $\gcd(n, \lceil d^{d/2} \rceil) = 1$, then n will be relatively prime to the determinant of any $d \times d$ binary matrix and we can apply theorem 2.5 above. ■

Bounds

The exact formula of $\alpha(n,d)$ for $d > \frac{n}{2}$ is known [10]. However, an exact formula of $\alpha(n,d)$ eludes us for now, the best we could hope for is to give some nice bounds. Before that the following definition is necessary.

Definition 12. Let $\mathcal{I}(n,d) = \{(x_1, ..., x_d) \,\epsilon\, \mathcal{G}(n,d) : gcd(x_1, ..., x_d, n) = 1\}$,$\mathcal{R}(n,d) = \mathcal{G}(n,d) \setminus \mathcal{I}(n,d)$, and $\beta(n,d) = |\mathcal{I}(n,d)|$. Then,

$$\mathcal{G}(n,d) = \mathcal{I}(n,d) \cup \mathcal{R}(n,d)\ ,\ \mathcal{I}(n,d) \cap \mathcal{R}(n,d) = \emptyset.$$

Proposition V.1.7. *We have* $0 \leq \beta(n,d) \leq \alpha(n,d) \leq (n-1)^{d-1}(n-2)$ *for all* $d \geq 3$ *and all* $n \geq 3$.

Proof. The relation $0 \leq \beta(n,d) \leq \alpha(n,d)$ is clear by definition. For the upper bound,let $(x_1, ..., x_d)$ be an element in $\mathcal{G}(n,d)$. Then we know that for all $1 \leq i \leq d-1$, we have $x \neq 0$ and $x_d \neq 0$ or $x_d \neq -(x_1 + ... + x_{d-1})$. So we have at most $n-2$ choices for x_d and $n-1$ for the rest. ■

If we restrict to a prime field, we can obtain a sharper bound as follows in the next proposition.

Proposition V.1.8. *For all primes $p \geq 3$ and all $d \geq 3$, we have* $\beta(p, d) \leq \alpha(p, d) \leq (p-1)(p-2)^{d-2}(p-3)$.

Proof. In $\mathbb{Z}_p^d$, every zero-sum-free $d-$tuple is also irreducible because the condition $\gcd(x_1, ..., x_d, p) = 1$ is automatically satisfied. For $1 \leq i \leq p-1$, let $\mathcal{G}(p, d; i)$ be the set of all $d-$tuples in $\mathcal{G}(p, d)$ where the first component is i. Since p is prime, multiplication by i induces a bijection between $\mathcal{G}(p, d; 1)$ and $\mathcal{G}(p, d; i)$. Thus, we have $|\mathcal{G}(p, d)| = (p-1)|\mathcal{G}(p, d; 1)|$. Note that

$$\mathcal{G}(p, d) \subseteq \{(1, x_2, ..., x_d) : x_1 \neq 0 \text{ or } -1\}$$

We have at most $p-2$ choices for each x_i. In addition,$x_d \neq -(1 + x_2 + ... + x_{d-1})$, and this quantity cannot be zero or -1 when $p \geq 3$. This completes the proof. ■

Now we consider lower bounds. The following observation can be used to get some recurrence relation for the lower bound of $\alpha(n, d)$. Let m be a divisor of n. Then the natural ring homomorphism $\mathbb{Z}_n \longrightarrow \mathbb{Z}_m$ extends to a ring homomorphism $\psi : \mathbb{Z}_n^d \longrightarrow \mathbb{Z}_m^d$. It is clear that $\mathbf{x} = (x_1, ..., x_d)$ is in $\mathcal{G}_m^d$.

Proposition V.1.9. *For all m and n such that m divides n, we have* $\alpha(n, d) \geq (n/m)^d \alpha(m, d)$.

Proof. The kernel of the homomorphism $\psi : \mathbb{Z}_n^d \longrightarrow \mathbb{Z}_m^d$ has order $(n/m)^d$. If (m) is the ideal generated by m in $\mathbb{Z}_n$, then the kernel is $(\mathbb{Z}_n/(m))^d$. So every zero-sum-free $d-$tuple in $\mathbb{Z}_m^d$ pulls back to $(n/m)^d$ zero-sum-free $d-$tuples in $\mathbb{Z}_n^d$. This completes the proof. ■

CHAPTER VI: CONCLUSION

In conclusion, this thesis has provided a moderate introduction to the highly vast and interesting topic of *Hyperplane Arrangement Over the Ring of Integers modulo n*. In particular, it has explored the braid, graphical, shi, threshold, and complete hyperplane arrangements by highlighting their individual characteristic polynomials, apart from the complete arrangement. It also shows how these arrangements relate to other interesting mathematical objects such as permutations, set partitions, threshold graphs and graph coloring. By adopting the finite field method to analyse these arrangements, this work further reveals a natural connection between the problem of counting connected regions created by hyperplane arrangements and the combinatorial problem of enumerating points in some finite field or ring.

REFERENCES

[1] R.P. Stanley. An Introduction to Hyperplane Arrangements. *Geometric Combinatorics* (E. Miller, V. Reiner, and B. turmfels, eds), IAS/Park City Mathematics Series, vol.13, Amer. Math.Soc., Providence, 2007, 389-496.

[2] L. Lionel. Algebraic Combinatorics. *Lecture Note by Jacob Bower*, (2011). Lecture 15, 18.312.

[3] A. Drew and R. Brendon. The Shi Arrangement and the Ish Arrangement. *Departement of Mathematics, University of Miami, Coral Gables, FL,33146; Departement of Mathematics, MIT, Cambridge, MA,02139*, (2010)

[4] D. Priyavrat, M. Krishna,S. Anurag. A Combinatorial Statistic for Labeled Threshold Graphs. *enumerative combinatorics and applications*, ECA **1:3** Article #S2R22.

[5] C.A. Athanasiadis. Characteristic Polynomial of Subspace Arrangements and Finite Fields. *Adv. Math*, 122 (1996), 193-233.

[6] C.A. Athanasiadis. On Free Deformations of the Braid Arrangements. *European J. Combin.*, 19 (1997), 7-18.

[7] H. Crapo, G.C. Rota. On the Foundations of Combinatorial Theory. *Combinatorial Geometries*, MIT Press, Cambridge, MA(1970). Preliminary edition.

[8] S. Seunghyun. The Catalan Threshold Arrangement. *Journal of Integer Sequences*, 20 (2017), Article 17.1.1.

[9] H.S. Wilf Generatingfunctionology. *Department of mathematics, University of Pennsylvania, Philadelphia*, Academic Press, Inc,(1990 and 1994).

[10] S.K. Chebolu and P.A. Sissokho. Zero-Sum-Free Tuples and Hyperplane Arrangements. *Integers* 22 (2022), Paper No. A12, 25 pp *math.NT*, arXiv:2201.01714v1.